Analysis of Implicit Uncertain Systems

Part I: Theoretical Framework

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Abstract

This paper introduces a general and powerful framework for the analysis of uncertain systems, encompassing linear fractional transformations, the behavioral approach for system theory and the integral quadratic constraint formulation. In this approach, a system is defined by implicit equations, and the central analysis question is to test for solutions of these equations. In Part I, the general properties of this formulation are developed, and computable necessary and sufficient conditions are derived for a robust performance problem posed in this framework.

1 Introduction

In the predominant viewpoint in systems and control theory, a system is an input-output (I/O) entity, where the variables are clearly separated in two groups, and a cause-effect relationship is established between them. This approach entails a "signal flow" conception, adequate for systems which are deliberately built to match the I/O philosophy, such as computers and amplifiers.

For many other physical systems this point of view often appears artificial; as an example, a mass or energy balance equation in a chemical process is more naturally thought of as an equation or constraint between variables than as a cause-effect law. While this observation

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will appear as no surprise to an engineer performing modeling of such a system, it is only recently that its theoretical implications have been extensively considered.

In a research program best summarized by the survey paper [19], Willems has advocated an approach to system theory where the central concept is the *behavior*, a set of allowed signal trajectories, and no input-output partition is a priori established between the variables. The corresponding theory of finite dimensional linear systems has been extensively developed [19].

This paper adopts the same philosophy and contends that this point of view is even more natural for systems involving uncertainty. If the relationship between the variables is not precisely known, the cause-effect point of view is itself suspect: it is more natural to think of an uncertain *implicit* relationship between variables.

It is noteworthy that partial versions of this viewpoint have been present in early work leading to current robust control theory. In the foundational paper [22], Zames states the basic stability theorems describing systems as *relations*, motivated by some nonlinear systems which do not fit the I/O concept; the same ideas are present in Safonov [16] in the early years of robust control. A further example which has led to powerful analysis techniques is the Integral Quadratic Constraint (IQC) formulation of Yakubovich [21] and Megretski [9, 10], where a component is described by constraints between the signals involved.

It might be argued that as long as the I/O framework involves no mathematical loss of generality, an argument of symmetry or esthetics is not sufficient to abandon an approach which is widespread. This paper shows evidence, however, that with respect to robustness analysis, an implicit approach strictly enhances the range of applications of existing theory for the I/O setting. The main extension which is provided is the ability to analyze over-constrained systems in a unified framework; these arise when superimposing an uncertain model and a number of constraints relevant to the analysis problem under consideration.

In Part I of this paper we propose a theoretical framework which encompasses the behavioral approach for system theory, the Linear Fractional Transformation (LFT) paradigm for uncertainty descriptions, and the IQC formulation. Section 3 introduces the framework and shows how it allows for the formulation of a general robust performance analysis problem, in terms of finding solutions to uncertain *equations*.

This problem leads to a notion of robust stability for implicit systems, presented in Section 4, which naturally extends the existing I/O theory. The general analysis problem is reduced to a canonical case to be considered in the rest of the paper.

Section 5 contains an analysis test for robust stability of an implicit system when the nominal equations are linear, time invariant, and the uncertainty is allowed to vary in the class of arbitrary norm bounded operators. This condition is a convex feasibility test on a constant scaling, which extends the scaled small-gain conditions for robust stability, and recent results [17, 10] on the necessity of these conditions for the standard I/O setting. The extension also includes δI operator blocks in the uncertainty description.

Preliminary versions of these results were presented in the conference papers [14, 15]. All proofs are collected in the Appendix.

2 Notation

In this paper we will consider vector spaces of signals $\mathcal{V} \subset (\mathbb{F}^n)^{\mathbb{T}}$, where $\mathbb{F}$ is the field $\mathbb{R}$ or $\mathbb{C}$, and the time index $\mathbb{T}$ can denote continuous time $\mathbb{R}$ or $\mathbb{R}^+$, or discrete time $\mathbb{Z}$ or $\mathbb{Z}^+$; for concreteness, the results will be presented in discrete time, but most of the theory extends to continuous time in a straightforward way. For $T \in \mathbb{T}$, define the truncation operator P^T as:

$$P^T : v(t) \rightarrow \begin{cases} v(t) & \text{if} \quad t \leq T \\ 0 & \text{if} \quad t > T \end{cases} \tag{1}$$

Signal norms can be introduced, choosing $\mathcal{V}$ as a Banach space of signals. Following [22, 5], we introduce extended signal spaces to include signals which "blow up" at infinite time, by defining (for positive time axis $\mathbb{T} = \mathbb{Z}^+$) $\mathcal{V}_e = \{v \in (\mathbb{F}^n)^{\mathbb{T}} : P^T(v) \in \mathcal{V} \;\; \forall T \in \mathbb{T}\}$. In this paper we will consider the 2-norm for signals; we will indicate later which parts can be extended to other signal norms. l_2^n will denote the Hilbert space of square summable, $\mathbb{F}^n$-valued sequences over $\mathbb{Z}$ or $\mathbb{Z}^+$; $\|v\|^2 = \sum_{t \in \mathbb{T}} |v(t)|^2$. l_{2e}^n is the corresponding extended space over $\mathbb{Z}^+$. The spatial dimension n will be dropped when clear from context.

The class of linear, bounded operators $G : l_2^n \rightarrow l_2^m$ is denoted $\mathcal{L}(l_2^n, l_2^m)$ or simply $\mathcal{L}(l_2)$. We are also interested in operators which can be extended to l_{2e}; in this respect it is convenient to consider the class of *causal* maps, which verify $P^T G P^T = P^T G$ for all T. Equivalently, a causal linear map can be characterized by having a lower triangular infinite matrix representation.

A causal linear map over l_{2e} has finite gain if there exists $\gamma < \infty$ such that $\|P^T G v\| \leq \gamma \|P^T v\|$ for all $v \in l_{2e}$, $T \in \mathbb{T}$. Equivalently, the restriction of G to l_2 is a bounded operator, of norm $\|G\| = \sup_{T \in \mathbb{Z}^+}(\bar{\sigma}(G^T))$ where G^T is the matrix representation of $P^T G P^T$,

$$G^T = \begin{bmatrix} G_{00} & 0 & 0 & \cdots \\ G_{10} & G_{11} & \cdot & \cdots \\ \vdots & \vdots & \vdots & \cdots \\ G_{T0} & G_{T1} & \cdots & G_{TT} \end{bmatrix} \tag{2}$$

and $\bar{\sigma}$ denotes maximum singular value. We will also use the minimum singular value of a matrix $\underline{\sigma}(G) = \min\{\|G\xi\| : \|\xi\| = 1\}$.

The set of all causal, finite gain operators $G : l_{2e}^n \rightarrow l_{2e}^m$ is denoted $\mathcal{L}_c(l_{2e}^n, l_{2e}^m)$ or simply $\mathcal{L}_c(l_{2e})$. The unit delay operator is denoted by λ. A map G is time invariant if $G\lambda = \lambda G$.

3 The Implicit Framework for Analysis

3.1 Implicitly Defined Systems

This paper deals with implicit characterizations of systems. Loosely speaking, this means that the laws governing a system and the constraints imposed on a problem under consideration are all expressed as *equations* on a specified set of variables. A formal definition follows.

Definition 1 *An implicit system (W, E, G) is defined by two vector spaces, the* **variable** *space W and the* **equation** *space E, and an* **equation operator** *$G : W \rightarrow E$. The* **behavior** *of the implicit system is the set $\mathcal{B} = Ker(G) = \{w \in W : Gw = 0\}$. The system is called linear if G is a linear map.*

The definition above is closely related to the *behavioral* approach to system theory, introduced by Willems [19]. In this type of formulation, all variables in a system are a priori on an equal footing, without a distinction between inputs and outputs. The system laws are constraints in the possible values of these variables, which define a set: the behavior.

In our case the *implicit equations* defined by G play a central role, not captured entirely by the behavior, since as we shall see equation *error* will be added for the analysis. Interconnections of subsystems is reduced to superimposing equations.

These descriptions arise naturally when modeling physical systems from first principles, where physical laws are more naturally thought of as equations between variables than as "signal-flow" maps. For instance, a resistor in a circuit is naturally modeled by the equation $v - Ri = 0$, and there is no need to specify a cause-effect relationship between the two variables v and i; this distinction is artificial and not available a priori. For further discussion of the features of this modeling paradigm, see [19, 2].

