# $\mathcal{H}_\infty$ Optimal Interconnections

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**Abstract**

In this paper, a general $\mathcal{H}_\infty$ problem for continuous time, linear time invariant systems is formulated and solved in a behavioral framework. This general formulation, which includes standard $\mathcal{H}_\infty$ optimization as a special case, provides added freedom in the design of sub-optimal compensators, and can in fact be viewed as a means of designing optimal systems. In particular, the formulation presented allows for singular interconnections, which naturally occur when interconnecting first principles models.

## 1 Introduction

Invariably, most of the tools developed in the field of optimal control have relegated the control design process to a secondary role in the design of systems: a control algorithm is only sought after the system to be controlled has already been designed, and the type and location of the actuators and sensors has been determined; equivalently, given both sensor variables and actuator variables, a viable control strategy consists of an algorithm which produces actuators signals from the measured variables, and results in a closed loop system which achieves certain performance objectives. These objectives may be input-output in nature (such as $\mathcal{H}_\infty$, $\mathcal{H}_2$, or $\mathcal{L}_\infty$), or transient oriented (such as $LQR$).

For most applications, however, the level of performance which can be attained by *any* control strategy is dictated by the dynamics of the plant [7]. Thus from a system level, the above standard approaches are not optimal, since the control design process is de-coupled from the design of the rest of the system. The result is that the control engineer is left with little freedom in how to control the system, or an iteration must take place between the design of the system and the design of the controller. Clearly, the optimal strategy would be

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to design the system *and* controller at the same time, or in other words, to view the design of the controller as part of the system design process.

In this paper, a general optimization problem is posed, where the objective is of finding optimal relations between a system's variables. This type of formulation is very closely related to the behavioral framework for describing systems, as proposed by Willems [22]; in this paradigm, a model for a system is characterized by its *behavior*, the set of all allowable trajectories for the system. The design process in this framework takes the form of finding additional constraints on a system's behavior such that the remaining allowable trajectories satisfy given a-priori requirements. The optimization criterion adopted in this paper is the rejection of $\mathcal{L}_2$ bounded disturbances, which leads to a general version of the $\mathcal{H}_\infty$ design problem for continuous time, linear time invariant systems described by implicit equations.

A preliminary version of the results in this paper appeared in [3]. Other research in this area includes the work of Trentelman and Willems [20], which consider a similar type of problem formulation from a polynomial representation standpoint. Skelton also addresses the problem of integrated system and controller design using a successive covariance approximation approach in [19], and provides an excellent summary of the key issues which will drive future research into integrated system design.

## 2 Background and Notation

Most of the notation in this paper is standard. The Lebesgue spaces $\mathcal{L}_2^q = \mathcal{L}_2(\mathbb{R}, \mathbb{R}^q)$ and $\mathcal{L}_{2+}^q = \mathcal{L}_2(\mathbb{R}^+, \mathbb{R}^q)$ consist, respectively, of square-integrable $q$ valued functions on $\mathbb{R}$ and $\mathbb{R}^+$. When the spatial dimension $q$ is clear from context, the shorthand notation $\mathcal{L}_2$ and $\mathcal{L}_{2+}$ will be used instead. Similarly, $C^\infty(\mathbb{R}, \mathbb{R}^q)$ denotes the set of infinitely differentiable functions from $\mathbb{R}$ to $\mathbb{R}^q$, with shorthand notation $C^\infty$. The $\mathcal{H}_\infty$ norm defined in the frequency-domain for a stable transfer function is

$$\|G\|_\infty := \sup_\omega \overline{\sigma}(G(j\omega)) \tag{1}$$

where $\overline{\sigma}$ := maximum singular value. A transfer function in terms of state space data is denoted

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] := D + C(sI - A)^{-1}B \tag{2}$$

The linear fractional transformation (LFT) between two transfer functions $G(s)$ and $K(s)$ is denoted $G \star K$, and is defined as:

$$G \star K := G_{11} + G_{12}K(I - G_{22}K)^{-1}G_{21} \tag{3}$$

where

$$G = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}$$

when the inverse of $(I - G_{22}K)$ exists. $\overline{\mathbb{C}}^+$ is the region in the complex plane for which the real component is positive or zero. $\mathbb{C}^-$ is the region for which the real component is negative. Given matrix $A$, Im $(A)$ and Ker $(A)$ denote, respectively, the image and null subspaces of $A$; $A^\perp$ is used to denote any basis for Ker $(A^{\mathbf{T}})$.

