$\mathcal{H}_\infty$ Control of Nonlinear Systems: A Class of Controllers

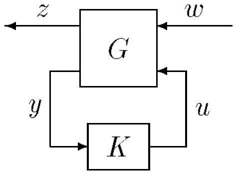
Wei-Min Lu* and John C. Doyle†

Abstract

The standard state space solutions to the $\mathcal{H}_\infty$ control problem for linear time invariant systems are generalized to nonlinear time-invariant systems. A class of nonlinear $\mathcal{H}_\infty$-controllers are parameterized as nonlinear fractional transformations on contractive, stable free nonlinear parameters. The investigation is carried out in a methodical fashion. As in the linear case, the $\mathcal{H}_\infty$ control problem is solved by its reduction to four simpler special problems, together with a separation argument. Another byproduct of this approach is that the sufficient conditions for $\mathcal{H}_\infty$-control problem to be solved are also derived with this machinery. The solvability for nonlinear $\mathcal{H}_\infty$-control problem requires positive definite solutions to two parallel decoupled Hamilton-Jacobi inequalities and these two solutions satisfy an additional coupling condition. An illustrative example, which deals with a passive plant, is given at the end.

1 Introduction

An important issue in control system synthesis is to design a control system which attenuates the effects of external disturbances on some desired signals. The standard configuration we will consider is



where G is the generalized plant and K is the controller to be designed; w is the vector of exogenous disturbance inputs and u is the vector of control inputs; z is the the vector of outputs to be regulated; and y is the vector of measured outputs based on which the control action is generated. A standard approach is to treat these signals in some normed space. If the related signals are in $\mathcal{L}_2$-space, the performance of a system is measured by $\mathcal{L}_2$-gain, or $\mathcal{H}_\infty$-norm for a linear system. The $\mathcal{H}_\infty$-control problem is to find the controller(s) which stabilizes the closed loop system and minimizes its $\mathcal{L}_2$-gain. The general linear time invariant case was first solved by Doyle (1984), but the resulting state-space formulas and derivations were substantially streamlined in Doyle et al. (1989). In particular, it was shown that the $\mathcal{H}_\infty$-control problem, which requires the $\mathcal{H}_\infty$-norm of the closed loop system less

*Electrical Engineering 116-81, California Institute of Technology, Pasadena, CA 91125; `wmlu@hot.caltech.edu`.

†Electrical Engineering/Control and Dynamical Systems 116-81, California Institute of Technology, Pasadena, CA 91125

than 1, is solvable if and only if the unique stabilizing solutions to two parallel algebraic Riccati equations are positive definite and the spectral radius of their product is less than 1. Furthermore, controllers which solve the $\mathcal{H}_\infty$ problem can be parameterized as a linear fractional transformation on a contractive, stable free parameter. The simplicity of this characterization together with its clear connections with traditional methods in optimal control have stimulated several attempts to generalize the $\mathcal{H}_\infty$ results in state space to nonlinear systems. (The reader is referred to Ball and Helton (1988) and Foias-Tannenbaum (1989) for the generalization in the context of operator theory.) We will abuse terminology and use the term nonlinear $\mathcal{H}_\infty$ to describe such efforts.

The study of dynamical systems which have finite $\mathcal{L}_2$-gains can be traced back to at least the early 70's. It is known that the finiteness of $\mathcal{L}_2$-gain and the dissipativity for a dynamical systems are strongly connected. A systematic exploration of general dissipative systems was performed by Willems (1972); some extensions can be found, for instance, in Moylan (1974) and Hill-Moylan (1980). Recently, in the context of $\mathcal{L}_2$-gain analysis, Van der Schaft (1992) extensively studied the $\mathcal{L}_2$-gain of nonlinear time-invariant (NLTI) control-affine systems by using dissipation theory. He reconfirmed that the $\mathcal{L}_2$-gains for a class of systems can be characterized by Hamilton-Jacobi equations (HJEs) or inequalities (HJIs). He also considered the solutions of HJEs in depth in terms of the related Hamiltonian vector field. Based on the analysis results, he investigated the $\mathcal{H}_\infty$ control problem in both state feedback and output feedback cases for a class of NLTI systems. He showed that a sufficient condition for the state feedback $\mathcal{H}_\infty$-control problem to be solvable is that the corresponding HJI has a positive solution. In the output feedback case, he asserted that the $\mathcal{H}_\infty$ control problem is locally solvable if it is solvable for the linearized system. But this assertion requires that the equilibrium point of the related Hamiltonian vector field be hyperbolic. Isidori and Astolfi (1992; Isidori, 1992) developed other sufficient conditions, which are less conservative, for the output control problem to be solvable from a game theoretical point of view. They showed that the solution to the $\mathcal{H}_\infty$-control problem requires the existence of positive definite solutions of two hierarchically coupled HJIs. They also parameterized a class of controllers for the full information (FI) structure (Isidori and Astolfi, 1992).

Ball et al. (1993) worked on the nonlinear $\mathcal{H}_\infty$-control problem from another direction. They derived the necessary conditions for the existence of an output feedback controller such that the HJI related to the closed loop system has a positive smooth solution (specifically, the $\mathcal{H}_\infty$-control problem is solvable). Just as neat as in the linear case, these conditions are that two HJIs have positive solutions and the solutions are coupled locally. They confirmed the separation principle for the nonlinear $\mathcal{H}_\infty$-control system, and also provided a recipe to construct the controllers from the necessary conditions. The reader is referred to Ball and Helton (1993), Didinsky et al. (1993), Helton and Zhan (1993), Isidori (1993), Lu (1993b, 1994), Lu and Doyle (1993b), and Van der Schaft (1993) for other treatments about nonlinear $\mathcal{H}_\infty$-control.

Our goal in this paper is to systematically examine the nonlinear $\mathcal{H}_\infty$-control problem in state space for a class of nonlinear systems and obtain an $\mathcal{H}_\infty$ controller parameterization. This investigation is carried out in a methodical fashion. We follow the techniques used in the linear case (Doyle et al., 1989), and propose a machinery for nonlinear $\mathcal{H}_\infty$-controller design. Four problems: full information (FI), disturbance feedforward (DF), full control (FC) and output estimation (OE), are also considered. These problems are essential for the constructions of the controllers, but structurally are not the special cases for the output feedback structure of the class of nonlinear systems considered in this paper; in addition, they are important in there own right. The sufficient conditions for solvability are obtained and a parameterized class of controllers are derived for each $\mathcal{H}_\infty$-control problems (note that the FI problem has been solved by Van der Schaft (1992) and Isidori-Astolfi (1992), see also Isidori (1992). Sufficient conditions for the output feedback $\mathcal{H}_\infty$-control problem to be solvable locally are also derived using this machinery. Like the conditions in the linear case, the solvability of the $\mathcal{H}_\infty$-control problem requires the positive definite solutions to two parallel decoupled HJIs with the same numbers of dependent parameters and these two solutions satisfy an additional condition. A class of $\mathcal{H}_\infty$-controllers are parameterized as a nonlinear fractional transformation on contractive, stable free nonlinear operators. It would be appropriate to emphasize at this point that similar sufficient conditions are also obtained by Isidori (1992) with different machinery. Any concept or result in this paper is local unless otherwise noted. Specifically, problems DF, OE and the output feedback (OF) are treated locally.

This paper is organized as follows: In section 2, some background material related to the $\mathcal{L}_2$-gains is given. In section 3, the $\mathcal{H}_\infty$-control problem is stated and the structure of the general system is simplified. We also give the four nonlinear structures of special problems related to the general system: FI, DF, FC and OE. In section 4, the $\mathcal{H}_\infty$-control problem for four special structures are considered, both the solvability conditions and controller parameterizations are given. In section 5, the main results of this paper, solutions to the output feedback $\mathcal{H}_\infty$-control problem, are given. The solvability of this problem requires the coupled positive definite solutions to two decoupled HJIs. The standard separation principle in this case is re-examined and a class of $\mathcal{H}_\infty$-controllers are parameterized. As an illustrative example, the $\mathcal{H}_\infty$ control design for a passive system is conducted.

Conventions

The following conventions are made in this paper. $\mathbf{R}$ is the set of real numbers, $\mathbf{R}^+ := [0, \infty) \subset \mathbf{R}$. $\mathbf{R}^n$ is n-dimensional real Euclidean space; if $u \in \mathbf{R}^n$, then $\|u\|$ is Euclidean norm of u. $\mathbf{R}^{n\times m}$ is the set of real $n \times m$ matrices; if $A \in \mathbf{R}^{n\times m}$, then $A^T \in \mathbf{R}^{m\times n}$ is the transpose of A. By $P > 0$ ($P \geq 0$) for some matrix $P = P^T \in \mathbf{R}^{n\times n}$ we mean that the matrix is (semi-)positive definite. A function is said to be of class $\mathbf{C}^k$ if it is continuously differentiable k times; so $\mathbf{C}^0$ stands for the class of continuous functions. $\|\cdot\|$ stands for the **Euclidean norm**. $\mathcal{B}_r := \{x \in \mathbf{R}^n \mid \|x\| < r, \text{for some integer } n > 0\}$; we shall not specify the dimension of the environmental space, and always use the same r to denote its radius if no confusion occurs. $\mathcal{L}_2[0,T], \mathcal{L}_2[0,\infty)$ are two standard Lebesgue Spaces; $\mathcal{L}_2^e[0,\infty)$ is the extended space of $\mathcal{L}_2[0,\infty)$. $\Omega(G,K)$ represents **fractional transformation** of operator G on operator K; $\Sigma(M_1, M_2)$ stands for the **Redheffer product** of operators M_1 and M_2 (see Redheffer, 1960, Lu and Doyle, 1993) for exact definitions).

2 $\mathcal{L}_2$-Gains of Nonlinear Systems

In this section, some background material about $\mathcal{L}_2$-gain analysis of nonlinear systems is provided. The reader is also referred Van der Schaft (1992) for the similar consideration. However, the treatment here is slightly more general than the one given in Van der Schaft (1992). The machinery used here is from Willems (1972).

Consider the following affine nonlinear time-invariant (NLTI) system:

$$G : \begin{cases} \dot{x} = f(x) + g(x)u \\ y = h(x) + k(x)u \end{cases} \tag{1}$$

where $x \in \mathbf{X} \subset \mathbf{R}^n$ is state vector, (where $\mathbf{X}$ is an open subset, and $0 \in \mathbf{X}$,) $u \in \mathbf{R}^p$ and $y \in \mathbf{R}^q$ are input and output vectors, respectively. We will assume $f, g, h, k \in \mathbf{C}^2$, and $f(0) = 0, h(0) = 0$. Therefore, $0 \in \mathbf{R}^n$ is the equilibrium of the system with $u = 0$. The state transition function $\phi : \mathbf{R}^+ \times \mathbf{X} \times \mathbf{R}^p \rightarrow \mathbf{R}^n$ is so defined that $x = \phi(T, x_0, u^*)$ means that system G evolves from initial state x_0 to state x in time T under the control action u^*.

Definition 2.1 *(i) A system G (or $[f(x), g(x)]$) is* **reachable** *from* 0 *if for all $x \in \mathbf{X}$, there exist $T \in R^+$ and $u^*(t) \in \mathcal{L}_2[0, T]$ such that $x = \phi(T, 0, u^*)$;*

(ii) A system G (or $[h(x), f(x)]$) is (zero-state) **detectable** *if for all $x \in \mathbf{X}$, $h(\phi(t, x, 0)) = 0 \Rightarrow \phi(t, x, 0) \rightarrow 0$ as $t \rightarrow \infty$.*

Definition 2.2 *A system G is said to have $\mathcal{L}_2$-gain less than or equal to* 1 *if*

$$\int_0^T \|y(t)\|^2 \, dt \leq \int_0^T \|u(t)\|^2 \, dt \tag{2}$$

for all $T \geq 0$ and $u(t) \in \mathcal{L}_2[0, T]$, and $y(t) = h(x(t)) + k(x(t))u(t)$, as long as $x(t) := \phi(t, 0, u(t) \in \mathbf{X}$.

Note that in the above definition, we take the initial state $x(0) = 0$. Define

$$V_a(x) := \sup_{u \in \mathcal{L}_2[0,\tau], \tau \geq 0, x(0)=x} - \int_0^\tau (\|u(t)\|^2 - \|y(t)\|^2) dt. \tag{3}$$

Note that $V_a(x) \geq 0$ for all $x \in \mathbf{X}$, and if the system has $\mathcal{L}_2$-gain ≤ 1, then $V_a(0) = 0$. We will assume $V_a(0) = 0$ from now on.