An important special case of Definition 1 is the class of *dynamical* implicit systems, where the sets W and E are vector-valued signal spaces. As an example, if $R(\xi)$ is a polynomial matrix, the differential equations $R(\frac{d}{dt})w = 0$ define an implicit system, where G is the differential operator $R(\frac{d}{dt})$, and W, E can be chosen as spaces of smooth functions (or alternatively distribution spaces).

The choice of the defining elements W, E, G is essentially determined by the type of analysis to be performed. In this paper we will consider two possible settings which are standard in robust control:

1. To formulate quantitative performance specifications, a signal normed space is required; we use the l_2 space $W = l_2^q$, $E = l_2^p$, with G a bounded mapping between them. This paper deals with linear equations $G \in \mathcal{L}(l_2)$.

2. In some cases in stability theory it cannot be assumed a priori that signals have finite norm, and the extended spaces $W = l_{2e}^q$, $E = l_{2e}^p$ are used, together with $G \in \mathcal{L}_c(l_{2e})$.

These two cases will be followed in parallel throughout this paper.

3.2 Uncertainty and LFTs

We now incorporate into the implicit paradigm deterministic descriptions of uncertainty in the style of robust control. The equation map G is replaced by a parameterized map $G(\Delta)$, where

Δ is an uncertainty operator. A general class of uncertainty descriptions can be parameterized by the class of "spatially" structured perturbations, of the form

$$\Delta = diag\,[\delta_1 I_{r_1}, \ldots, \delta_L I_{r_L}, \Delta_{L+1}, \ldots, \Delta_{L+F}] \tag{3}$$

The *diag* notation implies that $\Delta z = [\delta_1 z_1, \ldots, \delta_L z_L, \ldots \Delta_{L+F} z_{L+F}]$, where the vector z is broken into spatial components z_i of appropriate dimensions. The blocks $\delta_i I_{r_i}$, Δ_{L+j} can be used to describe real parameters, or dynamic (linear time invariant (LTI), linear time varying (LTV) or nonlinear) perturbations; the square $\delta_i I_{r_i}$ blocks correspond to an uncertainty perturbation which is repeated in the spatial dimension; the possibly non-square full blocks Δ_{L+j} are unrestricted maps. In each case, there is a restricted class $\mathbf{\Delta}$ of allowed perturbations. In this paper we restrict ourselves to linear uncertainty blocks, i.e $\mathbf{\Delta} \subset \mathcal{L}(l_2^n)$ or $\mathbf{\Delta} \subset \mathcal{L}_c(l_{2e}^n)$. The uncertainty will be normalized to a unit ball $\mathbf{B}_{\mathbf{\Delta}}$ in the operator norm, $\mathbf{B}_{\mathbf{\Delta}} = \{\Delta \in \mathbf{\Delta} : \|\Delta\| \le 1\}$.

For the parameterization $G(\Delta)$ of implicit uncertain systems we will adopt the linear fractional transformation (LFT) paradigm (see [11]) depicted in Figure 1, which provides rich uncertainty descriptions. Implicit LFT representations were first considered in [3].

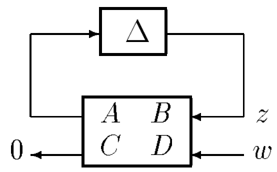

Figure 1: Implicit LFT system

In Figure 1, $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ is a nominal map in $\mathcal{L}(l_2^{n+q}, l_2^{m+p})$ or $\mathcal{L}_C(l_{2e}^{n+q}, l_{2e}^{m+p})$; an important special case is M time-invariant. The uncertainty Δ has the structure (3).

A remark regarding Figure 1 is that it contains remnants of the "signal-flow" approach, since the parameter Δ is depicted as an input-output map. This is done to highlight the connection with the standard LFT paradigm, but from an implicit point of view the system in Figure 1 is simply characterized by the *equations*

$$\varphi(\Delta, M)\begin{bmatrix} z \\ w \end{bmatrix} = 0, \quad \text{where} \quad \varphi(\Delta, M) \triangleq \begin{bmatrix} I - \Delta A & -\Delta B \\ C & D \end{bmatrix} \tag{4}$$

This description is "internal" since the signals z produced by the uncertainty operators are included in the variable space W. The parameterization $G(\Delta)$ is therefore *affine* in the parameter Δ. This simple form allows the representation of a rich variety of uncertain systems; in fact, all the complexity is captured by the structure of Δ. A standard input-output LFT uncertain system can be easily converted to this implicit form (see section 3.4 below).

3.3 Integral Quadratic Constraints

We will now present a feature of the implicit analysis framework which does not have a counterpart in the I/O setting: it allows for the representation of additional constraints in the signals of a robustness analysis problem.

In particular, the implicit formulation over l_2 permits the representation of Integral Quadratic Constraints (IQCs), which have been proposed by Yakubovich [21] and Megretski [9] as the basis of an alternative paradigm for robust control. IQCs are time-invariant quadratic forms in signal space, which in discrete time have the form

$$\langle \Pi z, z \rangle = \int_{-\pi}^{\pi} z(e^{j\omega})^* \Pi(e^{j\omega}) z(e^{j\omega})\, d\omega \quad \leq 0 \tag{5}$$

where $z(e^{j\omega})$ is the Fourier transform of an l_2 signal, $\Pi = \Pi^*$ is an LTI operator, assumed bounded on l_2 (i.e $\Pi(e^{j\omega})$ is in L_∞).

IQCs can be used to provide deterministic descriptions of an uncertain *component*, by defining a set of signals which captures the known information about the component (as in [9]). As will be explored in Part II of this paper, it may be of interest to describe properties of a *disturbance*, by constraining it in terms of IQCs; of such nature are "whiteness" constraints as in [12], which describe spectral properties of a disturbance.

It is now shown that these general IQCs can be captured by an uncertain implicit system. First choose P LTI (e.g. $P = kI$) such that $P^*P - \Pi > 0$. A spectral factorization gives

$P(e^{j\omega})^* P(e^{j\omega}) - \Pi(e^{j\omega}) = Q(e^{j\omega})^* Q(e^{j\omega})$, where Q can be chosen in $\mathcal{H}_\infty$. So $\Pi = P^*P - Q^*Q$, which reduces (5) to $\|Pz\|_2^2 \leq \|Qz\|_2^2$. We now introduce the following lemmas:

Lemma 1 *Let $z \in l_2^m, v \in l_2^n$. The following are equivalent:*

$$(i)\ \|v\|_2^2 - \|z\|_2^2 \geq 0 \tag{6}$$

$$(ii)\ \exists \Delta \in \mathcal{L}(l_2^n, l_2^m), \|\Delta\| \leq 1 :\ \Delta v = z \tag{7}$$

Lemma 2 *Let $z, v \in l_2^d$. The following are equivalent:*

$$(i)\ \int_{-\pi}^{\pi} v(e^{j\omega})v(e^{j\omega})^* - z(e^{j\omega})z(e^{j\omega})^*\ d\omega\ \geq 0 \tag{8}$$

$$(ii)\ \forall \eta \in \mathbf{C}^d, \|\eta^* v\|_2 \geq \|\eta^* z\|_2 \tag{9}$$

$$(iii)\ \exists \delta \in \mathcal{L}(l_2), \|\delta\| \leq 1 :\ \delta I_d v = z \tag{10}$$

From the above discussion and Lemma 1, for each z satisfying (5) there exists an operator Δ_C, $\|\Delta_C\| \leq 1$ such that $Pz = \Delta_C Qz$. So the set of $z \in l_2$ satisfying (5) can be described as the union over $\Delta_C, \|\Delta_C\| \leq 1$ of the behavior of the uncertain implicit system

$$(P - \Delta_C Q)w = 0 \tag{11}$$

Remarks:

- Although for each Δ_C the set $Ker(P - \Delta_C Q)$ is a linear subspace, the union of the parameterized behaviors describes a more complicated set given by (5).
- The set $\|\Delta_C\| \leq 1$ includes arbitrary time-varying *non-causal* operators. In this respect, implicit systems obtained from this procedure are *a priori* considered in the l_2 setting.
- A finite number of IQCs can be given a representation (11), where Δ_C is now a structured uncertainty operator.