### 2.1 Behavioral Representations

Systems for which the allowable trajectories are the solution set of the following set of differential equations will be considered:

$$R_L \frac{d^L w}{dt^L} + \cdots + R_0 w = 0 \tag{4}$$

where $R_0, \cdots, R_L$ are constant matrices. Defining

$$R(\xi) := R_L \xi^L + \cdots + R_0 \tag{5}$$

results in the shorthand notation $R(\frac{d}{dt})w = 0$ for equation (4). The above is referred to as an *autoregressive (AR)* representation.

Using the notation of Willems [22], a system will be denoted $\Sigma := \{\mathbb{R}, \mathbb{R}^q, \mathcal{B}\}$, where $\mathbb{R}$ and $\mathbb{R}^q$ correspond to $\mathbb{R}^q$ valued, bi-infinite, continuous time, trajectories, and $\mathcal{B}$ is the *behavior*, or the allowable trajectories:

$$\mathcal{B} := \left\{ w \in C^\infty(\mathbb{R}, \mathbb{R}^q) \mid R(\frac{d}{dt})w = 0 \right\} \tag{6}$$

The space of $C^\infty$ functions is chosen to simplify the development which follows. The larger class of locally square integrable functions could be adopted instead, with minor modifications.

The *interconnection* of two systems $\Sigma_1 = \{\mathbb{R}, \mathbb{R}^q, \mathcal{B}_1\}$ and $\Sigma_2 = \{\mathbb{R}, \mathbb{R}^q, \mathcal{B}_2\}$, possessing the same variables $w$, is defined to be

$$\Sigma_1 \wedge \Sigma_2 := \{\mathbb{R}, \mathbb{R}^q, \mathcal{B}_1 \cap \mathcal{B}_2\} \tag{7}$$

ie., the resulting behavior is simply the intersection of the two behaviors. Thus an allowable trajectory must satisfy the governing equations of both systems. Note that each of $\Sigma_1$ and $\Sigma_2$ can be trivially augmented to posses the same variables $w$.

## 3 Problem Formulation

We are given a system $\Sigma_p = \{\mathbb{R}, \mathbb{R}^{q_e+q_d+q_c+q_l}, \mathcal{B}_p\}$, ie., $w$ is partitioned into four parts, $w = (e, d, c, l)$:

**e**: error signals which are required to be small
**d**: exogenous disturbances, unexplained by the given model
**c**: variables which are accessible for control purposes
**l**: *latent* variables, auxiliary variables used when constructing $\Sigma_p$

The objective is to find system $\Sigma_c = \{\mathbb{R}, \mathbb{R}^{q_c}, \mathcal{B}_c\}$, acting on the variables $c$, such that $\Sigma := \Sigma_p \wedge \Sigma_c = \{\mathbb{R}, \mathbb{R}^{q_e+q_d+q_c+q_l}, \mathcal{B}\}$ satisfies the following:

(P1) **Unrestricted Disturbance:** For the interconnected system, $d$ is *free*:

$$\forall d \in C^{\infty},\ \exists\ e, c, l \in C^{\infty} s.t.\ w \in \mathcal{B} \tag{8}$$

Equivalently, system $\Sigma_c$ does not provide us with any additional information about the disturbance.

(P2) **Stability:**

$$d = 0, w \in \mathcal{B} \Longrightarrow \lim_{t\to\infty} e(t), c(t) = 0 \tag{9}$$

Thus if one stops exciting the system, the error and control signals decay to 0. Note that there is no such restriction on latent variables $l$; we will have more to say on this later on.

(P3) **Performance:**

$$\sup_{d\in(C^{\infty}\cap\mathcal{L}_2),\, \|d\|\leq 1, w\in\mathcal{B}} \|e\| < 1 \tag{10}$$

Note that the general performance specification $\|e\| < \gamma$ can be imposed by appropriately scaling $e$.

In general, a system $\Sigma_c$ which only has access to variables $c$ will be referred to as a *compensator*. If in addition $\Sigma$ satisfies constraints $P1$, $P2$, and $P3$, $\Sigma_c$ will be referred to as an *allowable* compensator.