As pointed out by Willems (1972), $V_a(x) < \infty$ if and only if there exists a solution $V : \mathbf{X} \rightarrow \mathbf{R}^+$ with $V(0) = 0$ to the *integral dissipation inequality (IDI)*:

$$\mathcal{D}_I(V, x, u) := V(x) - V(x_0) - \int_0^T (\|u(t)\|^2 - \|y(t)\|^2) dt \leq 0 \tag{4}$$

where $x = \phi(T, x_0, u(t))$ and $u(t) \in \mathcal{L}_2[0, T]$, i.e. system G is *dissipative* with respect to supply rate $\|u(t)\|^2 - \|y(t)\|^2$, and $V_a(\cdot)$ is also a solution. Moreover, the solutions to IDI (4) form a convex set, and any solution $V(x) \geq 0$ for $x \in \mathbf{X}$ with $V(0) = 0$ satisfies $V(x) \geq V_a(x)$.

Lemma 2.1 (Willems, 1971, 1972)

(i) System G (1) has $\mathcal{L}_2$-gain ≤ 1 **if** *$V_a(x) < \infty$, for all $x \in \mathbf{X}$.*

(ii) If system G (1) is reachable from 0*, then it has $\mathcal{L}_2$-gain ≤ 1* **only if** *$V_a(x) < \infty$, for all $x \in \mathbf{X}$.*

Proof **(i)** $V_a(x) < \infty$ satisfies IDI (4), so

$$\int_0^T (\|u(t)\|^2 - \|y(t)\|^2)dt \geq V_a(x(T)) - V_a(0) = V_a(x(T)) \geq 0$$

for all $T \in \mathbf{R}^+$.

(ii) Take $x \in \mathbf{X}$, by the reachability assumption, there exist $T \in \mathbf{R}^+$ and $u_1(t) \in \mathcal{L}_2[-T, 0)$ such that $x(0) = x$ for $x(-T) = 0$. Now take any $u_2(t) \in \mathcal{L}_2[0, \infty)$. Define $u \in \mathcal{L}_2[-T, \infty)$ as

$$u(t) = \begin{cases} u_1(t), & \text{if } t \in [-T, 0) \\ u_2(t), & \text{if } t \in [0, \infty) \end{cases}$$

Since the NLTI system (1) has $\mathcal{L}_2$-gain ≤ 1, for all $\tau \geq 0$, then

$$\int_{-T}^{\tau} \|y(t)\|^2 \, dt \leq \int_{-T}^{\tau} \|u(t)\|^2 \, dt$$

by time-invariance of (1), i.e.,

$$-\int_0^{\tau} (\|u_2(t)\|^2 - \|y(t)\|^2)dt \leq \int_{-T}^{0} (\|u_1(t)\|^2 - \|y(t)\|^2)dt$$

so

$$V_a(x) = \sup_{u_2 \in \mathcal{L}_2[0,\tau], \tau \geq 0, x(0)=x} -\int_0^{\tau} (\|u_2(t)\|^2 - \|y(t)\|^2)dt \leq \int_{-T}^{0} (\|u_1(t)\|^2 - \|y(t)\|^2)dt < \infty.$$

□

Thus, if the system is reachable from 0, then $\mathcal{L}_2$-gain ≤ 1 if and only if the system is dissipative with respect to supply rate $\|u(t)\|^2 - \|y(t)\|^2$; also $V_a(x) \geq 0$ is well-defined for all $x \in \mathbf{X}$, and there exists a solution $V : \mathbf{X} \rightarrow \mathbf{R}^+$ to IDI (4). Now define

$$\mathcal{H}(V, x) := \frac{\partial V}{\partial x}(x)(f(x) + g(x)R^{-1}(x)k^T(x)h(x)) + \frac{1}{4}\frac{\partial V}{\partial x}(x)g(x)R^{-1}(x)g^T(x)\frac{\partial V^T}{\partial x}(x) +$$
$$+ h^T(x)(I - k(x)k^T(x))^{-1}h(x) \tag{5}$$

The following lemma characterizes a class of nonlinear systems having $\mathcal{L}_2$-gain ≤ 1.

Lemma 2.2 *Consider a system G with $R(x) := I - k^T(x)k(x) > 0$ for all $x \in \mathbf{X}$, suppose G has $\mathcal{L}_2$-gain ≤ 1.*

i) If $V_a(x)$ defined by (3) is differentiable with respect to $x \in \mathbf{X}$, then it satisfies Hamilton-Jacobi equation (HJE)*: $\mathcal{H}(V, x) = 0$.*

ii) If $V : \mathbf{X} \rightarrow \mathbf{R}$ is differentiable on $\mathbf{X}$ and satisfies IDI (4), then it satisfies Hamilton-Jacobi inequality (HJI)*: $\mathcal{H}(V, x) \leq 0$.*

Proof Part (i) follows from the standard argument (cf. Kalman, 1963). As for Part (ii), since $V(x)$ is differentiable, the IDI (4) reduces to the following *differential dissipation inequality (DDI)*:

$$\begin{aligned}\mathcal{D}_D(V,x,u) &:= \frac{\partial V}{\partial x}(x)(f(x)+g(x)u) - \|u(t)\|^2 + \|y(t)\|^2 \\ &= \dot{V}(x) - \|u(t)\|^2 + \|y(t)\|^2 \leq 0 \end{aligned} \tag{6}$$

It follows that $\mathcal{D}_D(V,x,u) \leq 0$ for all $u \in \mathbf{R}^p$ if and only if

$$\sup_{u \in \mathbf{R}^p} \mathcal{D}_D(V,x,u) \leq 0$$

Let

$$\frac{\partial \mathcal{D}_D(V,x,u)}{\partial u}|_{u=u^*} = 0$$

then

$$u^* = R^{-1}(x)k^T(x)h^T(x) + \frac{1}{2}R^{-1}(x)g^T(x)\frac{\partial V^T}{\partial x}(x)$$

Thus,

$$\mathcal{D}_D(V,x,u^*) = \sup_{u \in \mathbf{R}^p} \mathcal{D}_D(V,x,u) \leq 0 \qquad \Longleftrightarrow \qquad \mathcal{H}(V,x) \leq 0.$$

□

It is easy to see that the converse results in the above lemma are also true (Kalman, 1963). The following statement follows from the above proof.

Corollary 2.3 *Suppose $\pi(x)$ is a function defined on $\mathbf{X}$ with $\pi(0) = 0$, then $V(x)$ with $V(0) = 0$ satisfies HJI: $\mathcal{H}(V,x) + \pi(x) \leq 0$ for $x \in \mathbf{X}$ if and only if it satisfies DDI: $\mathcal{D}_D(V,x,u) + \pi(x) \leq 0$, or*

$$\frac{\partial V}{\partial x}(x)(f(x)+g(x)u) \leq \|u(t)\|^2 - \|y(t)\|^2 - \pi(x)$$

for all $u(t) \in \mathbf{R}^p$. Moreover, if $\pi(x) \geq 0$ and $V(x) \geq 0$ for all $x \in \mathbf{X}$, then the $\mathcal{L}_2$-gain ≤ 1.

The above discussion can be summarized as following Theorem which slightly generalizes Theorem 2 in Van der Schaft (1992).

Theorem 2.4 *Consider system G (1) for $x \in \mathbf{X}$, suppose it is reachable from 0, then each of the following implications holds for all $x \in \mathbf{X}$ under the specified condition.*

$$\begin{tikzcd}
& \mathcal{L}_2\text{-}Gain < 1 \arrow[dl, shift left] \arrow[dr, "V_a \in \mathbf{C}^2"] & \\
\mathcal{D}_I(V,x,u) \leq 0 \arrow[ur, shift left, "V(x) \geq 0"] \arrow[d, shift right, "V \in \mathbf{C}^2"'] & & \mathcal{H}(V_a,x) = 0 \arrow[d, "V = V_a"] \\
\mathcal{D}_D(V,x,u) \leq 0 \arrow[u, shift right] \arrow[rr, shift left] & & \mathcal{H}(V,x) \leq 0 \arrow[ll, shift left]
\end{tikzcd}$$

Recall that $V : \mathbf{R}^n \rightarrow \mathbf{R}^+$ is locally positive-definite if there exists $r > 0$ such that for $x \in \mathcal{B}_r$, $V(x) = 0 \Rightarrow x = 0$; it is globally positive-definite if $V(x) = 0 \Rightarrow x = 0$, and $\lim_{x\rightarrow\infty} V(x) = \infty$. The following lemma, which is due to Hill and Moylan (1980; see also Van der Schaft, 1992), partially establishes the relationship between finite gain (stability) and asymptotic stability.

Lemma 2.5 *(i) Suppose system G with $u = 0$ is asymptotically stable at 0, then any function $V(x)$ on $\mathbf{X}$ with $V(0) = 0$ satisfying IDI: $\mathcal{D}_I(V, x, u) \leq 0$ is non-negative. Specially, if $V(x)$ satisfies HJI: $\mathcal{H}(V, x) \leq 0$ with $V(0) = 0$, then $V(x) \geq 0$ for all $x \in \mathbf{X}$.*

(ii) Assume system G is zero-state detectable. If there is a positive definite solution $V(x)$ on $\mathbf{X}$ to HJI: $\mathcal{H}(V, x) \leq 0$ for all $x \in \mathbf{X}$, then the system G with $u = 0$ is asymptotically stable at 0.

Definition 2.3 *The class $\mathcal{FG}$ of (affine) NLTI systems is defined as*

$$\mathcal{FG} := \{G | G \text{ is asymptotically stable and related HJI has a positive definite solution}\}.$$

Therefore, if $G \in \mathcal{FG}$ with state x, then there exists a positive definite $V(x)$ such that $\dot{V}(x) \leq \|u\|^2 - \|y\|^2$ (here $y = Gu$), so it has $\mathcal{L}_2$-gain ≤ 1.

We conclude this section with an observation. Consider the system G, define a new system G_N as

$$G_N : \begin{cases} \dot{x} = f_N(x) + g_N(x)u \\ y_N = h_N(x) \end{cases}$$

with

$$f_N(x) = f(x) - g(x)(I - k^T(x)k(x))^{-1}k^T(x)h(x),$$

$$g_N(x) = g(x)(I - k^T(x)k(x))^{-1/2}$$

and

$$h_N(x) = (I - k(x)k^T(x))^{-1/2}h(x).$$

System G can be simplified as G_N in the following sense.

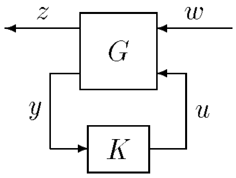
Theorem 2.6 *G_N is of $\mathcal{L}_2$-gain ≤ 1, and the related HJE (or HJI) has solution $V_a \geq 0$ (or $V \geq 0$) if and only if G has $\mathcal{L}_2$-gain ≤ 1, and its corresponding HJE (or HJI) also has the same solution. Moreover, G_N is zero-state detectable if and only if G is.*

Proof Simple algebra shows both systems correspond to the same HJE(or HJI). It is obvious that G_N is detectable if and only if G is. □

3 $\mathcal{H}_\infty$-Control: Problem Statement and Special Problems

3.1 $\mathcal{H}_\infty$-Control Problem

The basic block diagram considered in the $\mathcal{H}_\infty$-control synthesis problem is



where G is the nonlinear plant with two sets of inputs: the exogenous disturbance inputs w and the control inputs u, and two sets of outputs: the measured outputs y and the regulated outputs z. And K is the controller to be designed. Both G and K are nonlinear time-invariant and can be realized as affine state-space equations:

$$G : \left\{ \begin{array}{rclcccc} \dot{x} & = & f(x) & + & g_1(x)w & + & g_2(x)u \\ z & = & h_1(x) & + & k_{11}(x)w & + & k_{12}(x)u \\ y & = & h_2(x) & + & k_{21}(x)w & + & k_{22}(x)u \end{array} \right. \tag{7}$$

where $f, g_i, h_i, k_{ij} \in \mathbf{C}^2$, and $f(0) = 0, h_1(0) = 0, h_2(0) = 0$; x, w, u, z, and y are assumed to have dimensions n, p_1, p_2, q_1, and q_2, respectively.

$$K : \left\{ \begin{array}{rclcc} \dot{\hat{x}} & = & a(\hat{x}) & + & b(\hat{x})y \\ u & = & c(\hat{x}) & + & d(\hat{x})y \end{array} \right. \tag{8}$$

with $a, b, c, d \in \mathbf{C}^2$ and $a(0) = 0, c(0) = 0$.

The initial states for both plant and controller are $x(0) = 0$ and $\hat{x}(0) = 0$. The closed loop system will be denoted as nonlinear operator $\Omega(G, K)$ which represents the input/output relation: $z = \Omega(G, K)w$. In the following, we consider the following local $\mathcal{H}_\infty$-control problem.

$\mathcal{H}_\infty$-CONTROL PROBLEM: Find an input-affine feedback controller K (or a class such controllers) for system G (7) if any, such that the closed-loop system $\Omega(G, K)$ is locally asymptotically stable with $w = 0$ and has $\mathcal{L}_2$-gain ≤ 1, i.e.

$$\int_0^T (\|w(t)\|^2 - \|z(t)\|^2)dt \geq 0;$$

for all $T \in \mathbf{R}^+$, as long as the state of the closed loop system $(x, \hat{x}) \in \mathcal{B}_r \times \mathcal{B}_s$ for some $r, s > 0$.