- Analogously, the set $\bigcup_{\|\Delta_C\|\leq 1} Ker(P - \delta_C I\ Q)$ for scalar δ_C can be shown by means of Lemma 2 to correspond to the matrix-valued constraint

$$\int_{-\pi}^{\pi} P(e^{j\omega})z(e^{j\omega})z(e^{j\omega})^* P(e^{j\omega})^* - Q(e^{j\omega})z(e^{j\omega})z(e^{j\omega})^* Q(e^{j\omega})^*\, d\omega \ \leq 0 \qquad (12)$$

  Matrix valued constraints appear naturally in the context of characterizing multivariable white noise disturbances, as shown in Part II.

3.4 Robust performance analysis in the implicit setting

To illustrate how implicit descriptions might be used for robust performance analysis, consider the uncertain I/O system of Figure 2, where it is known that the (disturbance) input d satisfies a finite number of restrictions in terms of IQCs as in (5). We want to determine whether there exist signals d in the allowed class, and perturbations Δ_u such that the system l_2 gain is 1 or larger. This last requirement is captured by the extra "performance IQC" $\|d\|^2 - \|y\|^2 \leq 0$.

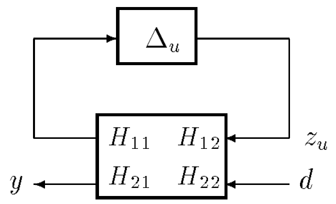

Figure 2: Input/Output LFT system

The implicit equations for the system, the IQCs on d and the performance IQC are captured respectively by (13),(14) and (15), where Δ_C and Δ_P are norm bounded operators (Δ_C is in general structured).

$$\begin{bmatrix} I - \Delta_u H_{11} & 0 & -\Delta_u H_{12} \\ H_{21} & -I & H_{22} \end{bmatrix} \begin{bmatrix} z_u \\ y \\ d \end{bmatrix} = 0 \qquad (13)$$

$$(P - \Delta_C Q)\, d = 0 \qquad (14)$$

$$\begin{bmatrix} \Delta_P & -I \end{bmatrix} \begin{bmatrix} y \\ d \end{bmatrix} = 0 \qquad (15)$$

The superposition of (13), (14) and (15) gives an implicit description for the robust performance analysis problem, which essentially reduces to the question:

> **Q**: "Does there exist a perturbation $(\Delta_u, \Delta_C, \Delta_P)$ such that (13-15) have non-trivial l_2 solutions?".

In **Q** the input/output partition has been eliminated from the problem, and the analysis is posed in terms of equations and solutions, rather than maps and gains; this allows for a natural incorporation of the constraints (14). Questions of this type are analyzed in the rest of this paper.

4 Stability in Implicit Systems

We begin by reviewing the concept of stability in standard system theory. Referring to the M-N feedback interconnection of Figure 3 (a), stability can be given two interpretations.

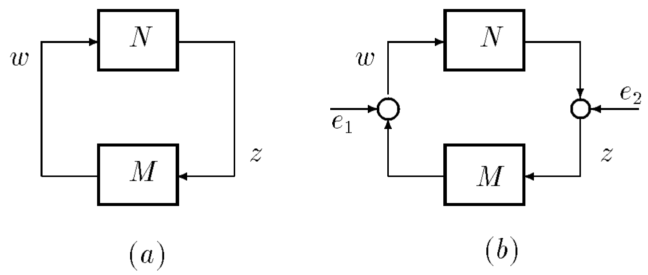

Figure 3: Stability in a standard feedback interconnection

In the first place, from the point of view of dynamical systems, stability ensures that solutions do not exist to the loop equations where the signals are unbounded (e.g. in $l_{2e} \backslash l_2$). This notion is usually considered for *causal* systems M, N.

Secondly, from an operator theoretic point of view, stability is sometimes interpreted as bounded *sensitivity* to small errors or disturbances (e_1, e_2 as in Figure 3 (b)) injected at the interconnection. In other words, the map $\begin{bmatrix} I & -M \\ -N & I \end{bmatrix}^{-1}$ between e_1, e_2 and signals w,z is a bounded operator in some normed space. This notion can be stated for non-causal systems

M and N, provided all signals are a priori constrained to the normed space (e.g. l_2) where the operators M and N are well defined.

Although these two versions are equivalent in many special cases of causal linear systems, we will find it useful to distinguish the two for the extension to the implicit framework, since we are led naturally to include non-causal perturbations as explained in Section 3.3. We will term the first notion "stability" and the second "l_2-stability". All the material in this section can be extended to other signal norms (e.g. l_∞) with minor changes.

4.1 Stability and l_2-Stability

Definition 2 *Let (W, E, G) be an implicit linear system over l_2 The system is l_2-stable if G is left invertible in $\mathcal{L}(l_2)$, i.e. $\exists L \in \mathcal{L}(l_2)$ such that $LG = I$.*

An equivalent characterization is

Proposition 3 *$Gw = 0$ is l_2-stable if and only if*

$$\inf\{\|Gw\| : w \in l_2, \|w\| = 1\} > 0 \tag{16}$$

Interpreting the definition, l_2-stability implies that the l_2 behavior $\mathcal{B}$ of the system is the trivial space, and that this property is not "sensitive" to equation error: an arbitrary small equation error e does not allow solutions w, $\|w\| = 1$, to the equation $Gw = e$.

An important remark is that in this framework, stability is only a property of systems with no free variables ("autonomous" systems, in the language of [19]), i.e. with at least as many effective equations as variables in w, so that only the trivial behavior is left inside l_2.

We will now compare this definition with the standard one, by considering the feedback interconnection of Figure 3. The maps M, N are (possibly non-causal) operators on l_2.

This interconnection can be represented by the following equations in the variables w, z:

$$\begin{bmatrix} I & -M \\ -N & I \end{bmatrix} \begin{bmatrix} w \\ z \end{bmatrix} = 0 \tag{17}$$

According to Definition 2, l_2-stability implies left invertibility of $G = \begin{bmatrix} I & -M \\ -N & I \end{bmatrix}$. This is slightly weaker than the usual definition, which requires G invertible (G left invertible and also G *onto* E). In other words, Definition 2 is weaker in the fact that equation errors are not required to be free to vary over l_2^p.

The reason for this weakened definition in the case of implicit systems is that we want to extend the notion to systems which are *over-constrained* (more equations than variables), such as the example considered in (13-15). In these cases, the operator will not be onto in general, and this should not be required: the equation errors need not be "free" since they are not physical noises (which should be included in the w variables); they just provide a means of testing sensitivity of equations.

We now turn to the notion of stability for causal systems. We consider the following definition, which extends the standard theory [22]:

Definition 3 *Let (W, E, G) be an implicit linear system over l_{2e}: $W = l_{2e}^q$, $E = l_{2e}^p$, $G : W \to E$ causal map . The system is said to be well posed if G has a causal left inverse $L : E \to W$. The system is stable if it is well posed and in addition, L has finite gain on the range of G, i.e. $\exists \gamma \in \mathbb{R}^+$ such that for every $e = Gw$, and every T, $\|P_T w\| \leq \gamma \|P_T e\|$.*

Remark: An immediate consequence of the definition is that a stable system, with errors $e \in l_2$, only allows signals $w \in l_2$ to satisfy the equation $Gw = e$.

Definition 3 can also be interpreted in terms of the infinite matrix representation (46) of the equation operator G, and its truncations G^T as in (2).

Proposition 4 *The implicit system (W, E, G) over l_{2e} is stable if and only if*

$$\inf_{T \geq 0} \underline{\sigma}\left(G^T\right) > 0 \tag{18}$$

To illustrate these definitions, we will consider the case of an "autoregressive" system (see [19]) defined by linear time-invariant equations of the form $G = R(\lambda)$, where $R(\xi)$ is a $p \times q$ polynomial matrix.

Proposition 5 *Let $R(\xi)$ be a $p \times q$ polynomial matrix. Then*

(i) The system $R(\lambda)w = 0$ is l_2-stable if and only if $R(\xi)$ has full column rank for all ξ in the unit circle ($|\xi| = 1$)

(ii) The system $R(\lambda)w = 0$ is stable if and only if $R(\xi)$ has full column rank for all ξ in the unit disk ($|\xi| \leq 1$)

Remark: The same argument carries through if $R(\xi)$ is a rational, rather than polynomial matrix, with no poles on the unit circle (respectively the unit disk).