It is useful to compare the above problem formulation to the standard, input-output $\mathcal{H}_\infty$ formulation of Figure 1. Variables $e$ and $d$ have the same interpretation, $c$ is a-priori partitioned into $y$ and $u$, and there are no latent variables $l$. In terms of $P1$ through $P3$, $P1$ is automatically satisfied by the imposed structure on $K$, $P2$ is usually replaced by requiring that the closed loop system be *internally stable* (see [9]), a more stringent requirement as we shall see later, and $P3$ reduces to $\|G \star K\|_\infty < 1$.

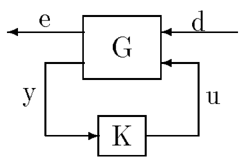


Figure 1: Standard input-output $\mathcal{H}_\infty$ formulation

## 3.1 Example

The following simple example can be used to illustrate the main differences between the problem formulation outlined above and standard $\mathcal{H}_\infty$ design. It consists of a one degree of freedom suspension design; an explicit solution to this problem is provided in Section 6. Consider the setup of Figure 2. The goal is to design system $\Sigma_c$, the suspension, in order to achieve certain performance objectives which will be described shortly. Variable $m$ denotes the sprung mass, or the mass of the cab where the passengers will ride. $\Sigma_c$ is the mechanism which we want to design; we restrict it to be a relation between $F_c$ and $z - r_0$. The spring and the damper model a tire, which is in contact with the road.

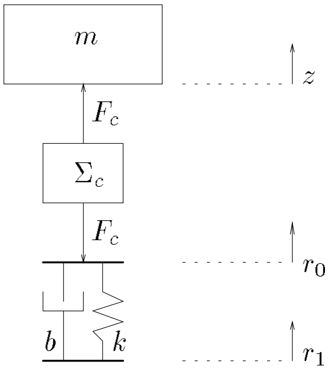


Figure 2: Suspension Design

The equations describing the system and the performance objectives are as follows:

$$
\begin{aligned}
0 &= F_c - m\ddot{z} \\
0 &= F_c + b(\dot{r_0} - \dot{r_1}) + k(r_0 - r_1) \\
c_1 &= F_c \\
c_2 &= z - r_0 \\
e_1 &= z - r_1 \quad \text{(tracking)} \\
e_2 &= \dddot{z} \qquad\quad \text{(comfort)} \\
d &= \ddot{r_1}
\end{aligned}
\tag{11}
$$

The first two equations are the equations of motion about an equilibrium point. The second two equations dictate which variables system $\Sigma_c$ has access to. The next two equations describe the performance objectives; we require that the sprung mass track the road, while simultaneously be subjected to small values of jerk (the jerk, or third derivative of position, is to a first approximation a good measure of passenger discomfort, and is in general a quantity which should be kept small in the design of mechanical systems [18]). The last equation models the allowable road disturbances; restricting $d$ to be an $\mathcal{L}_2$ disturbance of unit norm restricts $r_1$ to be small at high frequencies and allows $r_1$ to be large at low frequencies.

This corresponds to restricting large amplitude road disturbances to be gradual (hills), while allowing smaller amplitude disturbances to be sharper (potholes and speed bumps). Also note that when $d = 0$, $r_1(t) = C_0 + C_1 t$ for some constants $C_0$ and $C_1$; this corresponds to a constant climb, which should be allowed in the equations of motion.

It is clear from this example why the definition of stability should not encompass the latent variables: $r_1$ should not be restricted to decay to 0 when $d = 0$. In general, if one is concerned about the size of a latent variable, it could be penalized and be made a part of $e$.

There are several reasons why standard $\mathcal{H}_\infty$ design cannot directly handle this problem. The first is that there is no way to manipulate the above system into the form of Figure 1 with $G(s)$ proper. This precludes the use of standard state space methods for solving the problem. A further constraint is that the resulting design must result in a singular interconnection with the plant; equivalently, the interconnection imposes algebraic constraints on the states. This is not allowed in standard feedback control. It should be noted that by choosing appropriate weights for the various signals (for example, by first constructing a non-proper $G$, and then low-pass filtering all transfer functions which are not proper by a sufficient amount), one can approximate the problem with one which fits the setup of Figure 1. It is very unnatural to do so, however, and as shall be demonstrated, unnecessary as well.

# 4 Output Nulling Representations

The solution to the problem presented in Section 3 will be arrived at via state-space methods. The main reasons for adopting a state-space framework are the ease with which system representations may be manipulated, and the vast collection of state-space computational tools. In this section, we introduce *Output Nulling (ON)* representations for systems, and develop various tools for manipulating and analyzing systems in this form. These first order representations were extensively studied by Weiland [21]. Some of the results in this section and related algorithms found in the appendix first appeared in [5]. Related results on first order representations and various construction algorithms can be found in the work by Kuijper [13].