The following assumptions on system structure are made:

[A1]: $k_{11}(x) = 0, k_{22}(x) = 0$;

[A2]: $k_{12}^T(x) \left[\begin{array}{cc} h_1(x) & k_{12}(x) \end{array} \right] = \left[\begin{array}{cc} 0 & I \end{array} \right]$;

[A3]: $\left[\begin{array}{c} g_1(x) \\ k_{21}(x) \end{array} \right] k_{21}^T(x) = \left[\begin{array}{c} 0 \\ I \end{array} \right]$;

It is known that a large class of nonlinear systems can be simplified to satisfy the above assumption when $\mathcal{H}_\infty$ control problem is considered (Safonov et al., 1989).

3.2 Special Problems

As in the linear case (Doyle et al., 1989), we will consider four special problems which is essential to the construction of the $\mathcal{H}_\infty$-controllers for the output feedback problem considered in this paper, and will help us to examine the insights of the constructions and structures of nonlinear $\mathcal{H}_\infty$-controllers, especially to reveal the separation property for nonlinear $\mathcal{H}_\infty$-control systems. Structurally, those problem are not the special cases for the output feedback problem considered in this paper; they are also very important in their own right. The reader is referred to Doyle et al.(1989) for the motivations and interpretations about those problems.

- **Full Information (FI) Problem.**

  In this case, both state x and disturbance w are directly available to controller, the plant is

$$G_{FI}: \begin{cases} \dot{x} = f(x) + g_1(x)w + g_2(x)u \\ z = h_1(x) + k_{12}(x)u \\ y = \begin{bmatrix} x \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} w \end{cases}$$

  with

  [A2]: $k_{12}^T(x)\begin{bmatrix} h_1(x) & k_{12}(x) \end{bmatrix} = \begin{bmatrix} 0 & I \end{bmatrix}$;

- **Full Control (FC) Problem**

  The control action has full access to both state x through output injection and the regulated output.

$$G_{FC}: \begin{cases} \dot{x} = f(x) + g_1(x)w + \begin{bmatrix} I & 0 \end{bmatrix} u \\ z = h_1(x) + \begin{bmatrix} 0 & I \end{bmatrix} u \\ y = h_2(x) + k_{21}(x)w \end{cases}$$

  with

  [A3]: $\begin{bmatrix} g_1(x) \\ k_{21}(x) \end{bmatrix} k_{21}^T(x) = \begin{bmatrix} 0 \\ I \end{bmatrix}$;

- **Disturbance Feedforward (DF) Problem**

$$G_{DF}: \begin{cases} \dot{x} = f(x) + g_1(x)w + g_2(x)u \\ z = h_1(x) + k_{12}(x)u \\ y = h_0(x) + w \end{cases}$$

  where

  [A2]: $k_{12}^T(x)\begin{bmatrix} h_1(x) & k_{12}(x) \end{bmatrix} = \begin{bmatrix} 0 & I \end{bmatrix}$;

- **Output Estimation (OE) Problem**

$$G_{OE}: \begin{cases} \dot{x} &= f(x) &+ & g_1(x)w &+ & g_0(x)u \\ z &= h_1(x) &+ & & & u \\ y &= h_2(x) &+ & k_{21}(x)w & & \end{cases}$$

The structural assumptions for this structure are

[A3]: $\begin{bmatrix} g_1(x) \\ k_{21}(x) \end{bmatrix} k_{21}^T(x) = \begin{bmatrix} 0 \\ I \end{bmatrix}.$

In the linear case, the two pairs, FI and FC, DF and OE, are dual in structure. But it would be un-realistic to expect the same thing happens in the nonlinear case. In the next section, we will mainly consider those special problems.

4 $\mathcal{H}_\infty$-Control Synthesis: Solutions to Special Problems

Unlike linear case, the solutions for the special $\mathcal{H}_\infty$-control problems cannot be obtained by duality (if there is any). This section is devoted to the discussion of different special problems.

4.1 Full Information Problem

Consider

$$G_{FI}: \begin{cases} \dot{x} &= f(x) &+ & g_1(x)w &+ & g_2(x)u \\ z &= h_1(x) &+ & & & k_{12}(x)u \\ y &= \begin{bmatrix} x \\ 0 \end{bmatrix} &+ & \begin{bmatrix} 0 \\ I \end{bmatrix} w & & \end{cases} \tag{9}$$

with $x \in \mathbf{X}$. In addition to the structural constraints **[A2]**, we further assume

[A4]: $[h_1(x), f(x)]$ is zero-state detectable.

The $\mathcal{H}_\infty$-control problem for FI was first explicitly introduced and solved by Van der Schaft (1992; see also Isidori, 1992). The solutions to $\mathcal{H}_\infty$-control problem are related to the following HJI:

$$\mathcal{H}_{FI}(V,x) := \frac{\partial V}{\partial x}(x)f(x) + \frac{1}{4}\frac{\partial V}{\partial x}(x)(g_1(x)g_1^T(x) - g_2(x)g_2^T(x))\frac{\partial V^T}{\partial x}(x) + h_1^T(x)h_1(x) \le 0. \tag{10}$$

The following theorem reveals more properties related to HJI for FI.

Theorem 4.1 *(i) $\mathcal{H}_{FI}(V,x) \le 0$ ($x \in \mathbf{X}$) has a solution $V(x)$ with $V(0) = 0$ if and only if there is $F_0(x)$ such that*

$$\mathcal{H}_{SF}(V,F_0,x) := \frac{\partial V}{\partial x}(x)(f(x) + g_2(x)F_0(x)) + \frac{1}{4}\frac{\partial V}{\partial x}(x)g_1(x)g_1^T(x)\frac{\partial V^T}{\partial x}(x) +$$

$$+(h_1(x) + k_{12}(x)F_0(x))^T(h_1(x) + k_{12}(x)F_0(x)) \le 0.$$

Moreover, if $V(x)$ solves $\mathcal{H}_{FI}(V,x) \leq 0$ with $V(0)=0$, then $F_0(x)$ can be taken as

$$F_0(x) = -\frac{1}{2}g_2^T(x)\frac{\partial V^T}{\partial x}(x).$$

(ii) If $[h_1(x), f(x)]$ is assumed to be zero-state observable, then any solution $V(x) \geq 0$ to $\mathcal{H}_{FI}(V,x) \leq 0$ with $V(0)=0$ is positive definite.

Similar result to property (ii) also appears in Van der Schaft (1992). Property (ii) insures that the HJI corresponding to FI has a positive definite solution under some mild assumption (observability). This justifies the assumption that HJI has positive definite solutions in the next Theorem.

Proof (i) Note that there exists $F(x)$ such that

$$\mathcal{H}_{SF}(V, F_0, x) \leq 0$$

if and only if

$$0 \geq \inf_{F_0(x)} \mathcal{H}_{SF}(V, F_0, x) = \mathcal{H}_{SF}(V, F_0, x)\big|_{F_0(x) = -\frac{1}{2}g_2^T(x)\frac{\partial V^T}{\partial x}(x)} = \mathcal{H}_{FI}(V, x)$$

The latter can be confirmed by taking

$$\frac{\partial \mathcal{H}_{SF}}{\partial F_0}(V, F_0, x) = 0$$

for fixed x.

(ii) Suppose $V(x) \geq 0$ is such that $\mathcal{H}_{FI}(V,x) \leq 0$. Then by the completion of square technique, it follows that

$$\begin{aligned}\dot{V}(x) &= \frac{\partial V}{\partial x}(x)(f(x) + g_1(x)w + g_2(x)u) \\ &\leq \left\| u + \frac{1}{2}g_2^T(x)\frac{\partial V^T}{\partial x}(x)\right\|^2 - \left\| w - \frac{1}{2}g_1^T(x)\frac{\partial V^T}{\partial x}(x)\right\|^2 - \|z\|^2 + \|w\|^2\end{aligned}$$

If $u = -\frac{1}{2}g_2^T(x)\frac{\partial V^T}{\partial x}(x)$ then

$$\dot{V}(x) \leq -\|z\|^2 + \|w\|^2$$

Since $V(x) \geq 0$, the closed-loop system

$$\begin{cases} \dot{x} = f(x) - \frac{1}{2}g_2(x)g_2^T(x)\frac{\partial V^T}{\partial x}(x) + g_1(x)w \\ z = h_1(x) - \frac{1}{2}k_{12}(x)g_2^T(x)\frac{\partial V^T}{\partial x}(x) \end{cases}$$

is dissipative with respect to supply rate $\|w\|^2 - \|z\|^2$. Thus, for all $x \in \mathbf{R}^n$,

$$V(x) \geq \sup_{w \in \mathcal{L}_2[0,\tau], \tau \geq 0, x(0)=x} -\int_0^\tau (\|w(t)\|^2 - \|z(t)\|^2)dt$$

$$\geq -\int_0^\infty (0 - \|z(t)\|^2)dt = \int_0^\infty (\|z(t)\|^2)dt$$

$V(x) = 0 \Longrightarrow \|z\|^2 = \|h_1(x)\|^2 + \frac{1}{4}\left\|g_2^T(x)\frac{\partial V^T}{\partial x}(x)\right\|^2 = 0$, so

$$\left\{\begin{array}{l} \dot{x} = f(x) - \frac{1}{2}g_2(x)g_2^T(x)\frac{\partial V^T}{\partial x}(x) \\ z = h_1(x) - \frac{1}{2}k_{12}(x)g_2^T(x)\frac{\partial V^T}{\partial x}(x) = 0 \end{array}\right. \quad \Leftrightarrow \quad \left\{\begin{array}{l} \dot{x} = f(x) \\ h_1(x) = 0 \end{array}\right.$$

So $x = 0$ by observability assumption. (Moreover, the closed-loop system is also observable.) $V(x)$ is (locally) positive definite. □

The solutions to the $\mathcal{H}_\infty$-control problem for FI in the above setting are included here for completeness.

Theorem 4.2 (Van der Schaft, 1992; Isidori, 1992; Isidori and Astolfi, 1992) *Consider G_{FI} (9), suppose there exists a local positive definite function $V(x) \geq 0$ such that $\mathcal{H}_{FI}(V, x) \leq 0$, for $x \in \mathbf{B}_r$ with some $r > 0$, and $V(0) = 0$. Then the $\mathcal{H}_\infty$-control problem for FI is solvable, and such a FI $\mathcal{H}_\infty$-controller is provided by*

$$u = -\frac{1}{2}g_2^T(x)\frac{\partial V^T}{\partial x}(x).$$

Moreover, for all $Q \in \mathcal{FG}$, the controller

$$u = -\frac{1}{2}g_2^T(x)\frac{\partial V^T}{\partial x}(x) + Q(w - \frac{1}{2}g_1^T(x)\frac{\partial V^T}{\partial x}(x))$$

solves the FI $\mathcal{H}_\infty$ control problem.

Remark 4.1 If $V(x)$ is globally positive definite, and $H_{FI}(V, x) \leq 0$ for all $x \in \mathbf{R}^n$, then Theorem 4.1 gives the global solution. The same thing happens for FC $\mathcal{H}_\infty$-control problem to be investigated later.

4.2 Disturbance Feedforward Problem

Consider

$$G_{DF} : \left\{\begin{array}{lcccccc} \dot{x} & = & f(x) & + & g_1(x)w & + & g_2(x)u \\ z & = & h_1(x) & + & & & k_{12}(x)u \\ y & = & h_0(x) & + & w & & \end{array}\right. \tag{11}$$

where $f, g_1, g_2, h_1, h_0, k_{12} \in \mathbf{C}^2$ under the constraints **[A2]**.

The additional assumptions relevant to DF problem are as follows:

[A4]: $[h_1(x), f(x)]$ is (locally) zero-state detectable, and

[A6]: $\dot{x} = f(x) + g_1(x)h_0(x)$ is locally exponentially stable at 0.

By converse Lyapunov Theorem, assumption **[A6]** implies that there exists a locally positive definite function $U : \mathbf{R}^n \rightarrow \mathbf{R}^+$, such that

$$L_D(U, x) := \frac{\partial U}{\partial x}(x)(f(x) + g_1(x)h_0(x)) \tag{12}$$

is locally negative definite. Furthermore, the Hessian matrix of $L_D(U, x)$ is nonsingular at $x = 0$ by suitably choosing the Lyapunov function $U(x)$. And if $U(x)$ has these properties, so does $kU(x)$ for all constant $k > 0$.