Example 1 *Consider $G(\lambda) = 1 - 2\lambda$. Setting $W = E = l_{2e}$, we find that the implicit system over l_{2e} is not stable, since the signal $w(t) = 2^t, t \geq 0$, which is in $l_{2e} \backslash l_2$ gives $e = Gw \in l_2$ ($e(0) = 1$, $e(t) = 0$ for $t > 0$).*

However, the system over l_2 (with $W = E = l_2$) is l_2-stable since $\inf_{\|w\|=1} \|Gw\| = 1 > 0$.

4.2 Robust Stability and l_2-Stability

In this section we consider the case of implicit uncertain systems as in (4). For simplicity we will assume henceforth that the nominal equation map M is in $\mathcal{L}(l_2)$ for the l_2-stability case, or in $\mathcal{L}_c(l_{2e})$ for the stability case.

Definition 4 *Let $M \in \mathcal{L}(l_2)$, $\mathbf{\Delta} \subset \mathcal{L}(l_2)$. The implicit system* (4) *has robust l_2-stability if it is l_2-stable for each $\Delta \in \mathbf{B_\Delta}$.*

Robust l_2-stability implies that for *each* $\Delta \in \mathbf{B_\Delta}$, $\varphi(\Delta, M)$ has a bounded left inverse. In some treatments (e.g. [17]) a *uniform* notion of robust stability is employed, which in this context would imply that there is a uniform bound on the norms of the left inverses across $\mathbf{B_\Delta}$; we will not pursue this refinement here since it is appears to be no stronger in most cases of interest [1].

The constrained I/O robust performance problem posed in Section 3.4 converts to robust l_2-stability of the corresponding implicit system: a negative answer to **Q** implies a trivial

[1] We acknowledge G. Dullerud for pointing this out; see also the proof of theorem 10.

l_2-kernel for every Δ. Robust l_2 stability is slightly stronger in requiring this property to be insensitive to equation errors, but this is desirable anyway. This example will be reconsidered in 5.3.

The following proposition provides a simplified representation:

Proposition 6 *The implicit system* (4) *has robust l_2-stability if and only if*
(i) D is l_2-stable, with bounded left inverse L.
(ii) The implicit system $\begin{bmatrix} I - \Delta\hat{A} \\ \hat{C} \end{bmatrix} z = 0$, *where* $\hat{A} = A - BLC$, $\hat{C} = C - DLC$, *has robust l_2-stability.*

The previous result has reduced robust l_2 stability of (4) to a nominal l_2-stability condition (i) plus a robust l_2-stability condition (ii) in a simplified setup. We now turn to the situation of robust stability in systems over l_{2e}.

Definition 5 *Let* $M \in \mathcal{L}_c(l_{2e})$, $\Delta \in \mathcal{L}_c(l_{2e})$. *The implicit system* (4) *has robust stability if it is stable for each* $\Delta \in \mathbf{B_\Delta}$.

As is natural from the definition of stability, robust stability means that for each $\Delta \in \mathbf{B_\Delta}$, $\varphi(\Delta, M)$ admits a causal left inverse which has finite gain on the range of $\varphi(\Delta, M)$. For the reduction result corresponding to Proposition 6, the nominal stability condition will be slightly strengthened to

(i') D has causal left inverse L, which has finite gain over l_{2e}.

The strengthening comes from the fact that the finite gain of L is required over all l_{2e}, not just the range of D. This does not appear to be a major limitation, since it is no stronger for the case when D is LTI, as shown in the proof of Proposition 5.

Proposition 7 *Consider the system* (4) *where* $\Delta \in \mathcal{L}_c(l_{2e})$, *and* $M \in \mathcal{L}_c(l_{2e})$ *satisfies* (i'). *Then the system is robustly stable if and only if* $\begin{bmatrix} I - \Delta\hat{A} \\ \hat{C} \end{bmatrix} z = 0$ *is robustly stable, where* $\hat{A} = A - BLC$, $\hat{C} = C - DLC$.

The previous results show that for robustness analysis, it suffices to consider the canonical implicit system

$$\begin{bmatrix} I - \Delta A \\ C \end{bmatrix} z = 0 \tag{19}$$

where for simplicity we have renamed $\hat{A}, \hat{C}$ as A, C. This case will be considered in the rest of the paper.

It is useful to compare this setup with the question of robust stability in standard robust control, which specifies the invertibility of $(I - \Delta A)$. The main difference is that (19) allows for additional constraints defined by the C equations. A problem with more equations than variables such as the one considered in (13-15) will result in the presence of these additional equations.

A problem where the C equations do not appear will be termed the "unconstrained" case. For example, if the IQCs (14) are eliminated from (13-15), the problem can be reduced to the equations

$$\left(I - \begin{bmatrix} \Delta_u & 0 \\ 0 & \Delta_P \end{bmatrix} S \right) \begin{bmatrix} z_u \\ d \end{bmatrix} = 0 \tag{20}$$

which are in the standard form $(I - \Delta A)z = 0$.

In an unconstrained problem, the only apparent difference with the standard case is the fact that our definition of stability only specifies *left* invertibility of $(I - \Delta A)$. The following proposition shows that the two notions become in fact equivalent when they are considered across $\mathbf{B}_{\boldsymbol{\Delta}}$.

Proposition 8 *Let $A \in \mathcal{L}(l_2)$, $\boldsymbol{\Delta} \subset \mathcal{L}(l_2)$. Then $(I - \Delta A)$ has bounded left inverse $\forall \Delta \in \mathbf{B}_{\boldsymbol{\Delta}}$ if and only if $(I - \Delta A)$ has bounded inverse $\forall \Delta \in \mathbf{B}_{\boldsymbol{\Delta}}$.*

A similar result can be obtained for the case of robust stability of systems over l_{2e}. This shows nothing changes when using the left-invertibility notion for the unconstrained case; on the other hand, this is the only reasonable notion for constrained problems.

5 Robustness Analysis with LTV uncertainty

In this section we will focus on the class of implicit uncertain systems as in (19) where equation maps A, C are linear time invariant (LTI) and the uncertainty set $\mathbf{\Delta}$ consists of arbitrary linear-time varying perturbations with spatial structure as in (3).

The main result in this section is a necessary and sufficient condition for robust stability. This result extends the "scaled small gain" sufficient conditions for robust stability in the standard input output framework, and recent results on the necessity of the constant scales tests for linear time-varying perturbations, obtained by Shamma [17] and Megretski [9, 10] (and also previously in the l_∞ setting by Khammash [8]).

Once more the l_2 and l_{2e} settings are considered separately; we will concentrate, however, in the l_2 setting since the main motivation for over-constrained problems, given in section 3.4, refers to this case. These issues will be discussed further in section 5.3.

5.1 A Necessary and Sufficient Condition for Robust l_2-Stability

We now consider analysis in the l_2 setting; A ,C, Δ are in $\mathcal{L}(l_2)$ of appropriate dimensions (over $\mathbb{Z}$ or $\mathbb{Z}^+$, and are not restricted to be causal); A ,C are LTI.

In view of the duality between IQCs and implicit LFT descriptions which was shown in section 3.3, the theorem given below is a version in the implicit LFT framework of the S-procedure losslessness results of Megretski and Treil [10]. The main extension needed is to capture the δI blocks, or equivalently, to consider matrix-valued IQCs; this is done by extending the "∇ set" method in standard μ-analysis [11]. This extension has more than purely theoretical interest since these representations arise naturally in the context of set characterizations of white noise, as shown in Part II.

We begin with some further notation. For a delta structure $\mathbf{\Delta}$ of the form (3), let $\mathbf{Y}$ be the set of constant, hermitian scaling matrices $\mathbf{Y}$ that commute with the elements in $\mathbf{\Delta}$,

$$\mathbf{Y} = \{\mathrm{diag}\,[Y_1, \ldots, Y_L,\, y_{L+1} I_{m_1}, \ldots, y_{L+F} I_{m_F}]\,,\ Y_i = Y_i^*, i = 1 \ldots L,\, y_{L+j} \in \mathbf{R}, j = 1 \ldots F\} \qquad (21)$$

$\mathbf{Y}$ is a real vector space, and we can define an inner product

$$\langle Y, \bar{Y} \rangle = \sum_{i=1}^{L} \operatorname{tr}(Y_i \bar{Y}_i) + \sum_{j=1}^{F} y_{L+j} \bar{y}_{L+j} \tag{22}$$

Two important subsets of $\mathbf{Y}$ are $\mathbf{X} = \{Y \in \mathbf{Y}, Y > 0\}$, and $\bar{\mathbf{X}} = \{Y \in \mathbf{Y}, Y \geq 0\}$, the set of positive and nonnegative scalings, respectively. They are both convex cones in $\mathbf{Y}$.