Given the following set of equations,

$$\begin{bmatrix} \dot{x} \\ 0 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} =: M \begin{bmatrix} x \\ w \end{bmatrix} \tag{12}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times q}$, $C \in \mathbb{R}^{r\times n}$, $D \in \mathbb{R}^{r\times q}$, and $M \in \mathbb{R}^{(n+q)\times(n+r)}$, the behavior of a system $\Sigma = \{\mathbb{R}, \mathbb{R}^q, \mathcal{B}\}$ is defined to be

$$\mathcal{B} := \{w \in C^\infty(\mathbb{R}, \mathbb{R}^q) | (12) \text{ is satisfied for some } x \in C^\infty(\mathbb{R}, \mathbb{R}^n)\} \tag{13}$$

Matrix $M$ is referred to as as a *representation matrix* for $\Sigma$. Matrices $A$, $B$, $C$, and $D$ are uniquely specified for a given $M$ and $q$; since $q$ will usually be known from context and be constant, $M$ contains all the information required to characterize $\mathcal{B}$. A procedure for constructing an ON representation given an AR set of equations may be found in Appendix A.

Since there are many representations which yield the same behavior, it will be useful to define the following equivalence relation: given $M \in \mathbb{R}^{(n+r)\times(n+q)}$ and $\overline{M} \in \mathbb{R}^{(\overline{n}+\overline{r})\times(\overline{n}+q)}$, $M \sim \overline{M}$ if $M$ and $\overline{M}$ yield the same behavior.

Representation matrix $M$ is *observable* if $(C, A)$ is an observable pair. Given $M$, one can always construct observable $\overline{M} \sim M$ by eliminating the unobservable portion of $x$ in (12).

$M$ is termed *dependent* if there exists $\overline{M} \sim M$ such that $\overline{r} < r$. Thus a dependent representation has redundant equations.

$M$ is termed *minimal* if $\overline{M} \sim M \Rightarrow \overline{n} \geq n$, $\overline{r} \geq r$. The following Lemma is from [21]:

**Lemma 1** *Representation matrix $M$ is minimal if and only if it is observable and $D$ is full row rank.*

A procedure for constructing a minimal representation is outlined in Appendix C.

The following Lemma, also from [21], outlines the transformations which may be performed on $M$ to yield equivalent representations:

**Lemma 2** *Given a (minimal) representation matrix $M$, $\overline{M}$ is an equivalent representation matrix if (and only if)*

$$\overline{M} = \begin{bmatrix} T^{-1}(A + LC)T & T^{-1}(B + LD) \\ PCT & PD \end{bmatrix} \tag{14}$$

*where $L$ is any matrix, $P$ and $T$ are any square, invertible matrices.*

## 4.1 Input Output Maps

Given a minimal representation matrix $M \in \mathbb{R}^{(n+r)\times(n+q)}$, it is a straightforward matter to construct a proper, input-output parametrization of the behavior $\mathcal{B}$. Since $D$ is full row rank, there exists a re-ordering of variables $w = (y, u)$ such that $D = \begin{bmatrix} D_y & D_u \end{bmatrix}$ with $D_y$ square and invertible, and $B = \begin{bmatrix} B_y & B_u \end{bmatrix}$. By the transformations of Lemma 2 ($P = D_y^{-1}$, $L = -B_y D_y^{-1}$), it follows that the following is an equivalent parametrization of $\mathcal{B}$:

$$\begin{bmatrix} \dot{x} \\ y \end{bmatrix} = \begin{bmatrix} A - B_y D_y^{-1} C & B_u - B_y D_y^{-1} D_u \\ -D_y^{-1} C & -D_y^{-1} D_u \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \tag{15}$$

It can be shown that all proper input-output maps may be generated in this fashion. Note that the number of outputs is equal to $r$, and the number of inputs $q - r$. As shown in [22], these are integer invariants, thus all input-output maps (proper or not) must have $r$ outputs and $q - r$ inputs. It is also shown in [22] that the number of free variables must be equal to the number of input variables, $q - r$, and leads to the following corollary:

**Corollary 1** *Given that a representation matrix $M$ is not dependent, the number of outputs in any input-output map is $r$, and the number of free variables is at most $q - r$.*