Theorem 4.3 *Consider G_{DF} (11), suppose there exists a smooth solution $V(x) \geq 0$ with $V(0) = 0$ to $\mathcal{H}_{FI}(V, x) \leq 0$ for $x \in \mathbf{B}_r$. Then the $\mathcal{H}_\infty$-control problem for DF is solvable. Moreover, the controller given by*

$$\begin{cases} \dot{\tilde{x}} = f(\tilde{x}) - g_1(\tilde{x})h_0(\tilde{x}) + g_0(\tilde{x})F_0(\tilde{x}) + g_1(\tilde{x})y \\ u = F_0(\tilde{x}) \end{cases} \qquad F_0(\tilde{x}) := -\frac{1}{2}g_2^T(\tilde{x})\frac{\partial V^T}{\partial \tilde{x}}(\tilde{x})$$

solves the $\mathcal{H}_\infty$-control problem.

Define a system

$$P_{DF} : \begin{cases} \dot{\tilde{x}} & = & f(\tilde{x}) - g_1(\tilde{x})h_0(\tilde{x}) & + & g_1(\tilde{x})y & + & g_2(\tilde{x})u \\ u & = & & & & & u \\ y_0 & = & \begin{bmatrix} \tilde{x} \\ -h_0(\tilde{x}) \end{bmatrix} & + & \begin{bmatrix} 0 \\ I \end{bmatrix} y & & \end{cases}$$

We have the following lemma.

Lemma 4.4 *Given two functions $V, U : \mathbf{R}^n \to \mathbf{R}$, where the function $V(x) \geq 0$ with $V(0) = 0$ additionally satisfies HJI: $\mathcal{H}_{FI}(V, x) \leq 0$ for $x \in \mathbf{B}_r$. Let $x, \tilde{x}$ be states of systems G_{DF} and P_{DF}, $e := x - \tilde{x}$. Define*

$$S(U, e, \tilde{x}) := \frac{\partial U}{\partial e}(e)[(f(x) - f(\tilde{x})) - g_1(\tilde{x})(h_0(x) - h_0(\tilde{x}))];$$

$$R_1(e, \tilde{x}) := h_1^T(x)h_1(x) - h_1^T(\tilde{x})h_1(\tilde{x});$$

$$R_2(e, \tilde{x}) := \frac{1}{4}\left\|(g_1^T(x) - g_1^T(\tilde{x}))\frac{\partial U^T}{\partial e}(e)\right\|^2 + \frac{1}{2}\frac{\partial V}{\partial \tilde{x}}(\tilde{x})g_1(\tilde{x})(g_1^T(x) - g_1^T(\tilde{x}))\frac{\partial U^T}{\partial e}(e) +$$

$$-\frac{1}{2}\frac{\partial V}{\partial \tilde{x}}(\tilde{x})g_2(\tilde{x})(g_2^T(x) - g_2^T(\tilde{x}))\frac{\partial U^T}{\partial e}(e).$$

Then under the assumption **[A6]**, *there exists locally positive definite function $U(x)$ such that for all $x, \tilde{x} \in \mathcal{B}_r$ with some $r > 0$,*
(i) $S(U, e, \tilde{x}) \leq 0$,
(ii) $S(U, e, \tilde{x}) + R_1(e, \tilde{x}) + R_2(e, \tilde{x}) \leq 0$.

Proof By assumption **[A6]**, there exists locally positive definite function $U_o(x)$ such that $L_D(U_o, x)$ defined by (12) locally negative definite and the Hessian matrix of $L_D(U_o, x)$ is nonsingular at $x = 0$. Note that $S(U_o, e, \tilde{x})$ has the same Hessian matrix with respect to e at $(e, \tilde{x}) = 0$ as does $L_D(U_o, e)$ at $e = 0$. Therefore, for $\tilde{x} \in \mathcal{B}_r$ for some $r > 0$ small enough, $S(U_o, e, \tilde{x})$ is dominated by its quadratic term of e which is the same as the one of $L_D(U_o, e)$, so $S(U_o, e, \tilde{x}) \leq 0$ for $x, \tilde{x} \in \mathcal{B}_r$ with the adjusted $r > 0$. This confirms (i). Also note that if $U_o(x)$ is such that $L_D(U_o, e)$ has negative definite Hessian matrix with respect to e at 0, so is $kU_o(x)$ for all constant $k > 0$. Thence, we can choose $k > 0$ large enough such that $S(kU_o, e, \tilde{x}) + R_1(e, \tilde{x})$ also has negative Hessian matrix with respect to e at $(e, \tilde{x}) = 0$; it is dominated by its quadratic term of e. By adding the higher order term $R_2(e, \tilde{x})$, we can conclude that $S(U, e, \tilde{x}) + R_1(e, \tilde{x}) + R_2(e, \tilde{x}) \leq 0$ with $U(x) := kU_o(x)$ for $x, \tilde{x} \in \mathcal{B}_r$ with adjusted $r > 0$. This confirms (ii). Note that the new U also works for (i). □

Proof **[Theorem 4.3]** Consider $\Sigma(G_{DF}, P_{DF})$, which has following realization

$$\begin{cases} \dot{x} &= f(x) &+ \; g_1(x)w &+ \; g_2(x)u \\ \dot{\tilde{x}} &= (f(\tilde{x}) - g_1(\tilde{x})h_0(\tilde{x}) + g_1(\tilde{x})h_0(x)) &+ \; g_1(\tilde{x})w &+ \; g_2(\tilde{x})u \\ z &= h_1(x) &+ & \; k_{12}(x)u \\ y &= \begin{bmatrix} \tilde{x} \\ h_0(x) - h_0(\tilde{x}) \end{bmatrix} &+ \; \begin{bmatrix} 0 \\ I \end{bmatrix} w & \end{cases}$$

Let $e := x - \tilde{x}$, define

$$W(e, \tilde{x}) := U(e) + V(\tilde{x})$$

where $U(x)$ is given in the above lemma, then $W(e, \tilde{x}) \geq 0$, and $W(0,0) = U(0) + V(0) = 0$. For all $(x, \tilde{x}) \in \mathcal{B}_r$.

$$\begin{aligned} \dot{W}(e, \tilde{x}) &= \dot{U}(e) + \dot{V}(\tilde{x}) \\ &= \frac{\partial U}{\partial e}(e)((f(x) - f(\tilde{x})) - g_1(\tilde{x})(h_0(x) - h_0(\tilde{x}))+ \\ &\quad + \frac{\partial U}{\partial e}(e)((g_1(x) - g_1(\tilde{x}))w + (g_2(x) - g_2(\tilde{x}))u)+ \\ &\quad + \frac{\partial V}{\partial \tilde{x}}(\tilde{x})(f(\tilde{x}) + g_1(\tilde{x})w + g_2(\tilde{x})u) + \frac{\partial V}{\partial \tilde{x}}(\tilde{x})g_1(\tilde{x})(h_0(x) - h_0(\tilde{x})) \\ &\leq S(U, e, \tilde{x}) + \frac{\partial U}{\partial e}(e)(g_1(x) - g_1(\tilde{x}))w + \frac{\partial U}{\partial e}(e)(g_2(x) - g_2(\tilde{x}))u+ \\ &\quad - \frac{1}{4}\frac{\partial V}{\partial e}(e)(g_1(\tilde{x})g_1^T(\tilde{x}) - g_2(\tilde{x})g_2^T(\tilde{x}))\frac{\partial V^T}{\partial e}(e) - h_1^T(\tilde{x})h_1(\tilde{x})+ \\ &\quad + \frac{\partial V}{\partial \tilde{x}}(\tilde{x})(g_1(\tilde{x})w + g_2(\tilde{x})u) + \frac{\partial V}{\partial \tilde{x}}(\tilde{x})g_1(\tilde{x})(h_0(x) - h_0(\tilde{x})) \\ &= S(U, e, \tilde{x}) + R_1(e, \tilde{x}) + R_2(e, \tilde{x}) \\ &\quad + \left\| u + \frac{1}{2}g_2^T(\tilde{x})\frac{\partial V^T}{\partial \tilde{x}}(\tilde{x}) + \frac{1}{2}(g_2^T(x) - g_2^T(\tilde{x}))\frac{\partial U^T}{\partial e}(e) \right\|^2 - \frac{1}{4}\left\| (g_2^T(x) - g_2^T(\tilde{x}))\frac{\partial U^T}{\partial e}(e) \right\|^2 + \\ &\quad - \left\| w - \frac{1}{2}g_1^T(\tilde{x})\frac{\partial V^T}{\partial \tilde{x}}(\tilde{x}) - \frac{1}{2}(g_1^T(x) - g_1^T(\tilde{x}))\frac{\partial U^T}{\partial e}(e) \right\|^2 - \|z\|^2 + \|w\|^2 \\ &\leq \left\| u + \frac{1}{2}g_2^T(\tilde{x})\frac{\partial V^T}{\partial \tilde{x}}(\tilde{x}) + \frac{1}{2}(g_2^T(x) - g_2^T(\tilde{x}))\frac{\partial U^T}{\partial e}(e) \right\|^2 - \frac{1}{4}\left\| (g_2^T(x) - g_2^T(\tilde{x}))\frac{\partial U^T}{\partial e}(e) \right\|^2 + \\ &\quad - \left\| w - \frac{1}{2}g_1^T(\tilde{x})\frac{\partial V^T}{\partial \tilde{x}}(\tilde{x}) - \frac{1}{2}(g_1^T(x) - g_1^T(\tilde{x}))\frac{\partial U^T}{\partial e}(e) \right\|^2 - \|z\|^2 + \|w\|^2 \end{aligned}$$

If we take

$$u = -\frac{1}{2} g_2^T(\tilde{x}) \frac{\partial V^T}{\partial \tilde{x}}(\tilde{x}) = F_0(\tilde{x}),$$

then

$$\dot{W}(e, \tilde{x}) \leq -\|z\|^2 + \|w\|^2$$

so

$$\int_0^T (\|w\|^2 - \|z\|^2) dt \geq W(e(T), \tilde{x}(T)) - W(0,0) = W(e(T), \tilde{x}(T)) \geq 0.$$

for all $T \geq 0$ and $(x, \tilde{x}) \in \mathcal{B}_r$.

Next, consider the asymptotic stability of the closed-loop system. It is sufficient to show if $(x(0), \tilde{x}(0)) \in \mathcal{B}_r$, then $(x(t), \tilde{x}(t)) \to 0$ or $(e(t), x(t)) \to 0$ as $t \to \infty$. Take

$$u = -\frac{1}{2} g_2^T(\tilde{x}) \frac{\partial V^T}{\partial \tilde{x}}(\tilde{x}) \quad \text{and} \quad w = 0.$$

$V(\tilde{x})$ and $U(e)$ is positive definite by assumptions. Therefore $W(e, \tilde{x}) = U(e) + V(\tilde{x})$ is also positive definite, and it can be used as a Lyapunov function.

We already have

$$\dot{W}(e, \tilde{x}) \leq -\|z\|^2 + \|w\|^2 = -\|z\|^2 \leq 0$$

Now $\dot{W}(e, \tilde{x}) = 0 \Rightarrow z = 0 \quad \Rightarrow \quad x(t) \to 0$ as $t \to \infty$ by assumption **[A4]**. On the other hand, $\dot{U}(e) = T(U, e, \tilde{x})$ is negative definite for $(x, \tilde{x}) \in \mathcal{B}_r$ by previous lemma, then $e(t) \to 0$ as $t \to \infty$. By LaSalle's Theorem, the closed loop system is (locally) asymptotically stable.

Finally, the DF controller $u = \Omega(P_{DF}, K_{FI})y$ is recovered as

$$\begin{cases} \dot{\tilde{x}} = f(\tilde{x}) - \frac{1}{2} g_2(\tilde{x}) g_2^T(\tilde{x}) \frac{\partial V^T}{\partial \tilde{x}}(\tilde{x}) - g_1(\tilde{x}) h_0(\tilde{x}) - g_1(\tilde{x}) y \\ u = -\frac{1}{2} g_2^T(\tilde{x}) \frac{\partial V^T}{\partial \tilde{x}}(\tilde{x}) \end{cases}$$

□

Next, we consider the $\mathcal{H}_\infty$-controller parameterization. Denote

$$\epsilon_u(x, \tilde{x}) := \frac{1}{2}(g_2^T(x) - g_2^T(\tilde{x})) \frac{\partial U^T}{\partial e}(e), \qquad \epsilon_w(x, \tilde{x}) := \frac{1}{2}(g_1^T(x) - g_1^T(\tilde{x})) \frac{\partial U^T}{\partial e}(e)$$

Define

$$u^* := -\frac{1}{2} g_2^T(\tilde{x}) \frac{\partial V^T}{\partial \tilde{x}}(\tilde{x}) - \epsilon_u(x, \tilde{x})$$

$$w^* := \frac{1}{2} g_1^T(\tilde{x}) \frac{\partial V^T}{\partial \tilde{x}}(\tilde{x}) + \epsilon_u(x, \tilde{x})$$

From the previous proof,

$$\dot{W}(e, \tilde{x}) \leq \|u - u^*\|^2 - \|w - w^*\|^2 - \|z\|^2 + \|w\|^2$$

If

$$u = u^* + Q(w - w^*)$$

with $Q \in \mathcal{FG}$, let ξ be the state variable for Q, and U_Q is a solution to the HJI with respect to Q, then

$$\dot{U}_Q(\xi) \leq \|w - w^*\|^2 - \|u - u^*\|^2$$

then

$$\dot{W}(e, \tilde{x}) + \dot{U}_Q(\xi) \leq -\|z\|^2 + \|w\|^2$$

so

$$\int_0^T (\|w\|^2 - \|z\|^2)dt \geq W(e(T), \tilde{x}(T)) + U_Q(\xi(T)) \geq 0.$$

It seems that the controller recovered from $u = u^* + Q(w - w^*)$ solves the $\mathcal{H}_\infty$-control problem for DF. However, it can hardly be physically implemented, since (i) Q requires some extra high order terms $\epsilon_u(x, \tilde{x})$ and $\epsilon_w(x, \tilde{x})$ as parts of its input and output, but they can not be provided by the closed loop system; and (ii) the term $h_0(x) - h_0(\tilde{x})$, which is a part of the measured output, does not appear in the required input for Q. But, fortunately, the terms $\epsilon_u(x, \tilde{x})$ and $\epsilon_w(x, \tilde{x})$ can be eliminated by the assumption **[A6]**; and $h_0(x) - h_0(\tilde{x})$ is actually the measured noise introduced by the controller. It is reasonable to take $w_{DF} = w + h_0(x) - h_0(\tilde{x})$ as the total disturbance.