Given a vector $z(t) \in l_2^n$, the block structure introduces a natural partition of z,

$$z = \left[z_1' \ldots z_L' \; z_{L+1}' \ldots z_{L+F}'\right]'$$

Given an LTI map $A \in \mathcal{L}(l_2)$, an analogous notation is used for the partition of $Az \in l_2^n$. Consider the following quadratic functions of $z \in l_2^n$, where $\xi = Az$:

$$\begin{aligned}
\Phi_i(z) &= \sum_{t=-\infty}^{\infty} \xi_i(t)\xi_i(t)^* - z_i(t)z_i(t)^* \quad i = 1 \ldots L \\
\sigma_{L+j}(z) &= \|(Az)_{L+j}\|^2 - \|z_{L+j}\|^2 = \sum_{t=-\infty}^{\infty} \xi_{L+j}(t)^*\xi_{L+j}(t) - z_{L+j}(t)^* z_{L+j}(t) \quad j = 1 \ldots F \\
\Lambda(z) &= \operatorname{diag}\left[\Phi_1(z), \ldots, \Phi_L(z),\, \sigma_{L+1}(z) I_{m_1}, \ldots, \sigma_{L+F}(z) I_{m_F}\right]
\end{aligned} \tag{23}$$

Now let $C \in \mathcal{L}(l_2)$ be LTI and $\epsilon > 0$. Define a subset of $\mathbf{Y}$,

$$\nabla^\epsilon = \{\Lambda(z) : \|z\| = 1, \|Cz\| < \epsilon\} \tag{24}$$

Lemma 9 *The closure $\bar{\nabla}^\epsilon$ of ∇^ϵ is convex and compact in $\mathbf{Y}$.*

Theorem 10 *Let A, C be LTI stable systems, ∇^ϵ as in (24). Assume $\mathbf{\Delta}$ is the set of structured, otherwise arbitrary linear operators in l_2^n. The following are equivalent:*

$$\begin{aligned}
&(i) \quad \textit{The implicit uncertain system } (19) \textit{ has robust } l_2-\textit{stability.} \\
&(ii) \quad \exists\, \epsilon > 0 \textit{ such that } \bar{\nabla}^\epsilon \cap \bar{\mathbf{X}} = \emptyset \\
&(iii) \quad \exists\, X \in \mathbf{X} \textit{ such that } A^*XA - X - C^*C < 0
\end{aligned} \tag{25}$$

In (iii) above, condition (25) is of the form $\Psi < 0$ where Ψ is a self-adjoint operator on l_2; this must be interpreted as a strong version of negative definiteness, $\langle \Psi u, u \rangle \leq -\rho \|u\|^2$

for some $\rho > 0$. For the case of A,C finite dimensional, this condition can be tested in the frequency domain in the form

$$\exists\, X \in \mathbf{X}, \text{ such that } A(e^{j\omega})^* X A(e^{j\omega}) - X - C(e^{j\omega})^* C(e^{j\omega}) < 0 \;\; \forall \omega \in [-\pi, \pi] \tag{26}$$

This test is a Linear Matrix Inequality (LMI, [1]) evaluated over frequency, and lends itself to available convex optimization tools.

5.2 Necessary, Sufficient Conditions in the l_{2e} setting

As remarked before, the motivation we have provided for constrained problems refers mainly to the l_2 setting. It is well known, however, that in the standard unconstrained case, the condition $A^* X A - X < 0$, (or equivalently $\left\| X^{\frac{1}{2}} A X^{-\frac{1}{2}} \right\| < 1$) is sufficient for robust stability in the l_{2e} sense (this is a consequence of the small gain theorem [22]). Also, [17, 9] show it is a necessary condition.

It therefore seems natural to explore this issue in the constrained case of (19). We first consider necessity:

Theorem 11 *Let A, C be LTI in $\mathcal{L}_c(l_{2e})$, $\mathbf{\Delta}$ is the set of structured, otherwise arbitrary causal linear operators in $\mathcal{L}_c(l_{2e})$. If (19) over l_{2e} is robustly stable then (25) holds.*

For the proof, it clearly suffices to prove condition (ii) in Theorem 10; the only required modification in the argument presented for Theorem 10, is that *causality* of the destabilizing perturbation must be ensured. This can be done following the lines of the proof in [17] for the I/O case; for reasons of brevity this construction will not be developed here.

Regarding the issue of sufficiency of condition (25), we consider the following example.

Example 2 *Let $A = 2$, $C = 1 - 2\lambda$. $\Delta \in \mathcal{L}_c(l_{2e})$. For any $X \in \mathbb{R}^+$,*

$$\langle (A^* X A - X - C^* C)v, v\rangle = 3X \left\|v\right\|^2 - \left\|(1-2\lambda)v\right\|^2 \le (3X - 1)\left\|v\right\|^2 \tag{27}$$

so for $0 < X < \frac{1}{3}$ (25) is satisfied. But the perturbation $\Delta = \lambda$ gives $\begin{bmatrix} I - \Delta A \\ C \end{bmatrix} = \begin{bmatrix} 1 - 2\lambda \\ 1 - 2\lambda \end{bmatrix}$ which is unstable, as shown in Example 1.

This example shows that in general, condition (25), (or (26)) is *not* sufficient for robust stability, even for time invariant perturbations: (25) does not provide information of the behavior outside l_2. One could think of strengthening condition (26) to include frequency points inside the disk; this would eliminate the previous counterexample, and in fact guarantee robust stability with LTI perturbations (see Part II), but it is still not sufficient for the LTV case. A counterexample for this is the system $A = 4\lambda - 8\lambda^2$, $C = 1 - 4\lambda + 4\lambda^2$; we omit the rather lengthy verification [2].

We can state, however, a partial result which is applicable in many cases.

Theorem 12 *Let A be LTI in $\mathcal{L}_c(l_{2e})$, and C be a static map. If (25) holds, then system (19) is robustly stable over the class $\mathbf{\Delta}_c$ of structured, otherwise arbitrary operators in $\mathcal{L}_c(l_{2e})$.*

Although this result is not a major extension of the standard small gain theorem of [22], since only static additional constraints are allowed, it is still quite rich from the point of view of posing constrained robustness analysis problems, as will be shown below.

5.3 Analysis for Robust Performance Revisited

To conclude Part I of this paper, we review the example considered in section 3.4 using the results of this section. Referring to (13-15) and Figure 2 we assume that H, P and Q are causal, LTI, stable maps. We recall from section 3.4 that without loss of generality in the analysis, P can be chosen to be a static map kI. Also the perturbations Δ_P, Δ_C vary in the class of arbitrary time-varying operators.

The reduction procedure from Propositions 3, 4, yields a system of the form (19), where

$$\Delta = \begin{bmatrix} \Delta_u & 0 & 0 \\ 0 & \Delta_P & 0 \\ 0 & 0 & \Delta_C \end{bmatrix}, \quad A = \begin{bmatrix} H_{11} & H_{12} & 0 \\ H_{21} & H_{22} & 0 \\ 0 & Q & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & P & -I \end{bmatrix}, \quad z = \begin{bmatrix} z_u \\ z_p \\ z_c \end{bmatrix} \tag{28}$$

If P is static, we observe that C is static, as in the assumption of Theorem 12. Therefore if Δ_u is LTV uncertainty, condition (25) is necessary and sufficient for

[2] This counterexample can be related to an analogous situation in [20], involving a frequency domain condition over the right half plane.

- robust l_2-stability in the case Δ_u, Δ_P, Δ_C in $\mathcal{L}(l_2)$,

- robust stability in the case Δ_u, Δ_P, Δ_C in $\mathcal{L}_c(l_{2e})$,

If we review the problem statement that let to equations (13-15), the l_2 version seems more appropriate, but we might also be interested in a "hybrid" problem, by testing strong stability in the z_u variables. More precisely, Δ_u may be considered a causal perturbation, and it must be guaranteed in the first place that the z_u variable does not "blow up" (it remains in l_2 if d and the equation errors are in l_2). Once this is known, the analysis can be restricted to l_2 and the (possibly non-causal) perturbations Δ_P, Δ_C can be considered, casting the robust performance analysis as a robust l_2 stability question, as argued in section 3.4.