## 4.2 Stability

A system $\Sigma$ with variables $w$ is said to be *stable* if $w \in \mathcal{B} \Rightarrow w(t) \longrightarrow 0$ as $t \longrightarrow \infty$. Let $M$ be an observable representation matrix for $\Sigma$. The following is a characterization of stability:

**Lemma 3** *Given an observable representation matrix $M$, $\Sigma$ is stable if and only if*

$$\begin{bmatrix} A - sI & B \\ C & D \end{bmatrix} \quad \text{is full column rank } \forall s \in \overline{\mathbb{C}}^{+} \tag{16}$$

**Proof:** Assume that (16) is not full column rank for some $s_0 \in \overline{\mathbb{C}}^{+}$. Then there exist complex vectors $v_1$ and $v_2$ such that $x(t) = v_1 e^{s_0 t}$ and $w(t) = v_2 e^{s_0 t}$ satisfy the ON equations. Note that by observability, $v_2 \neq 0$. Furthermore, since Re $(s_0) \geq 0$, $w(t)$ does not decay to zero. If $s_0$ is purely real, $v_1$ and $v_2$ can be taken to be real, implying that $w(t)$ is real. If $s_0$ has an imaginary component, the real parts of $x(t)$ and $w(t)$ are non-zero and will also satisfy the ON equations.

Now assume that (16) is full column rank. $D$ is not necessarily full row rank, but by applying the reduction procedure of Section C, it can be shown that the resulting minimal representation will also satisfy the rank condition of (16). This implies that the resulting $D$ matrix is square and invertible. Setting $P = D^{-1}$, $L = -BD^{-1}$, and applying the transformations of Lemma 2, the following is an equivalent representation matrix for $\Sigma$:

$$\begin{bmatrix} A - BD^{-1}C & 0 \\ D^{-1}C & I \end{bmatrix} =: \begin{bmatrix} \overline{A} & 0 \\ \overline{C} & I \end{bmatrix} \tag{17}$$

where the rank condition implies that $\overline{A}$ is Hurwitz. The only solutions to these equations are $w(t) = \overline{C} e^{\overline{A}t} x_0$, which decay to zero.

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Note that the above notion of stability is not input-output in nature, since all elements of $w$ must decay to zero. Stability requirement $P2$ is equivalent to requiring that the system obtained by setting $d = 0$ and eliminating latent variables $l$ be stable.

## 4.3 Interconnection

There are several integer invariants associated with a system $\Sigma$ (see [22]). One is $p^*(\Sigma)$, the number of outputs in any input-output map; given a representation which is not dependent, this invariant is equal to $r$. Another is the minimum number of states required to describe $\Sigma$ in ON form, $n^*(\Sigma)$; given a minimal representation, this invariant is equal to $n$.

As defined by Willems [23], $\Sigma_1 \wedge \Sigma_2$ will be termed a *feedback interconnection* if

$$p^*(\Sigma_1 \wedge \Sigma_2) = p^*(\Sigma_1) + p^*(\Sigma_2) \tag{18}$$

An interpretation of the above is that the laws of the systems can be viewed as independent. A feedback interconnection will be termed *regular* if

$$n^*(\Sigma_1 \wedge \Sigma_2) = n^*(\Sigma_1) + n^*(\Sigma_2) \tag{19}$$

If $n^*(\Sigma_1 \wedge \Sigma_2) < n^*(\Sigma_1) + n^*(\Sigma_2)$, the interconnection will be called *singular*. Regular feedback interconnections are the standard ones considered in feedback control. Singular feedback interconnections differ in that the interconnection results in algebraic constraints on the states; thus the states of the individual systems must be matched before interconnection can take place. Perhaps the simplest example of a singular interconnection is connecting two capacitors in parallel; the voltages across each capacitor must be the same before interconnection, else an infinite (in practice, "large") current will flow between the two components. In terms of the invariant $n$, one state (the voltage across the capacitor) is required to describe each component, but only one state is required to describe the two capacitors in parallel, not two, since the voltages across each capacitor are required to be the same. The following Lemma can be used to construct a representation matrix for the interconnection of two systems:

**Lemma 4** *Given minimal representations matrices $M_1$ and $M_2$ for systems $\Sigma_1$ and $\Sigma_2$,*

$$M := \begin{bmatrix} A_1 & 0 & B_1 \\ 0 & A_2 & B_2 \\ C_1 & 0 & D_1 \\ 0 & C_2 & D_2 \end{bmatrix} \tag{20}$$

*is a representation matrix for $\Sigma = \Sigma_1 \wedge \Sigma_2$. The interconnection is a feedback interconnection if and only if $M$ is not dependent; the feedback interconnection is regular if and only if $M$ is minimal.*

**Proof:** The definition of interconnection immediately implies that $M$ is a representation matrix for $\Sigma$. The equivalence between the interconnection being a feedback interconnection and $M$ not being dependent is a direct consequence of Corollary 1. The equivalence between regularity and minimality of $M$ is a direct consequence of $(A, C)$ being an observable pair.