The DF $\mathcal{H}_\infty$-control problem can be modified as follows.

DF $\mathcal{H}_\infty$ CONTROL PROBLEM: Find a class of controllers such that the closed loop system is locally asymptotically stable and satisfies

$$\int_0^T (\|z\|^2 - \|w_{DF}\|^2)dt \leq 0.$$

for all $T \geq 0$, as long as the state of the closed loop system $x, \tilde{x} \in \mathcal{B}_r$; i.e., the controllers attenuate the external disturbance and the measured noise introduced by itself.

Theorem 4.5 *Under the assumptions of previous Theorem, define*

$$F_0(\tilde{x}) := -\frac{1}{2} g_2^T(\tilde{x}) \frac{\partial V^T}{\partial \tilde{x}}(\tilde{x}), \qquad F_1(\tilde{x}) := \frac{1}{2} g_1^T(\tilde{x}) \frac{\partial V^T}{\partial \tilde{x}}(\tilde{x}).$$

The controller $u = \Omega(M_{FD}, Q)y$ with M_{DF} given by

$$\begin{cases} \dot{\tilde{x}} &= f(\tilde{x}) + g_2(\tilde{x})F_0(\tilde{x}) - g_1(\tilde{x})h_0(\tilde{x}) &+ & g_1(\tilde{x})y &+& g_2(\tilde{x})u_0 \\ u &= F_0(\tilde{x})) & & & & u_0 \\ y_0 &= -h_0(\tilde{x}) - F_1(\tilde{x}) &+& y & & \end{cases}$$

for all $Q \in \mathcal{FG}$ also solves the DF $\mathcal{H}_\infty$-control problem.

Proof Consider system $\Omega(G_{DF}, \Omega(M_{DF}, Q))$, assume ξ is the state of Q. Take $W(e, \tilde{x})$ the same as in previous them, the same arguments in previous proof yield that, for all $(x, \tilde{x}, \xi) \in \mathcal{B}_r$.

$$\dot{W}(e,\tilde{x}) \leq \left\| u + \frac{1}{2} g_2^T(\tilde{x}) \frac{\partial V^T}{\partial \tilde{x}}(\tilde{x}) \right\|^2 - \left\| w_{DF} - \frac{1}{2} g_1^T(\tilde{x}) \frac{\partial V^T}{\partial \tilde{x}}(\tilde{x}) \right\|^2 - \|z\|^2 + \|w_{DF}\|^2$$

Since $Q \in \mathcal{FG}$, then there is a positive definite $U_Q(\xi)$ related to Q such that

$$\dot{U}_Q(\xi) \leq - \left\| u + \frac{1}{2} g_2^T(\tilde{x}) \frac{\partial V^T}{\partial \tilde{x}}(\tilde{x}) \right\|^2 + \left\| w_{DF} - \frac{1}{2} g_1^T(\tilde{x}) \frac{\partial V^T}{\partial \tilde{x}}(\tilde{x}) \right\|^2$$

So

$$\dot{W}(e,\tilde{x}) + \dot{U}_Q(\xi) \leq - \|z\|^2 + \|w_{DF}\|^2$$

It can be concluded that

$$\int_0^T (\|w_{DF}\|^2 - \|z\|^2) dt \geq 0, \forall T \in R^+.$$

Next, consider the stability of the closed loop system whose state is $(e, \tilde{x}, \xi)$, set $w = 0$, then it is of the form

$$\begin{cases} \dot{e} = \alpha(t, e) \\ \dot{\tilde{x}} = \beta(t, e, \tilde{x}, \xi) \\ \dot{\xi} = a(\xi) + b(\xi) y_0 \end{cases}$$

where $y_0|_{(e,\tilde{x})=0} = 0$. Notice that the subsystems with states e, $\tilde{x}$ and ξ can be viewed as being hierarchically interconnected[1]. Assume the closed loop system evolves in $\mathcal{B}_r$. $e \to 0$ as $t \to \infty$ by the similar argument in the proof of previous Theorem. Consider the connected system with state $(\tilde{x}, \xi)$; $L_{DF}(\tilde{x}, \xi) := W(e, \tilde{x})|_{e=0} + U_Q(\xi)$ is positive definite, and can be used as the Lyapunov function. Let $w = 0$. Since $e = \tilde{x} - x = 0$, $w_{DF} = 0$.

$$\dot{L}_{DF}(\tilde{x}, \xi) \leq - \|z\|^2 \leq 0$$

$\dot{L}_{DF}(\tilde{x}, \xi) = 0$ implies $z = 0$, so $\tilde{x}(t) = x(t) \to 0$ as $t \to \infty$. On the other hand, $\tilde{x} = 0 \to y_0 = 0$, this also implies $\xi(t) \to 0$ as $t \to \infty$. LaSalle's Theorem implies interconnected system with states $(\tilde{x}, \xi)$ is asymptotically stable. By the stability Theorem of hierarchical systems (Vidyasagar, 1980), The closed loop system is locally asymptotically stable. □

4.3 Full Control Problem

The main purpose of considering FC problem is to provide some required tools for the construction of output feedback $\mathcal{H}_\infty$-controllers. The system is as follows.

$$G_{FC}: \begin{cases} \dot{x} &= f(x) + g_1(x) w + \begin{bmatrix} I & 0 \end{bmatrix} u \\ z &= h_1(x) + \begin{bmatrix} 0 & I \end{bmatrix} u \\ y &= h_2(x) + k_{21}(x) w \end{cases} \tag{13}$$

[1] Strictly speaking, the subsystems are not hierarchically interconnected; this issue is discussed in detail in Lu (1994a).

with constraint **[A3]**. An additional assumption for this structure is as follows.

[A5]: $[h_2(x), f(x)]$ is zero-state detectable..

The solvability of $\mathcal{H}_\infty$-control problem to FC is also related to the HJI:

$$\mathcal{H}_{FC}(U,x) := \frac{\partial U}{\partial x}(x)f(x) + \frac{1}{4}\frac{\partial U}{\partial x}(x)g_1(x)g_1^T(x)\frac{\partial U^T}{\partial x}(x) + h_1^T(x)h_1(x) - h_2^T(x)h_2(x) \leq 0. \tag{14}$$

Theorem 4.6 *(i) If $U(x)$ with $U(0) = 0$ satisfies*

$$\mathcal{H}_{OI}(U, L_0, x) := \frac{\partial U}{\partial x}(x)(f(x) + L_0(x)h_2(x)) +$$

$$+\frac{1}{4}\frac{\partial U}{\partial x}(x)(g_1(x) + L_0(x)k_{21}(x))(g_1(x) + L_0(x)k_{21}(x))^T\frac{\partial U^T}{\partial x}(x) + h_1^T(x)h_1(x) \leq 0$$

for some $L_0(x)$, then $U(x)$ satisfies $\mathcal{H}_{FC}(U,x) \leq 0$ with $U(0) = 0$ as well.

Conversely, if $U(x)$ satisfies $\mathcal{H}_{FC}(U,x) \leq 0$ with $U(0) = 0$, and $L_0(x)$ is such that

$$\frac{\partial U}{\partial x}(x)L_0(x) = -2h_2^T(x),$$

then $\mathcal{H}_{OI}(U, L_0, x) \leq 0$.

(ii) If system $[h_2(x), f(x)]$ is zero-state observable. Suppose $U(x) \geq 0$ solves $\mathcal{H}_{FC}(U,x) \leq 0$ with $U(0) = 0$. If

$$\frac{\partial U}{\partial x}(x)L_0(x) = -2h_2^T(x),$$

has a solution $L_0(x)$, then $U(x)$ is positive definite.

Proof (i) Notice that there exists $L_0(x)$ such that

$$\mathcal{H}_{OI}(U, L_0, x) \leq 0$$

only if

$$0 \geq \inf_{\frac{\partial U}{\partial x}(x)L_0(x)} \mathcal{H}_{OI}(U, L_0, x) = \mathcal{H}_{OI}(U, L_0, x)|_{\frac{\partial U}{\partial x}(x)L_0(x) = -2h_2^T(x)} = \mathcal{H}_{FC}(U,x)$$

And it is also sufficient if $\frac{\partial U}{\partial x}(x)L_0(x) = -2h_2^T(x)$ has a solution $L_0(x)$.

(ii) Will be proved during the proof of the following Theorem. □

Theorem 4.7 *Suppose $U : \mathbf{R}^n \to \mathbf{R}^+$ is locally positive definite such that $\mathcal{H}_{FC}(U,x) \leq 0$. If $L_0(x)$ satisfies*

$$\frac{\partial U}{\partial x}(x)L_0(x) = -2h_2^T(x), \tag{15}$$

then the controller given by "output injection"

$$u(x) = \begin{bmatrix} L_0(x) \\ 0 \end{bmatrix} y$$

solves the $\mathcal{H}_\infty$-control problem for FC.

Proof (i) $\mathcal{L}_2$**-gain** ≤ 1. Just $U(x) \geq 0$ is assumed.

$$\begin{aligned}
\dot{U}(x) &= \frac{\partial U}{\partial x}(x)(f(x) + g_1(x)w + \left[\begin{array}{cc} I & 0 \end{array}\right] u) \\
&\leq -\frac{1}{4}\frac{\partial U}{\partial x}(x)g_1(x)g_1^T(x)\frac{\partial U^T}{\partial x}(x) - h_1^T(x)h_1(x) + h_2^T(x)h_2(x) + \\
&\quad + \frac{\partial U}{\partial x}(x)g_1(x)w + \left[\begin{array}{cc} \frac{\partial U}{\partial x}(x) & 0 \end{array}\right] u \\
&= \left\| \left[\begin{array}{cc} 0 & I \end{array}\right] u \right\|^2 - \left\| w - \frac{1}{2}g_1^T(x)\frac{\partial U^T}{\partial x}(x) + k_{21}^T(x)h_2(x) \right\|^2 - \|z\|^2 + \|w\|^2 + \\
&\quad + (\left[\begin{array}{cc} \frac{\partial U}{\partial x}(x) & 2h_1^T(x) \end{array}\right] u + 2h_2^T(x)(h_2(x) + k_{21}(x)w))
\end{aligned}$$

Note that $k_{21}(x)k_{21}^T(x) = I$ for all $x \in \mathbf{R}^n$ by assumption **[A3]**, so $\|k_{21}(x)v\| \leq \|v\|$ for all $v \in \mathbf{R}^{p_1}$.

Observe that $y = h_2(x) + k_{21}(x)w = k_{21}(x)(w - \frac{1}{2}g_1^T(x)\frac{\partial U^T}{\partial x}(x) + k_{21}^T(x)h_2(x))$, then

$$\|y\|^2 = \|h_2(x) + k_{21}(x)w\|^2 \leq \left\| w - \frac{1}{2}g_1^T(x)\frac{\partial U^T}{\partial x}(x) + k_{21}^T(x)h_2(x) \right\|^2$$

So

$$\dot{U}(x) \leq \left\| \left[\begin{array}{cc} 0 & I \end{array}\right] u \right\|^2 - \|y\|^2 - \|z\|^2 + \|w\|^2 + (\left[\begin{array}{cc} \frac{\partial U}{\partial x}(x) & 2h_1^T(x) \end{array}\right] u + 2h_2^T(x)y)$$

Note that

$$\dot{U}(x) \leq -\|y\|^2 - \|z\|^2 + \|w\|^2 \leq -\|z\|^2 + \|w\|^2$$

if u is such that $\left[\begin{array}{cc} 0 & I \end{array}\right] u = 0$ and $\left[\begin{array}{cc} \frac{\partial U}{\partial x}(x) & 2h_1^T(x) \end{array}\right] u + 2h_2^T(x)y = 0$; but the latter is guaranteed by taking the controller as the given "output injection":

$$u(x) = \left[\begin{array}{c} L_0(x) \\ 0 \end{array}\right] y$$

where $L_0(x)$ solves

$$\frac{\partial U}{\partial x}(x)L_0(x) = -2h_2^T(x).$$

It follows that

$$\int_0^T (\|w\|^2 - \|z\|^2)dt \geq U(x(T)) - U(0) = U(x(T)) \geq 0$$

for all $T \geq 0$.