This hybrid question is also answered by the test (25); in fact, since the first block of C is zero, the upper portion of $A^*XA - X - C^*C$ is $H_{11}^* X_u H_{11} - X_u$, which provides the standard robust stability test in the z_u variables, in addition to the robust l_2 stability test on all the variables.

6 Conclusions

The work reported in Part I of this paper provides the foundation for a more general robustness analysis theory, which extends the standard theory based on the small gain theorem. In this approach, we abandon the concepts of "input-output maps" and "gains" in favor of *equations* and *signal constraints*, and the central analysis question is to test whether there exist solutions to these equations.

The results in this paper demonstrate that nothing is lost, from a mathematical point of view, by adopting this approach for *analysis* instead of the standard input-output formulation; on the contrary, the analysis setup presented in section 3.4, further developed in Part II, shows evidence of substantial advantages.

There are still reasons to preserve the standard "signal-flow" approach, which has led to a large body of knowledge, since some of its intuitive value for design is lost in the implicit formulation. The conclusion is, however, that if research is not confined to the traditional paradigm, the potential of the resulting theory will be enhanced.

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Appendix: Proofs

Lemmas 1 and 2

The only non-trivial implication is $(ii) \Rightarrow (iii)$ in Lemma 2.

If $\hat{v}_1, \ldots \hat{v}_r$, is an orthonormal basis of the subspace of l_2 spanned by the coordinates $v_1, \ldots v_d$ of v, we write $\hat{v} = Pv$, where P is an invertible matrix, $\hat{v} = [\hat{v}_1, \ldots \hat{v}_r, 0, \ldots 0]'$. Let $\hat{z} = Pz$, then (ii) implies $\hat{z} = [\hat{z}_1, \ldots \hat{z}_r, 0, \ldots 0]'$, and

$$\|\eta_1 \hat{v}_1 + \ldots + \eta_r \hat{v}_r\| \geq \|\eta_1 \hat{z}_1 + \ldots + \eta_r \hat{z}_r\| \quad \forall \eta_1, \ldots \eta_r \in \mathbb{C} \tag{29}$$

Now define $\delta : u \mapsto \sum_{i=1}^{r} \langle u, \hat{v}_i \rangle \hat{z}_i$; then $\delta : \hat{v}_i \mapsto \hat{z}_i, i = 1 \ldots r$, so $\delta I \hat{v} = \hat{z}$ which implies $\delta I v = z$. Also, by (29) and the Bessel inequality, $\|\delta u\| \leq \|\sum_{i=1}^{r} \langle u, \hat{v}_i \rangle \hat{v}_i\| \leq \|u\|$. So $\|\delta\| \leq 1$.

$\square$

Proposition 3

For the necessity, note that since $LG = I$, $\|L\| \, \|Gw\| \geq \|w\|$. For the sufficiency, condition (16) implies that G is injective, and its range is closed. An application of the open mapping theorem [4] implies that G has a bounded inverse on its range, which can be extended to a bounded operator on E, resulting in a left inverse for G.

$\square$

Proposition 4

[Necessity] the definition of stability gives $\|G^T\xi\| \geq \frac{1}{\gamma}\|\xi\|$ from which (18) follows.

[Sufficiency] Let $\epsilon = \inf_{T\geq 0}\underline{\sigma}(G^T) > 0$. Then a causal left inverse L can be constructed by induction over T: given a lower triangular left inverse L^T of G^T, consider the matrix $G^{T+1} = \begin{bmatrix} G^T & 0 \\ \bar{G} & G_{T+1,T+1} \end{bmatrix}$. Since $\underline{\sigma}(G^{T+1}) > 0$, $G_{T+1,T+1}$ has left inverse $L_{T+1,T+1}$, then

$$L_{T+1} = \begin{bmatrix} L^T & 0 \\ -L_{T+1,T+1}\bar{G}L^T & L_{T+1,T+1} \end{bmatrix}$$

is a left inverse of G_{T+1}. This procedure produces a causal operator $L : E \rightarrow W$. If $e = Gw$, then $P_Te = G^TP_Tw$ therefore $\|P_Te\| \geq \epsilon\|P_TLe\|$ so L has finite gain in the range of G.

□

Proposition 5

(i) If $R(e^{j\omega_0})\bar{w} = 0$, $\bar{w} \neq 0$, it is easy to construct l_2 signals $w^{(k)}$ with norm 1, in the direction $\bar{w}$, and spectrum supported in $[\omega_0 - \frac{1}{k}, \omega_0 - \frac{1}{k}]$, so that $\|R(\lambda)w^{(k)}\| \stackrel{k\rightarrow\infty}{\longrightarrow} 0$,violating l_2-stability. Conversely if $R(\xi)$ has full column rank on the unit circle, $\min_\omega \underline{\sigma}(R(e^{j\omega})) > 0$ and it is easy to construct pointwise an $\mathcal{L}_\infty$ left inverse for $R(\lambda)$.

(ii) If $R(0)$ has a kernel, then $R(\lambda)$ cannot have a causal left inverse, so well posedness fails. If $R(\xi_0)\bar{w} = 0$, $0 \neq |\xi_0| \leq 1$ then the signal $w(t) = (\frac{1}{\xi_0})^t\bar{w}$, $t > 0$ is in $l_{2e}\backslash l_2$, but $R(\lambda)w \in l_2$, violating stability.

If $R(\xi)$ has full column rank over the unit disk, then the theory of coprime factorizations over the stable ring (see e.g. [18]) implies that $R(\xi)$ has a left inverse in $\mathcal{RH}_\infty$. This is a causal left inverse which has finite gain (over all l_{2e}) so stability is satisfied.

□

Proposition 6

[Necessity] If (4) is robust l_2-stable, setting $\Delta = 0$, $\varphi(0, M) = \begin{bmatrix} I & 0 \\ C & D \end{bmatrix}$ has bounded left inverse, therefore D has bounded left inverse L. Fix $\Delta \in \mathbf{B}_{\boldsymbol{\Delta}}$; the following identity holds:

$$\varphi(\Delta, M)\, U = V\, \varphi(\Delta, \hat{M}) \tag{30}$$

where U and V are the invertible operators

$$U = \begin{bmatrix} I & 0 \\ -LC & I \end{bmatrix}, \quad V = \begin{bmatrix} I & -\Delta BL \\ 0 & I \end{bmatrix}, \quad U^{-1} = \begin{bmatrix} I & 0 \\ LC & I \end{bmatrix}, \quad V^{-1} = \begin{bmatrix} I & \Delta BL \\ 0 & I \end{bmatrix}, \tag{31}$$

and

$$\hat{M} = \begin{bmatrix} \hat{A} & 0 \\ \hat{C} & D \end{bmatrix}, \quad \varphi(\Delta, \hat{M}) = \begin{bmatrix} I - \Delta\hat{A} & 0 \\ \hat{C} & D \end{bmatrix} \tag{32}$$

Also denote $\Psi(\Delta, \hat{M}) = \begin{bmatrix} I - \Delta\hat{A} \\ \hat{C} \end{bmatrix}$. If $T(\Delta)$ is a bounded left inverse for $\varphi(\Delta, M)$, then $U^{-1}T(\Delta)V$ is a left inverse for $\varphi(\Delta, \hat{M})$, which implies $\Psi(\Delta, \hat{M})$ is left invertible.

[Sufficiency] If $\hat{T}(\Delta) = \begin{bmatrix} T_1 & T_2 \end{bmatrix}$ is a bounded left inverse for $\Psi(\Delta, \hat{M})$ then (note that by definition of $\hat{C}$, $L\hat{C} = 0$),

$$\begin{bmatrix} T_1 & T_2(I - DL) \\ 0 & L \end{bmatrix} \begin{bmatrix} I - \Delta\hat{A} & 0 \\ \hat{C} & D \end{bmatrix} = I \tag{33}$$

therefore $\varphi(\Delta, \hat{M})$ is left invertible, and so is $\varphi(\Delta, M)$ by (30).