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In Section 5, a solution to the problem of Section 3 will be presented with the assumption that the compensator $\Sigma_c$ forms a feedback interconnection with $\Sigma_p$. It is shown below, however, that a pre-compensating system can be first interconnected with $\Sigma_p$ to make this assumption unrestrictive.

**Theorem 1** *Let $\Sigma_c$ be an allowable compensator. There exist systems $\overline{\Sigma}_c$ and $\hat{\Sigma}_c$ such that*

1. *$\overline{\Sigma}_c \wedge \Sigma_c$ is an allowable compensator.*
2. *$\hat{\Sigma}_c$ and $\Sigma_p \wedge \overline{\Sigma}_c$ form a feedback interconnection.*
3. *$\Sigma_p \wedge \overline{\Sigma}_c \wedge \Sigma_c = \Sigma_p \wedge \overline{\Sigma}_c \wedge \hat{\Sigma}_c$.*

**Proof:** Let $R_p(\xi) = [\ R_p^e(\xi)\ \ R_p^d(\xi)\ \ R_p^l(\xi)\ \ R_p^c(\xi)\ ]$ be an AR representation for $\Sigma_p$. Using the Smith form decompostion for polynomial matrices (see [12], for example), and the equivalence of AR representations under left multiplication by unimodular matrices [22], it follows that $R_p(\xi)$ can be assumed to have the form

$$R_p(\xi) = \begin{bmatrix} R^e_{p1}(\xi) & R^d_{p1}(\xi) & R^l_{p1}(\xi) & R^c_{p1}(\xi) \\ 0 & 0 & 0 & D(\xi)R^c_{p2}(\xi) \end{bmatrix} \tag{21}$$

where $[\ R^e_{p1}(\xi) \ \ R^d_{p1}(\xi) \ \ R^l_{p1}(\xi)\ ]$ is full normal row rank, $R^c_{p2}(\xi)$ is right invertible in the ring of polynomial matrices, and $D(\xi)$ is square and full normal rank.

Define $\overline{\Sigma}_c$ by AR representation $R^c_{p2}$. Interconnecting $\Sigma_p$ with $\overline{\Sigma}_c$ results in the following AR representation for the interconnected system:

$$\begin{bmatrix} R^e_{p1}(\xi) & R^d_{p1}(\xi) & R^l_{p1}(\xi) & R^c_{p1}(\xi) \\ 0 & 0 & 0 & R^c_{p2}(\xi) \end{bmatrix} \tag{22}$$

In the language of [22], this pre-compensator has the effect of removing the finite dimensional *uncontrollable behavior* which involves only variables $c$. Since the behavior of $\Sigma_p \wedge \overline{\Sigma}_c \wedge \Sigma_c$ is a subset of the behavior of $\Sigma_p \wedge \Sigma_c$, it follows that requirements $P2$ and $P3$ are satisfied for compensator $\overline{\Sigma}_c \wedge \Sigma_c$. Furthermore, since the two closed loop behaviors differ only by a finite dimensional subspace, requirement $P1$ must be satisfied as well. This proves part 1.

Let $R_c(\xi)$ be an AR representation for $\Sigma_c$. An AR representation for $\Sigma_p \wedge \overline{\Sigma}_c \wedge \Sigma_c$ is

$$\begin{bmatrix} R^e_{p1}(\xi) & R^d_{p1}(\xi) & R^l_{p1}(\xi) & R^c_{p1}(\xi) \\ 0 & 0 & 0 & R^c_{p2}(\xi) \\ 0 & 0 & 0 & R_c(\xi) \end{bmatrix} \tag{23}$$