(ii) **Proof of Theorem 4.6(ii)**. It is assumed $[h_2(x), f(x)]$ is observable in this proof.

If the above controller is taken, then the closed loop system is

$$\begin{cases} \dot{x} = f(x) + L_0(x)h_2(x) + (g_1(x) + L_0(x)k_{21}(x))w \\ z = h_1(x) \\ y = h_2(x) + k_{21}(x)w \end{cases}$$

From the previous proof,

$$\dot{U}(x) \leq -\|y\|^2 - \|z\|^2 + \|w\|^2$$

Since $U(x) \geq 0$, the closed loop system is dissipative with respect to $\|w(t)\|^2 - \|z(t)\|^2 - \|y(t)\|^2$. For all $x \in \mathbf{R}^n$,

$$\begin{aligned} U(x) &\geq \sup_{w \in \mathcal{L}_2[0,\infty), x(0)=x} - \int_0^\infty (\|w(t)\|^2 - \|z(t)\|^2 - \|y(t)\|^2)dt \\ &\geq -\int_0^\infty (0 - \|z(t)\|^2 - \|y(t)\|^2)dt = \int_0^\infty (\|z(t)\|^2 + \|y(t)\|^2)dt \end{aligned}$$

$U(x) = 0 \Longrightarrow y(t) = h_2(x(t)) = 0$ for $w(t) = 0$.

$$\begin{cases} \dot{x} = f(x) + L_0(x)h_2(x) \\ h_2(x) = 0 \end{cases} \Leftrightarrow \begin{cases} \dot{x} = f(x) \\ h_2(x) = 0 \end{cases}$$

By the observability assumption $x = 0$, $U(x)$ is (locally) positive definite.

(iii) **Asymptotic Stability**
$U(x)$ is positive definite by assumption, it can be used as a Lyapunov function.
Set $w = 0$,

$$\dot{U}(x) \leq \|w(t)\|^2 - \|z(t)\|^2 - \|h_2(x(t))\|^2 = -\|z(t)\|^2 - \|h_2(x(t))\|^2 \leq 0.$$

So $\dot{U}(x) = 0 \quad \Rightarrow \quad h_2(x(t)) = 0 \quad \Rightarrow \quad x(t) \rightarrow 0$ as $t \rightarrow \infty$ by assumption **[A5]**. LaSalle's Theorem implies $\dot{x} = f(x) + L_0(x)h_2(x)$ is asymptotically stable.

□

From the above proof,

$$\dot{U}(x) \leq \|u_2\|^2 - \|y\|^2 - \|z\|^2 + \|w\|^2 + (\frac{\partial U}{\partial x}(x)u_1 + 2h_1^T(x)u_2 + 2h_2^T(x)y)$$

Assume $L_1(x)$ is such that

$$\frac{\partial U}{\partial x}(x)L_1(x) = -2h_1^T(x).$$

Take $u_1 = L_0(x)y + L_1(x)u_2$, then

$$\frac{\partial U}{\partial x}(x)u_1 + 2h_1^T(x)u_2 + 2h_2^T(x)y = 0$$

Therefore

$$\dot{U}(x) \leq \|u_2\|^2 - \|y\|^2 - \|z\|^2 + \|w\|^2$$

Let $u_2 = Qy$ with $Q \in \mathcal{FG}$ (so $u_1 = L_0(x)y + L_2(x)Qy$ then), Q can be assumed to have the following realization

$$\begin{cases} \dot{\xi} = a(\xi) + b(\xi)y \\ u_2 = c(\xi) \end{cases}$$

Then there exists $U_c(\xi) \geq 0$ positive definite such that

$$\dot{U}_c(\xi) \leq \|y\|^2 - \|u_2\|^2$$

and $\dot{\xi} = a(\xi)$ is asymptotically stable.

Define $W(x,\xi) = U(x) + U_c(\xi)$ for $(x,\xi) \in \mathbf{R}^n \times \mathbf{R}^n$, then $W(x,\xi) \geq 0$ is positive definite.

$$\begin{aligned} \dot{W}(x,\xi) &= \dot{U}(x) + \dot{U}_c(\xi) \\ &\leq (\|u_2\|^2 - \|y\|^2 - \|z\|^2 + \|w\|^2) + (\|y\|^2 - \|u_2\|^2) \\ &= \|w\|^2 - \|z\|^2 \end{aligned}$$

So

$$\int_0^T (\|w\|^2 - \|z\|^2)dt \geq W(x(T),\xi(T)) - W(0,0) = W(x(T),\xi(T)) \geq 0.$$

for all $T \geq 0$.

Thus, we motivated the characterization of a class of controllers.

Theorem 4.8 *The assumptions are the same as in the last Theorem. If in addition, U is such that $\mathcal{H}_{FC}(U,x)$ is negative definite, and $L_1(x)$ also satisfies:*

$$\frac{\partial U}{\partial x}(x)L_1(x) = -2h_1^T(x), \tag{16}$$

then

$$u = \begin{bmatrix} L_0(x) + L_1(x)Q \\ Q \end{bmatrix} y$$

for all $Q \in \mathcal{FG}$ also solves the $\mathcal{H}_\infty$-control problem for FC.

Proof We only need to consider the stability. Since $\mathcal{H}_{FC}(U,x)$ is negative definite, then there exist a positive definite $\pi(x)$, such that

$$\mathcal{H}_{FC}(U,x) + \pi(x) \leq 0$$

By corollary 2.3,

$$\dot{U}(x) \leq \|u_2\|^2 - \|y\|^2 - \|z\|^2 + \|w\|^2 - \pi(x)$$

$W(x,\xi) = U(x) + U_c(\xi)$ is positive definite, as $U(x)$ and $U_c(\xi)$ are. It can be taken as Lyapunov function, let $w = 0$, then

$$\dot{W}(x,\xi) \leq (\|u_2\|^2 - \|y\|^2 - \|z\|^2 - \pi(x)) + \|y\|^2 - \|u_2\|^2) = -\|z\|^2 - \pi(x) \leq 0$$

Note that $\dot{W}(x,\xi) = 0$ implies $\pi(x) = 0$, $x = 0$ by assumption. If $x = 0$ then $\dot{\xi} = a(\xi) + b(\xi)y = a(\xi)$ is asymptotically stable, so $\xi(t) \to 0$ as $t \to \infty$. LaSalle's Theorem implies the asymptotic stability. □

Remark 4.2 Notice that the suggested $\mathcal{H}_\infty$-controllers for FC problems characterized in terms of HJI (14) in general depend also on the full state x. This is because the solutions $L_0(x)$ and $L_1(x)$ of equations (15) and (16) in general depend on the state x. The pure full control solution is possible when equations (15) and (16) both have constant solutions; however this condition is restrictive. At this point, we would satisfy with the way the controllers are constructed. It might be argued that the obvious controller u with u_1 stabilizes the system, say $u_1 = -f(x) - x$, and $u_2 = -h_1(x)$ can do this job with resulting $\mathcal{L}_2$-gain $= 0$, since $z = 0$ in this case. Nevertheless, the main purpose of considering the FC problem in terms of HJI (14) is to build up the required machinery for the output feedback problem. The current treatment indeed achieves this goal. This point will be clear soon.

4.4 Output Estimation Problem

Consider

$$G_{OE} : \begin{cases} \dot{x} &=& f(x) &+& g_1(x)w &+& g_0(x)u \\ z &=& h_1(x) &+& & & u \\ y &=& h_2(x) &+& k_{21}(x)w & & \end{cases} \tag{17}$$

where $g_1, g_0, h_1, h_2, k_{21} \in \mathbf{C}^2$ with constraint **[A3]**. An additional assumption for this structure is as follows.

[A5]: $[h_2(x), f(x)]$ is (locally) zero-state detectable.

Theorem 4.9 *Consider G_{OE} (17), suppose there exists a local positive definite solution $U(x)$ to HJI: $\mathcal{H}_{FC}(U,x) \leq 0$, with $U(0) = 0$; and $U(x)$ makes the Hessian matrix of $\mathcal{H}_{FC}(U,x)$ with respect to $x \in \mathbf{R}^n$ be negative definite at 0. If $L_0(x)$ satisfies*

$$\frac{\partial U}{\partial x}(x) L_0(x) = -2h_2^T(x)$$

then there exists a controller such that the closed loop system has $\mathcal{L}_2$-gain ≤ 1 as long as the state of the closed loop system does not leave $\mathcal{B}_r$. And such a controller can be given by

$$K_{OE} : \begin{cases} \dot{\tilde{x}} &=& f(\tilde{x}) - g_0(\tilde{x})h_1(\tilde{x}) + L_0(\tilde{x})h_2(\tilde{x}) &-& L_0(\tilde{x})y \\ u &=& -h_1(\tilde{x}) & & \end{cases}$$

Furthermore, the closed loop system is also locally asymptotically stable at 0, if in addition,

[A7]: *$\dot{x} = f(x) - g_0(x)h_1(x)$ is locally asymptotically stable.*

Lemma 4.10 *Suppose there exists a locally positive definite function $U(x) \geq 0$ such that $\mathcal{H}_{FC}(U,x)$ is locally negative definite. Let $x, \tilde{x}$ be states of systems G_{OE} and K_{OE}, $e = \tilde{x} - x$. Define*

$$\begin{aligned} \mathcal{H}_e(e,\tilde{x}) := & \frac{\partial U}{\partial e}(e)(f(\tilde{x}) - f(x) + L_0(\tilde{x})(h_2(\tilde{x}) - h_2(x)))+ \\ & + \frac{1}{4}\frac{\partial U}{\partial e}(e)(g_1(x) + L_0(\tilde{x})k_{12}(x))(g_1(x) + L_0(\tilde{x})k_{12}(x))^T \frac{\partial U^T}{\partial e}(e)+ \\ & +(h_1^T(\tilde{x}) - h_1^T(x))(h_1(\tilde{x}) - h_1(x)) - \frac{\partial U}{\partial e}(e)(g_0(x) - g_0(\tilde{x}))h_1(\tilde{x}) \end{aligned}$$

with $L_0(\tilde{x})$ defined as in previous Theorem. Then for all $(x, \tilde{x}) \in \mathcal{B}_r$ with some $r > 0$, $\mathcal{H}_e(e,\tilde{x}) \leq 0$. Moreover, there exists a locally positive definite function $\pi(e)$ such that $\mathcal{H}_e(e,\tilde{x}) + \pi(e) \leq 0$.

Proof Recall that $\mathcal{H}_{OI}(U, L_0, e) = \mathcal{H}_{FC}(U, e)$, where

$$\mathcal{H}_{OI}(U, L_0, e) := \frac{\partial U}{\partial e}(e)(f(e) + L_0(e)h_2(e)) +$$

$$+\frac{1}{4}\frac{\partial U}{\partial e}(e)(g_1(e) + L_0(e)k_{21}(e))(g_1(e) + L_0(e)k_{21}(e))^T \frac{\partial U^T}{\partial e}(e) + h_1^T(e)h_1(e)$$

Also note that the Hessian matrix of $\mathcal{H}_e(e, \tilde{x})$ with respect to e at 0 is the same as the one of $\mathcal{H}_{OI}(U, L_0, e) = \mathcal{H}_{FC}(U, e)$ with respect to e at 0. The conclusion follows by the similar argument in the proof of Lemma 4.4. □

Remark 4.3 If we take $u = \begin{bmatrix} L_0(\tilde{x}) \\ 0 \end{bmatrix} y$ with $y_0 = h_2(\tilde{x}) - h_2(x) - k_{12}(x)w$, let $z = h_1(x) - h_1(\tilde{x}) + \begin{bmatrix} 0 & I \end{bmatrix} u$, then

$$\dot{U}(e) = \frac{\partial U}{\partial e}(e)(f(\tilde{x}) - f(x)) + g_1(x)w + \begin{bmatrix} I & 0 \end{bmatrix} u)$$

$$\leq -\|y_0\|^2 + \|w\|^2 - \|z\|^2 - \frac{\partial U}{\partial e}(e)(g_0(x) - g_0(\tilde{x}))h_1(\tilde{x})$$

for all $(x, \tilde{x}) \in \mathcal{B}_r$.