□

Proposition 7:

Defining U, V as in (31), by assumption (i') U, V, U^{-1}, V^{-1} are in $\mathcal{L}_c(l_{2e})$. From (30), robust stability of (4) reduces to that of $\varphi(\Delta, \hat{M}) \begin{bmatrix} z \\ w \end{bmatrix} = 0$. Clearly this implies stability of $\Psi(\Delta, \hat{M})z = 0$, It remains to show the converse implication.

Fix $\Delta \in \mathbf{B}_{\boldsymbol{\Delta}}$, let $\begin{bmatrix} T_1 & T_2 \end{bmatrix}$ be a causal left inverse of $\Psi(\Delta, \hat{M})$, with finite gain on its range, and construct a causal left inverse for $\varphi(\Delta, \hat{M})$ as in (33). The bottom portion $\begin{bmatrix} 0 & L \end{bmatrix}$ has finite gain by (i'). The top portion is

$$\begin{bmatrix} T_1 & T_2(I - DL) \end{bmatrix} = \begin{bmatrix} T_1 & T_2 \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & (I - DL) \end{bmatrix} \tag{34}$$

where $\begin{bmatrix} I & 0 \\ 0 & (I-DL) \end{bmatrix}$ has finite gain and maps a vector in the range of $\varphi(\Delta, \hat{M})$, to the range of $\Psi(\Delta, \hat{M})$ where $\begin{bmatrix} T_1 & T_2 \end{bmatrix}$ has finite gain, therefore the top portion has finite gain in the range of $\varphi(\Delta, \hat{M})$.

$\square$

Proposition 8

The result follows from some results in spectral theory of operators. For simplicity we consider the complex field $\mathbb{F} = \mathbb{C}$, but a similar proof can be written for the real case. For an operator $M : l_2(\mathbb{C}^q) \rightarrow l_2(\mathbb{C}^q)$, the spectrum is defined as $\varsigma(M) = \{\zeta \in \mathbb{C}, (\zeta I - M) \text{ not invertible}\}$. A subset of the spectrum is the approximate point spectrum, defined as

$$\varsigma_{ap}(M) = \{\zeta \in \mathbb{C}, (\zeta I - M) \text{ not left invertible}\} = \{\zeta \in \mathbb{C}, \inf_{\|z\|=1} \|(\zeta I - M)(z)\| = 0\} \quad (35)$$

It is known (see [4]) that the boundary of the spectrum is in the approximate point spectrum. Therefore the spectral radius $\rho(M) = \max\{|\zeta| : \zeta \in \varsigma(M)\}$ is achieved at $\zeta_0 \in \varsigma_{ap}(M)$.

We now prove the only if portion of the proposition (the other direction is trivial). Assume there exists $\Delta_0 \in \mathbf{B_\Delta}$, such that $I - \Delta_0 A$ not invertible. Therefore $\rho(\Delta_0 A) \geq 1$. This implies there exists $\zeta_0 \in \varsigma_{ap}(\Delta_0 A)$, with $|\zeta_0| \geq 1$. Therefore $\zeta_0 I - \Delta_0 A$ is not left invertible, which gives $I - \Delta A$ not left invertible, with $\Delta = \frac{1}{\zeta_0}\Delta \in \mathbf{B_\Delta}$, a contradiction.

$\square$

Lemma 9

Since $A \in \mathcal{L}(l_2)$, ∇^ϵ is bounded, therefore $\bar{\nabla}^\epsilon$ is compact. For the convexity proof, it suffices to show that $co(\nabla^\epsilon) \subset \bar{\nabla}^\epsilon$, where $co(\nabla^\epsilon)$ is the convex hull of ∇^ϵ. Consider the convex combination $\alpha\Lambda(z) + (1-\alpha)\Lambda(v)$ of two elements of ∇^ϵ (a finite number of terms can be handled in a similar way): ($\|z\| = \|v\| = 1$, $\|Cz\| < \epsilon$, $\|Cv\| < \epsilon$, $0 < \alpha < 1$). Define $v^{(k)} = (\alpha)^{\frac{1}{2}} z + (1-\alpha)^{\frac{1}{2}} \lambda^k v$. Then

$$\left\| v^{(k)} \right\|^2 = \alpha \|z\|^2 + (1-\alpha)\|v\|^2 + 2(\alpha)^{\frac{1}{2}}(1-\alpha)^{\frac{1}{2}}\langle z, \lambda^k v\rangle \quad (36)$$

$$\left\|Cv^{(k)}\right\|^2 \;=\; \alpha\left\|Cz\right\|^2 + (1-\alpha)\left\|Cv\right\|^2 + 2(\alpha)^{\frac{1}{2}}(1-\alpha)^{\frac{1}{2}}\langle Cz, \lambda^k Cv\rangle \tag{37}$$

where (37) uses the time invariance of C. The last terms in (36), (37) have limit 0 as $k\to\infty$. Therefore $\|v^{(k)}\|^2 \overset{k\to\infty}{\longrightarrow} \alpha\|z\|^2 + (1-\alpha)\|v\|^2 = 1$, and $\|Cv^{(k)}\| < \epsilon$ for large k, and therefore $\left\|C\frac{v^{(k)}}{\|v^{(k)}\|}\right\| < \epsilon$ for $k \geq k_0$. A similar argument using the time invariance of A, shows that

$$\Phi_i(v^{(k)}) \overset{k\to\infty}{\longrightarrow} \alpha\Phi_i(z) + (1-\alpha)\Phi_i(v) \qquad i = 1\ldots L \tag{38}$$

$$\sigma_{L+j}(v^{(k)}) \overset{k\to\infty}{\longrightarrow} \alpha\sigma_{L+j}(z) + (1-\alpha)\sigma_{L+j}(v) \quad j = 1\ldots F \tag{39}$$

This gives $\Lambda(\frac{v^{(k)}}{\|v^{(k)}\|}) \overset{k\to\infty}{\longrightarrow} \alpha\Lambda(z) + (1-\alpha)\Lambda(v) \in \bar{\nabla}^{\epsilon}$.

□

Theorem 10

$(i) \Rightarrow (ii)$ By contradiction, assume that for all $\epsilon > 0$, $\bar{\nabla}^{\epsilon} \cap \bar{\mathbf{X}} \neq \emptyset$. For a fixed $\epsilon > 0$, we can therefore find $z \in l_2$ such that $\|z\| = 1$, $\|Cz\| < \epsilon$, and $\Lambda(z) + \epsilon^2 I \geq 0$. Let $\xi = Az$. We have

$$\|\xi_{L+j}\|^2 - \|z_{L+j}\|^2 + \epsilon^2 \geq 0 \qquad j = 1\ldots F \tag{40}$$

$$\sum_{t=-\infty}^{\infty} \xi_i(t)\xi_i(t)^* - z_i(t)z_i(t)^* + \epsilon^2 I \geq 0 \qquad i = 1\ldots L \tag{41}$$

Focusing on (40), a slight extension of Lemma 1 shows that there exists an LTV operator Δ_{L+j}, $\|\Delta_{L+j}\| \leq 1$ and an error signal e_{L+j}, $\|e_{L+j}\| = O(\epsilon)$ such that $\Delta_{L+j}\xi_{L+j} + e_{L+j} = z_{L+j}$.

A similar argument (extending Lemma 2) is used for (41). The result is a structured LTV operator Δ, $\|\Delta\| \leq 1$ and an error signal e, $\|e\| = O(\epsilon)$ such that $(I - \Delta A)z = e$. Therefore

$$\left\|\begin{bmatrix} I - \Delta A \\ C \end{bmatrix} z\right\| = O(\epsilon). \tag{42}$$

Since this holds for any ϵ, there exist sequences of signals z^k and perturbations Δ_k such that

$$\|z^k\| = 1, \quad \left\|\begin{bmatrix} I - \Delta_k A \\ C \end{bmatrix} z^k\right\| \overset{k\to\infty}{\longrightarrow} 0 \tag{43}$$

It remains to show that a *single* perturbation Δ and a sequence $\bar{z}^k$ can be found with the same property. The following construction, (which shows that the concept of *uniform* robust stability is no stronger in this case) is due to Dullerud [7]. For simplicity we consider $\mathbb{T} = \mathbb{Z}^+$.