By the rank conditions on $[\ R^e_{p1}(\xi) \ \ R^d_{p1}(\xi) \ \ R^l_{p1}(\xi)\ ]$ and $R^c_{p2}$, there must exist polynomial matrices $U_1(\xi)$ and $U_2(\xi)$ such that

$$\begin{bmatrix} I & 0 \\ U_1(\xi) & U_2(\xi) \end{bmatrix} \begin{bmatrix} R^c_{p2}(\xi) \\ R_c(\xi) \end{bmatrix} = \begin{bmatrix} R^c_{p2}(\xi) \\ \hat{R}_c(\xi) \\ 0 \end{bmatrix} \tag{24}$$

where $U_2(\xi)$ is a unimodular matrix and $\begin{bmatrix} R^c_{p2}(\xi) \\ \hat{R}_c(\xi) \end{bmatrix}$ is full normal row rank. Thus system $\hat{\Sigma}_c$ defined by AR representation $\hat{R}_c(\xi)$ forms a feedback interconnection with $\Sigma_p \wedge \overline{\Sigma}_c$, and $\Sigma_p \wedge \overline{\Sigma}_c \wedge \Sigma_c = \Sigma_p \wedge \overline{\Sigma}_c \wedge \hat{\Sigma}_c$, proving parts 2 and 3.

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Thus by first interconnecting the given system $\Sigma_p$ with pre-compensator $\overline{\Sigma}_c$ and forming $\hat{\Sigma}_p := \Sigma_p \wedge \overline{\Sigma}_c$, one need only consider compensators $\hat{\Sigma}_c$ which form feedback interconnections with $\hat{\Sigma}_p$. Furthermore, if $\hat{\Sigma}_c$ is an allowable compensator for $\hat{\Sigma}_p$, $\overline{\Sigma}_c \wedge \hat{\Sigma}_c$ is an allowable compensator for $\Sigma_p$.

### 4.4 Dual Representations

It will be useful to introduce the notion of a dual ON representation for a system $\Sigma$. Given an observable representation $M$, where it can be assumed without loss of generality that the $A$ matrix is invertible by Lemma 2, the following equations capture the behavior $\mathcal{B}$:

$$\begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} A^{-1} & -A^{-1}B \\ CA^{-1} & D - CA^{-1}B \end{bmatrix} \begin{bmatrix} \dot{x} \\ w \end{bmatrix} =: \begin{bmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{bmatrix} \begin{bmatrix} \dot{x} \\ w \end{bmatrix} =: \tilde{M} \begin{bmatrix} \dot{x} \\ w \end{bmatrix} \tag{25}$$

It can be verified that the behavior preserving transformations of Lemma 2 apply to dual ON representations as well. The definitions of observable, dependent, and minimal can be applied to dual ON representations; it can be shown that these definitions are satisfied for an ON representation if and only if they are satisfied for its dual.

#### 4.4.1 Stability Conditions

By the following identities

$$\begin{aligned} \begin{bmatrix} A - sI & B \\ C & D \end{bmatrix} &= \begin{bmatrix} sI & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \tilde{A} - s^{-1}I & \tilde{B} \\ \tilde{C} & \tilde{D} \end{bmatrix} \begin{bmatrix} A & B \\ 0 & I \end{bmatrix}, \quad s \neq 0 \\ \begin{bmatrix} A & B \\ C & D \end{bmatrix} &= \begin{bmatrix} I & 0 \\ \tilde{C} & \tilde{D} \end{bmatrix} \begin{bmatrix} A & B \\ 0 & I \end{bmatrix} \end{aligned} \tag{26}$$

and Lemma 3, we have the following characterization of stability:

**Corollary 2** *Given observable dual representation matrix $\tilde{M}$, $\Sigma$ is stable if and only if*

$$\begin{bmatrix} \tilde{A} - sI & \tilde{B} \\ \tilde{C} & \tilde{D} \end{bmatrix} \text{ is full column rank } \forall s \in \overline{\mathbb{C}}^{+}/\{0\}, \; \tilde{D} \text{ is full column rank.} \tag{27}$$

## 5 Problem Conversion

One of the major complication which arises in the problem formulation of Section 3 is allowing singular interconnections, since they result in algebraic constraints on the states. This problem is circumvented by working with dual representations; as will be shown, algebraic constraints take the form of 0 uncontrollable modes for a related input-output state space representation. This characterization of singular interconnections vastly simplifies the problem at hand, and allows one to convert it to an almost standard $\mathcal{H}_\infty$ problem with minor modifications. The solution to this associated $\mathcal{H}_\infty$ problem may be found in Appendix E.