Proof **[Theorem 4.9]** Consider $\Omega(G_{OE}, K_{FC})$ which has following realization

$$\begin{cases} \dot{x} &= f(x) - g_0(x)h_1(\tilde{x}) & + & g_1(x)w \\ \dot{\tilde{x}} &= (f(\tilde{x}) - g_0(\tilde{x})h_1(\tilde{x})) + L_0(\tilde{x})(h_2(\tilde{x}) - h_2(x)) & - & L_0(\tilde{x})k_{21}(x)w \\ z &= h_1(x) - h_1(\tilde{x}) \end{cases}$$

Let $e = \tilde{x} - x$, for $(x, \tilde{x}) \in \mathcal{B}_r$.

$$\dot{U}(e) = \frac{\partial U}{\partial e}(e)(f(\tilde{x}) - f(x) + L_0(\tilde{x})(h_2(\tilde{x}) - h_2(x)) +$$

$$-(L_0(\tilde{x})k_{21}(x) + g_1(x))w) - \frac{\partial U}{\partial e}(e)(g_0(\tilde{x}) - g_0(x))h_1(\tilde{x})$$

$$\leq \|w\|^2 - \|z\|^2 - \|y_0\|^2 \leq -\|z\|^2 + \|w\|^2$$

so

$$\int_0^T (\|w\|^2 - \|z\|^2)dt \geq U(e(T)) - U(0) = U(e(T)) \geq 0.$$

for all $T \geq 0$.

Next, the asymptotic stability of the closed loop system is considered. It has the realization as follows.

$$\begin{cases} \dot{\tilde{x}} = f(\tilde{x}) - g_0(\tilde{x})h_1(\tilde{x}) + L_0(\tilde{x})(h_2(\tilde{x}) - h_2(x)) \\ \dot{e} = \eta(t, e) \end{cases}$$

Since $\dot{U}(e)$ is positive definite from the negative definiteness of (Hessian matrix of) $\mathcal{H}_{FC}(U,e)$. Therefore $e(t) \to 0$ as $t \to \infty$. Note also that $\dot{\tilde{x}} = f(\tilde{x}) - g_0(\tilde{x})h_1(\tilde{x})$ is asymptotically stable. Since the two systems are hierarchically interconnected, and asymptotically stable, the interconnected system is asymptotically stable (cf. Vidyasagar, 1980). $\square$

Remark 4.4 It is observed that the requirement for system $\dot{x} = f(x) - g_0(x)h_1(x)$ to be (locally) asymptotically stable is necessary for the zero-dynamics of the closed loop system to be (locally) asymptotically stable (Van der Schaft, 1993b).

Theorem 4.11 *Under the assumption of the previous Theorem, if in addition, $L_1(x)$ is such that*

$$\frac{\partial U}{\partial x}(x)L_1(x) = -2h_1^T(x),$$

then the controller $u = \Omega(M_{OE}, Q)y$ with M_{OE} given by

$$\left\{\begin{array}{lllllll} \dot{\tilde{x}} & = & f(\tilde{x}) - g_0(\tilde{x})h_1(\tilde{x}) + L_0(\tilde{x})h_2(\tilde{x}) & - & L_0(\tilde{x})y & + & (g_2(\tilde{x}) + L_1(\tilde{x}))u_2 \\ u & = & -h_1(\tilde{x}) & + & & & u_2 \\ y_0 & = & h_2(\tilde{x}) & - & y & & \end{array}\right.$$

for all $Q \in \mathcal{FG}$ also (locally) solves $\mathcal{H}_\infty$-control problem for OE.

Proof Consider $\Omega(G_{OE}, \Omega(M_{OE}, Q))$ for $Q \in \mathcal{FG}$ which has following realization.

$$\left\{\begin{array}{l} \dot{\xi} = a(\xi) + b(\xi)y \\ u_2 = c(\xi) \end{array}\right.$$

The similar argument shows that there exists $r > 0$, for $(x, \hat{x}, \xi) \in \mathcal{B}_r$,

$$\dot{U}(e) \le \|w\|^2 - \|z\|^2 - \|y_0\|^2 + \|u_2\|^2 - \pi(e)$$

for some locally positive definite $\pi(e)$.

And U_Q is a solution to the HJI with respect to Q with state ξ, then

$$\dot{U}_Q(\xi) \le \|y_0\|^2 - \|u_2\|^2$$

So

$$\dot{U}(e) + \dot{U}_Q(\xi) \le -\|z\|^2 + \|w\|^2 - \pi(e) \le -\|z\|^2 + \|w\|^2$$

Therefore,

$$\int_0^T (\|z\|^2 - \|w\|^2)dt \le U(0) - U(e(T)) = -U(e(T)) \le 0.$$

As for the stability, let $w = 0$ then the closed loop system has following hierarchical structure.

$$\left\{\begin{array}{l} \dot{e} = \eta(t, e) \\ \dot{\xi} = a(\xi) + b(\xi)(h_2(\tilde{x}) - h_2(x)) \\ \dot{\tilde{x}} = f(\tilde{x}) - g_0(\tilde{x})h_1(\tilde{x}) + L_0(\tilde{x})(h_2(\tilde{x}) - h_2(x)) + (g_2(\tilde{x}) + L_1(\tilde{x}))c(\xi) \end{array}\right.$$

Take $L_{OE}(e,\xi) = U(e) + U_Q(\xi)$ as the Lyapunov function of the interconnected system with state (e,ξ), then $\dot{L}_{OE}(e,\xi) \le -\|z\|^2 - \pi(e)$. Now $\dot{L}_{OE}(e,\xi) = 0 \Rightarrow \pi(e) = 0$, so $e = 0$. So in this case $\dot{\xi} = a(\xi)$; but it is asymptotically stable, so $\xi(t) \to 0$ as $t \to \infty$. The interconnected (e,ξ) is locally asymptotically stable by LaSalle's Theorem. Now if $(e,\xi) = 0$ then $\dot{\tilde{x}} = f(\tilde{x}) - g_0(\tilde{x})h_1(\tilde{x})$; but it is locally asymptotically stable at 0 by assumption **[A7]**. Thus, we can conclude that this closed loop system is asymptotically stable by the stability Theorem for hierarchical systems (Vidyasagar, 1980). $\square$

5 $\mathcal{H}_\infty$-Control: Output Feedback Problems

We now consider the output feedback $\mathcal{H}_\infty$ control problem. The solutions to this problem are based on the results in the last section.

5.1 Solutions to Output Feedback Problems

The NLTI plant is realized as affine state-space equation:

$$G : \begin{cases} \dot{x} &= f(x) + g_1(x)w + g_2(x)u \\ z &= h_1(x) + k_{12}(x)u \\ y &= h_2(x) + k_{21}(x)w \end{cases} \tag{18}$$

where $f, g_i, h_i, k_{ij} \in \mathbf{C}^2$ and $f(0) = 0, h_1(0) = 0, h_2(0) = 0$; x, w, u, z, and y are assumed to have dimensions n, p_1, p_2, q_1, and q_2, respectively.

The following assumptions are made:

[A2]: $k_{12}^T(x)\begin{bmatrix} h_1(x) & k_{12}(x) \end{bmatrix} = \begin{bmatrix} 0 & I \end{bmatrix}$;

[A3]: $\begin{bmatrix} g_1(x) \\ k_{21}(x) \end{bmatrix} k_{21}^T(x) = \begin{bmatrix} 0 \\ I \end{bmatrix}$;

[A4]: $[h_1(x), f(x)]$ is zero-state detectable.

The main idea of construction of the $\mathcal{H}_\infty$-controller is to convert the general problem OF into the simpler problems which have been solved.

Let $V(x) \ge 0$ be the solution of $\mathcal{H}_{FI}(V,x) \le 0$. Define

$$F_0(x) := -\frac{1}{2}g_2^T(x)\frac{\partial V^T}{\partial x}(x), \qquad F_1(x) := \frac{1}{2}g_1^T(x)\frac{\partial V^T}{\partial x}(x), \tag{19}$$

and new variables $r := w - F_1(x)$ and $v := u - F_0(x)$. We get a new system

$$G_a : \begin{cases} \dot{x} &= f_a(x) + g_1(x)r + g_2(x)u \\ v &= h_a(x) + u \\ y &= h_2(x) + k_{21}(x)r \end{cases} \tag{20}$$

where $f_a(x) := f(x) + g_1(x)F_1(x), h_a(x) := F_0(x)$.

We have the following lemma.

Lemma 5.1 *Consider systems G (18) and G_a (20). If the controller K makes $\Omega(G_a, K)$ have $\mathcal{L}_2$-gain ≤ 1, it also results in $\Omega(G,K)$ having $\mathcal{L}_2$-gain ≤ 1.*

Proof Note that $z = \Omega(G, K)w$ and $r = \Omega(G_t, K)v$.

Since $V(x) \geq 0$ solves $\mathcal{H}_{FI}(V, x) + \psi(x) = 0$, then

$$\begin{aligned}
\dot{V}(x) &= \frac{\partial V}{\partial x}(x)(f(x) + g_1(x)w + g_2(x)u) \\
&= -\|z\|^2 + \|w\|^2 - \left\| w - \frac{1}{2} g_1^T(x) \frac{\partial V^T}{\partial x}(x) \right\|^2 + \left\| u + \frac{1}{2} g_2^T(x) \frac{\partial V^T}{\partial x}(x) \right\|^2 - \psi(x) \\
&\leq -\|z\|^2 + \|w\|^2 - \|r\|^2 + \|v\|^2
\end{aligned}$$

So for all $T \geq 0$,

$$\int_0^T (\|w\|^2 - \|z\|^2)dt \geq \int_0^T (\|r\|^2 - \|v\|^2)dt + V(x(T)) \geq \int_0^T (\|r\|^2 - \|v\|^2)dt.$$

$$\int_0^T (\|r\|^2 - \|v\|^2)dt \geq 0 \quad \Rightarrow \quad \int_0^T (\|w\|^2 - \|z\|^2)dt \geq 0.$$

□

Define

$$\mathcal{H}_a(W, x) := \frac{\partial W}{\partial x}(x) f_a(x) + \frac{1}{4} \frac{\partial W}{\partial x}(x) g_1(x) g_1^T(x) \frac{\partial W^T}{\partial x}(x) + h_a^T(x) h_a(x) - h_2^T(x) h_2(x).$$

Take $W(x) = U(x) - V(x)$ with $W(0) = U(0) - V(0)$ where $V(x) \geq 0$ is given just now. Note that

$$\mathcal{H}_a(W, x) = \mathcal{H}_{FC}(U, x) - \mathcal{H}_{FI}(V, x) = \mathcal{H}_{FC}(U, x) + \psi(x),$$

where $\psi(x) \geq 0$ is such that $\mathcal{H}_{FI}(V, x) + \psi(x) = 0$. Thus, $\mathcal{H}_a(W, x) \leq 0$ if and only if $\mathcal{H}_{FC}(U, x) + \psi(x) \leq 0$. Assume $U(x)$ is such that $\mathcal{H}_{FC}(U, x) + \psi(x) \leq 0$ has a positive definite Hessian matrix at $x = 0$, then $\mathcal{H}_a(W, x)$ also has negative definite Hessian matrix at 0. Suppose $L_0(x)$ is such that $\frac{\partial W}{\partial x}(x) L_0(x) = -2h_2^T(x)$. The controller K for the new OE structure given by Theorem 4.9 is

$$K : \begin{cases} \dot{\hat{x}} = f_a(\hat{x}) + g_2(\hat{x}) h_a(\hat{x}) + L_0(\hat{x}) h_2(\hat{x}) - L(\hat{x}) y \\ u = h_a(\hat{x}) \end{cases}$$

is such that system $\Omega(G_a, K)$ locally has $\mathcal{L}_2$-gain ≤ 1.

By lemma 5.1, $\Omega(G, K)$ has $\mathcal{L}_2$-gain ≤ 1. Next, we examine the stability of the closed loop system $\Omega(G, K)$ which has the following realization,

$$\begin{cases} \dot{x} = f(x) + g_2(x) F_0(\hat{x}) + g_1(x) w \\ \dot{\hat{x}} = f_K(\hat{x}) + L_0(\hat{x})(h_2(\hat{x}) - h_2(x)) + L_0(\hat{x}) k_{21}(x) w \\ z = h_1(x) + k_{12}(x) F_0(\hat{x}) \end{cases}$$

where

$$f_K(\hat{x}) := f(\hat{x}) + g_1(\hat{x}) F_1(\hat{x}) + g_2(\hat{x}) F_0(\hat{x}). \tag{21}$$

Take $e = \hat{x} - x$. Note that $\mathcal{H}_a(W, \cdot)$ has negative definite Hessian matrix as does $\mathcal{H}_{FC}(U, \cdot)$. Using the same technique as in the proof of Theorem 4.7, it can be concluded that for some locally positive definite $\pi : \mathbf{R}^n \to \mathbf{R}^+$, such that if $(x, \hat{x}) \in \mathcal{B}_s$ for some $s > 0$

$$\dot{W}(e) \leq \|r\|^2 - \|v\|^2 - \pi(e)$$

Let $L_{OF}(x, e) = V(x) + W(e)$ with $e = \hat{x} - x$. By assumption $V(x)$ and $W(e)$ are positive definite, so is $L(x, e)$, and it can be used as a Lyapunov function. Take $w = 0$,

$$\dot{V}(x) \leq -\|z\|^2 + \|v\|^2 - \|r\|^2$$

$$\dot{L}_{OF}(x, e) = \dot{V}(x) + \dot{W}(e) \leq -\|z\|^2 - \pi(e) \leq 0.$$

Then $\dot{L}_{OF}(x, e) = 0 \Rightarrow z = 0$ and $\pi(e) = 0 \Rightarrow x = 0$ and $e = 0$. Therefore, $\dot{L}_{OF}(x, e)$ is locally negative definite, the closed loop system is locally asymptotically stable.