Starting from (43), let $\|(I-\Delta_k A)z^k\| < \epsilon_k$ with $\epsilon_k \stackrel{k\to\infty}{\longrightarrow} 0$. For fixed k we can find a truncation time T_k such that

$$\|(I - P^{T_k}\Delta_k P^{T_k} A)z^k\| < \epsilon_k \;\text{ and }\; \|(I - P^{T_k})Az^k\| < \epsilon_k \tag{44}$$

This follows from the fact that $P^T\Delta_k P^T v \stackrel{T\to\infty}{\longrightarrow} \Delta_k v$ for any v; $P^T\Delta_k P^T$ is identified with the truncated matrix Δ_k^T as in (2). Now define the operator Δ with infinite matrix representation

$$\Delta = \begin{bmatrix} \Delta_0^{T_0} & 0 & 0 & \cdots \\ 0 & \Delta_1^{T_1} & 0 & \cdots \\ \vdots & \vdots & \vdots & \cdots \\ 0 & \cdots & \Delta_k^{T_k} & \cdots \\ \vdots & \vdots & \vdots & \cdots \end{bmatrix} \tag{45}$$

Let τ_k be defined by $\tau_0 = 0$, $\tau_{k+1} = \tau_k + T_k$ for $k > 0$, and $\bar{z}^k \triangleq \lambda^{\tau_k} z^k$. From the time invariance of A, $(I-\Delta A)\bar{z}^k = \lambda^{\tau_k} z^k - \Delta\lambda^{\tau_k} A z^k = \lambda^{\tau_k}(I - \hat{\Delta}^k A)z^k$ where $\hat{\Delta}^k$ has infinite matrix representation

$$\hat{\Delta}^k = \begin{bmatrix} \Delta_k^{T_k} & 0 & 0 & \cdots \\ 0 & \Delta_{k+1}^{T_{k+1}} & 0 & \cdots \\ \vdots & \vdots & \vdots & \cdots \end{bmatrix} \tag{46}$$

It follows that

$$\begin{aligned} \|(I-\Delta A)\bar{z}^k\| &= \left\|(I - P^{T_k}\Delta_k P^{T_k} A)z^k + (P^{T_k}\Delta_k P^{T_k} - \hat{\Delta}^k)Az^k\right\| \\ &\le \|(I - P^{T_k}\Delta_k P^{T_k} A)z^k\| + \|(I-P^{T_k})Az^k\| < 2\epsilon_k \stackrel{k\to\infty}{\longrightarrow} 0 \end{aligned} \tag{47}$$

Since $\|\bar{z}^k\| = 1$, and $\|C\bar{z}^k\| \stackrel{k\to\infty}{\longrightarrow} 0$ from the time invariance of C, $\begin{bmatrix} I-\Delta A \\ C \end{bmatrix}$ is not l_2-stable.

□

$(ii) \Rightarrow (iii)$

$\bar{\nabla}^\epsilon$ and $\bar{\mathbf{X}}$ are disjoint convex sets in the inner product space $\mathbf{Y}$, $\bar{\nabla}^\epsilon$ is compact and $\bar{\mathbf{X}}$ is closed. We can use a hyperplane separation argument to find $X \in \mathbf{Y}$, $\eta > 0$ such that

$$\langle X, \Lambda\rangle \le \alpha - \eta < \alpha \le \langle X, Y\rangle \;\; \forall\, \Lambda \in \bar{\nabla}^\epsilon,\;\; Y \in \bar{\mathbf{X}} \tag{48}$$

Since $\bar{\mathbf{X}}$ is a cone, α can be chosen to be 0. It is easy to show that

$$\langle X, Y\rangle \geq 0 \ \ \forall Y \in \mathbf{Y},\ Y \geq 0 \ \implies X \geq 0 \tag{49}$$

A small perturbation of X ensures $X > 0$ (therefore $X \in \mathbf{X}$), and by continuity and compactness of $\bar{\nabla}^{\epsilon}$ we can modify η to achieve

$$\langle X, \Lambda\rangle \leq -\eta < 0 \ \ \forall \Lambda \in \bar{\nabla}^{\epsilon} \tag{50}$$

for the new X. Furthermore, by scaling down X (and η) we can ensure (50) holds and

$$\|A^* X A - X\| = \gamma^2 < \epsilon^2. \tag{51}$$

Now for any $z = l_2$, $\|z\| = 1$, $\xi = Az$,

$$\begin{aligned} \langle X, \Lambda(z)\rangle &= \sum_{i=1}^{L} \operatorname{tr}(X_i \Phi_i(z)) + \sum_{j=1}^{F} x_{L+j}\sigma_{L+j}(z) \\ &= \langle XAz, Az\rangle - \langle Xz, z\rangle = \langle (A^* X A - X)z, z\rangle \end{aligned} \tag{52}$$

Let $\Psi = A^* X A - X - C^* C$.

If $\|Cz\| < \epsilon$, then $\Lambda(z) \in \nabla^{\epsilon}$ so $\langle \Psi z, z\rangle \leq \langle (A^* X A - X)z, z\rangle = \langle X, \Lambda(z)\rangle \leq -\eta < 0$ by (50).

If $\|Cz\| \geq \epsilon$, then $\langle \Psi z, z\rangle \leq \langle (A^* X A - X)z, z\rangle - \|Cz\|^2 \leq \gamma^2 - \epsilon^2 < 0$ using (51).

Therefore $\langle \Psi z, z\rangle \leq max(-\eta, \gamma^2 - \epsilon^2) < 0$ for $\|z\| = 1$, which implies $\Psi < 0$.

$(iii) \Rightarrow (i)$

Fix $X > 0$ which solves (25). Without loss of generality it can be assumed that $X = I$. This results from the fact that invertible operations yield

$$\begin{bmatrix} X^{\frac{1}{2}} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} I - \Delta A \\ C \end{bmatrix} X^{\frac{1}{2}} = \begin{bmatrix} I - \Delta \bar{A} \\ \bar{C} \end{bmatrix} \tag{53}$$

where $\bar{A} \triangleq X^{\frac{1}{2}} A X^{-\frac{1}{2}}$, $\bar{C} \triangleq C X^{-\frac{1}{2}}$ verify (25) with $X = I$. The notion of negative definitenes allows by continuity to find $\alpha^2 < 1$ such that

$$A^* A - \alpha^2 I - C^* C < 0. \tag{54}$$

Let $e, z \in l_2$ ($\|z\| = 1$) and Δ ($\|\Delta\| \leq 1$) satisfy

$$\begin{bmatrix} I - \Delta A \\ C \end{bmatrix} z = e = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \tag{55}$$

$$\|\Delta Az\|^2 \leq \|Az\|^2 = \langle A^* Az, z\rangle \leq \alpha^2 \langle z, z\rangle + \langle C^* Cz, z\rangle = \alpha^2 + \|e_2\|^2 \tag{56}$$

Therefore $\|\Delta Az\| \leq \alpha + \|e_2\|$, and $\|e_1\| = \|z - \Delta Az\| \geq (1-\alpha) - \|e_2\|$ This shows that $\|e\| \geq (\|e_1\| + \|e_2\|)/\sqrt{2} \geq (1-\alpha)/\sqrt{2}$, which proves robust l_2-stability by Proposition 3.

□

Theorem 12

By the same argument given above, assume without loss of generality that $X = I$, and let α satisfy (54). Let T be fixed, z, e satisfy (55), with truncation $P^T z$ of norm 1. From causality

$$P^T \begin{bmatrix} I - \Delta A \\ C \end{bmatrix} P^T z = P^T e = \begin{bmatrix} P^T e_1 \\ P^T e_2 \end{bmatrix} \tag{57}$$

Since $P^T z \in l_2$, from (54) we obtain

$$\left\|AP^T z\right\|^2 < \alpha^2 \left\|P^T z\right\|^2 + \left\|CP^T z\right\|^2 \tag{58}$$

Since C is static, $CP^T z = P^T e_2$; also $\|P^T \Delta A P^T z\|^2 \leq \|AP^T z\|^2$, leading to

$$\left\|P^T e_1\right\| = \left\|P^T (I - \Delta A) P^T z\right\| \geq 1 - \alpha - \left\|P^T e_2\right\| \tag{59}$$

We conclude that $\|P^T e\| \geq \epsilon$ for a fixed $\epsilon > 0$, independent of T, Δ, which implies that $\inf_T \underline{\sigma}\left(\begin{bmatrix} I - \Delta A \\ C \end{bmatrix}^T\right) > 0$ for every Δ giving robust stability from Proposition 4.

□

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