Therefore, we have the following results about output $\mathcal{H}_\infty$-control problem,

Theorem 5.2 *Consider G (7), if there is some $\psi(x) \geq 0$ with $\psi(0) = 0$ such that*

(i) there exists a locally positive definite $V(x)$ which solves the HJE: $\mathcal{H}_{FI}(V, x) + \psi(x) = 0$ with $V(0) = 0$.

(ii) there exists a locally positive definite $U(x)$ which satisfies the HJI: $\mathcal{H}_{FC}(U, x) + \psi(x) \leq 0$ with $U(0) = 0$. And $\mathcal{H}_{FC}(U, x) + \psi(x)$ has nonsingular Hessian matrix at 0.

(iii) $U(x) - V(x) \geq 0$ is locally positive definite. And

$$\left(\frac{\partial U}{\partial x}(x) - \frac{\partial V}{\partial x}(x)\right)L_0(x) = -2h_2^T(x),$$

has a solution $L_0(x)$. Then the output feedback $\mathcal{H}_\infty$-control problem is solvable. Furthermore,

$$K : \begin{cases} \dot{\hat{x}} = f_K(\hat{x}) + L_0(\hat{x})h_2(\hat{x}) - L_0(\hat{x})y \\ u = F_0(\hat{x}) \end{cases}$$

is such a controller.

Similar results to the above theorem are also obtained by Isidori (1992). Note that $\mathcal{H}_\infty$-controllers have separation structures. The separation principle for the $\mathcal{H}_\infty$-performance in nonlinear systems was first confirmed by Ball-Helton-Walker (1993; see also Isidori, 1992). Similar arguments to Theorems 4.11 and 5.2 can be also used to construct the controller parameterization as follows (see also Lu and Doyle, 1993a).

Theorem 5.3 *Consider a system G (18) satisfying the condition in Theorem 5.1. If in addition $L_1(x)$ satisfies*

$$\left(\frac{\partial U}{\partial x}(x) - \frac{\partial V}{\partial x}(x)\right)L_1(x) = -2h_1^T(x),$$

then the controller $u = \Omega(M, Q)y$ with M given by

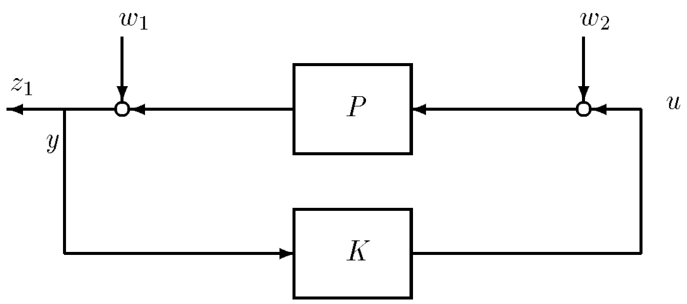
$$\begin{cases} \dot{\tilde{x}} &= f_K(\tilde{x}) - L_0(\tilde{x})y + (g_2(\tilde{x}) + L_1(\tilde{x}))u_0 \\ u &= F_0(\tilde{x}) + u_0 \\ y_0 &= h_2(\tilde{x}) - y \end{cases}$$

for all $Q \in \mathcal{FG}$ also solves output feedback $\mathcal{H}_\infty$-control problem.

Proof Consult Lu and Doyle (1993a). □

5.2 Examples

In this subsection, we will examine an example from Doyle et al. (1993). The basic block diagram is as follows



Where P is a passive nonlinear plant; K is the controller to be designed such that the output z_1 is regulated; y is the measured output, based on which the control action u is produced; w_2 is the disturbance from the actuator; and w_1 is the noise from the sensor. The **control problem** is to design the controller K such that the influence of the noises w_1 and w_2 on the regulated output z_1 can be reduced to the minimal with the reasonable effort (control action should not be too large).

To formulate this problem, all the signals are considered in space $\mathcal{L}_2[0,\infty)$. The $\mathcal{H}_\infty$-control problem in this setting is to find a controller K (if any) for given $\gamma > 0$, such that

$$\int_0^T (\|z_1\|^2 + r\|u\|^2)dt \leq \gamma^2 \int_0^T (\|w_1\|^2 + \|w_2\|^2)dt, \forall T \in \mathbf{R}^+; w_1, w_2 \in \mathcal{L}_2[0,\infty).$$

where $r > 0$ is weighting factor. Let $\gamma^*(r)$ be the smallest γ satisfying the above inequality.

The case where P is a general lossless plant is considered in Van der Schaft (1993). In the following, a passive plant which has the following realization is considered.

$$\begin{cases} \dot{x} = e^x(w_2 + u) \\ z_1 = x + w_1 \\ y = x + w_1 \end{cases}$$

We will consider two cases with different weighting factors $r = 0$ and $r = 1$. In both cases, since the stability of the resulting closed loop systems can be easily checked by using the corresponding Theorems, we just consider the $\mathcal{H}_\infty$-performance problem in a local sense.

Case I: $r = 0$

Consider the control problem that a controller K is designed such that:

$$\int_0^T \|z_1\|^2 dt \leq \gamma_0^2 \int_0^T (\|w_1\|^2 + \|w_2\|^2)dt, \forall T \in \mathbf{R}^+$$

where $\gamma_0 = 1/(1-\epsilon)$ for some $0 < \epsilon < 1$.

To standardize the problem, take

$$w := \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \qquad z := (1-\epsilon)z_1$$

Thus, the state-space realization is:

$$\begin{cases} \dot{x} &= & & \begin{bmatrix} 0 & e^x \end{bmatrix} w & + & e^x u \\ z &= & (1-\epsilon)x & + \begin{bmatrix} 1-\epsilon & 0 \end{bmatrix} w \\ y &= & x & + \begin{bmatrix} 1 & 0 \end{bmatrix} w \end{cases}$$

Change the variable $u' = e^x u$. Now it looks like the output-injection control problem. But it is not standard, since we need to get ride of the term $\begin{bmatrix} 1-\epsilon & 0 \end{bmatrix} w$ in the regulated output z. By using Safonov et al's simplification method (Safonov et al, 1989), we have the following simplified system:

$$\begin{cases} \dot{x} &= & & \begin{bmatrix} 0 & e^x \end{bmatrix} w & + & u' \\ z &= & \frac{1-\epsilon}{\sqrt{2\epsilon-\epsilon^2}}x \\ y_N &= & \frac{1}{\sqrt{2\epsilon-\epsilon^2}}x & + \begin{bmatrix} 1 & 0 \end{bmatrix} w \end{cases}$$

with $y_N = \sqrt{2\epsilon - \epsilon^2}y$. Now the system has a required output-injection structure.

Consider the HJI with respect to this structure:

$$\mathcal{H}_{FC}(U, x) = \frac{\partial U}{\partial x}(x) \cdot 0 + \frac{1}{4}e^{2x}(\frac{\partial U}{\partial x}(x))^2 + (\frac{1-\epsilon}{\sqrt{2\epsilon-\epsilon^2}}x)^2 - (\frac{1}{\sqrt{2\epsilon-\epsilon^2}}x)^2 \leq 0$$

A class of local positive solutions $U(x)$ are such that

$$\frac{\partial U}{\partial x}(x) = 2\rho e^{-x}x$$

for $0 \leq \rho \leq 1$. Take $\rho = 1$, the solution $L(x)$ to

$$\frac{\partial U}{\partial x}(x)L(x) = \frac{2x}{\sqrt{2\epsilon - \epsilon^2}}$$

is

$$L(x) = -\frac{e^x}{\sqrt{2\epsilon - \epsilon^2}}$$

It follows that the controller is

$$u' = L(x)y_N = -\frac{e^x}{\sqrt{2\epsilon-\epsilon^2}} \cdot \sqrt{2\epsilon - \epsilon^2}y = -e^x y$$

or the output-injection can be recovered as $u = -y$. Note that it is independent of ϵ.

This $\mathcal{H}_\infty$ controller is identity ($K = -1$). Actually, we have following general result.

Theorem 5.4 (Doyle et al., 1993) *Consider the feedback system as shown. Suppose the plant P has the same number of inputs and outputs, and is passive, i.e.*

$$\int_0^T (Pe)^T e dt \geq 0, \qquad \forall T \in \mathbf{R}^+$$

and $K = -1$, then

$$\int_0^T \|z_1\|^2 \, dt \leq \int_0^T (\|w_1\|^2 + \|w_2\|^2)dt, \forall T \in R^+$$

Case II: $r = 1$

Consider the control problem that a controller K is designed such that:

$$\int_0^T (\|z_1\|^2 + \|u\|)dt \le \gamma_0^2 \int_0^T (\|w_1\|^2 + \|w_2\|^2)dt, \forall T \in \mathbf{R}^+$$

where $\gamma_0 = \sqrt{2}/(1-\epsilon)$ for some $0 < \epsilon < 1$.

Take

$$w := \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \qquad z := \frac{1-\epsilon}{\sqrt{2}} \begin{bmatrix} z_1 \\ u \end{bmatrix}$$

Thus, the state-space realization is:

$$\begin{cases} \dot{x} = \begin{bmatrix} 0 & e^x \end{bmatrix} w + e^x u \\ z = \begin{bmatrix} \frac{1-\epsilon}{\sqrt{2}} x \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{1-\epsilon}{\sqrt{2}} & 0 \\ 0 & 0 \end{bmatrix} w + \begin{bmatrix} 0 \\ \frac{1-\epsilon}{\sqrt{2}} \end{bmatrix} u \\ y = x + \begin{bmatrix} 1 & 0 \end{bmatrix} w \end{cases}$$

To standardize this structure, we can get the simplified system by the method introduced in Safonov et al (1989).

$$\begin{cases} \dot{x} = \begin{bmatrix} 0 & e^x \end{bmatrix} w + \frac{\sqrt{2}}{1-\epsilon} e^x u_N \\ z = \begin{bmatrix} \frac{1-\epsilon}{\sqrt{1+2\epsilon-\epsilon^2}} x \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_N \\ y_N = \sqrt{\frac{2}{1+2\epsilon-\epsilon^2}} x + \begin{bmatrix} 1 & 0 \end{bmatrix} w \end{cases}$$

with $u_N = \frac{1-\epsilon}{\sqrt{2}} u$ and $y_N = \sqrt{\frac{1+2\epsilon-\epsilon^2}{2}} y$. Now the system has a required output feedback structure.

Now take $\psi(x) = 0$ as in Theorem 5.2, consider HJE:

$$\mathcal{H}_{FI}(V, x) = \frac{\partial V}{\partial x}(x) \cdot 0 + \frac{1}{4}(e^{2x} - \frac{2e^{2x}}{(1-\epsilon)^2})(\frac{\partial V}{\partial x}(x))^2 + (\frac{1-\epsilon}{\sqrt{1+2\epsilon-\epsilon^2}} x)^2 = 0.$$

The positive solution $V(x)$ is such that

$$\frac{\partial V}{\partial x}(x) = \frac{2(1-\epsilon)^2}{1+2\epsilon-\epsilon^2} e^{-x} x$$

Also consider HJI:

$$\mathcal{H}_{FC}(U, x) = \frac{\partial U}{\partial x}(x) \cdot 0 + \frac{1}{4} e^{2x} (\frac{\partial U}{\partial x}(x))^2 + (\frac{1-\epsilon}{\sqrt{1+2\epsilon-\epsilon^2}} x)^2 - (\sqrt{\frac{2}{1+2\epsilon-\epsilon^2}} x)^2 \le 0$$

A class of positive definite solutions $U(x)$ are such that

$$\frac{\partial U}{\partial x}(x) = 2\rho e^{-x} x$$

for $0 \le \rho < 1$. They make $\mathcal{H}_{FC}(U, x)$ have negative Hessian matrix at 0.

Now it can be easily checked that $U(x) - V(x)$ is positive definite if ρ is taken to be close enough to 1. And $L_0(x)$ can be solved by:

$$(\frac{\partial U}{\partial x}(x) - \frac{\partial V}{\partial x}(x)) L_0(x) = -2\sqrt{\frac{2}{1+2\epsilon-\epsilon^2}} x$$

Then the controller can be constructed by Theorem 5.2.

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