It will be assumed that pre-compensator $\overline{\Sigma}_c$ has already been applied to the given system, as per Theorem 1. The reader is referred to [12] for details on how to construct an AR representation for $\overline{\Sigma}_c$ as per equation (21). It will also be assumed that $\Sigma_p$ has no latent variables. The removal of these variables may be accomplished with the algorithm in Section D.

The starting point will thus be a representation matrix for $\Sigma_p$. Let $\tilde{M}_p \in \mathbb{R}^{(n_p+r_p)\times(n_p+q_e+q_d+q_c)}$ be a minimal dual representation matrix for $\Sigma_p$ and $\tilde{M}_c \in \mathbb{R}^{(n_c+r_c)\times(n_c+q_c)}$ be a dual representation matrix for the candidate compensator $\Sigma_c$. Let $\Sigma = \Sigma_p \wedge \Sigma_c$ be the resulting feedback interconnection. By Lemma 4 and equation (25), the following is a dual representation matrix for $\Sigma$:

$$\tilde{M} = \begin{bmatrix} \tilde{A}_p & 0 & \tilde{B}_p^e & \tilde{B}_p^d & \tilde{B}_p^c \\ 0 & \tilde{A}_c & 0 & 0 & \tilde{B}_c \\ \tilde{C}_p & 0 & \tilde{D}_p^e & \tilde{D}_p^d & \tilde{D}_p^c \\ 0 & \tilde{C}_c & 0 & 0 & \tilde{D}_c \end{bmatrix} \tag{28}$$

The following Lemma is central to converting the problem data to a more usable form:

**Lemma 5** *Given that an allowable compensator $\Sigma_c$ exists, the behavior $\mathcal{B}$ of $\Sigma$ is captured by the following equations*

$$\begin{aligned} \begin{bmatrix} x_p \\ e \\ y \end{bmatrix} &= M_p^{IO} \begin{bmatrix} \dot{x}_p \\ d \\ u \end{bmatrix}, \quad M_p^{IO} := \begin{bmatrix} A & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & 0 \end{bmatrix} \\ \begin{bmatrix} x_c \\ u \end{bmatrix} &= M_c^{IO} \begin{bmatrix} \dot{x}_c \\ y \end{bmatrix}, \quad M_c^{IO} := \begin{bmatrix} \overline{A} & \overline{B} \\ \overline{C} & \overline{D} \end{bmatrix} \\ c &= V^{-1} \begin{bmatrix} y \\ u \end{bmatrix} \end{aligned} \tag{29}$$

*where $V$ and $M_p^{IO}$ can be determined from $\tilde{M}_p$, and $M_c^{IO}$ can be determined from $\tilde{M}_c$.*

**Proof:** By Lemma 4, $\tilde{M}$ is not a dependent representation. By requirement $P1$, the number of free variables must be at least $q_d$; by Corollary 1 this implies that $r_p + r_c \leq q_e + q_c$. Stability requirement $P2$, on the other hand, implies that $r_p + r_c \geq q_e + q_c$ by the rank condition of Corollary 2. Thus $r_c = q_e + q_c - r_p$, and by Corollary 2, $\begin{bmatrix} \tilde{D}_p^e & \tilde{D}_p^c \\ 0 & \tilde{D}_c \end{bmatrix}$ must be square and invertible. By the behavior preserving transformations of Lemma 2, it can be assumed that $\tilde{D}_p^e = \begin{bmatrix} I \\ 0 \end{bmatrix}$, since $\tilde{D}_p^e$ must have full column rank. This induces partition $\tilde{D}_p^c = \begin{bmatrix} \tilde{D}_{p_1}^c \\ \tilde{D}_{p_2}^c \end{bmatrix}$, where $\tilde{D}_{p_2}^c$ must be full row rank. There exists, therefore, invertible matrix $V$ such that $\tilde{D}_{p_2}^c V^{-1} = \begin{bmatrix} I & 0 \end{bmatrix}$. Define the following change of co-ordinates for variables $c$:

$$\begin{bmatrix} y \\ u \end{bmatrix} = \overline{c} := Vc =: \begin{bmatrix} V_y \\ V_u \end{bmatrix} c \tag{30}$$

where the size of variable $y$ is equal to the number of rows of $\tilde{D}_{p_2}^c$.

Finally, partition $\tilde{D}_{p_2}^{\overline{c}} = \left[ \begin{array}{cc} I & 0 \end{array} \right]$ induces partition $\tilde{D}_{\overline{c}} = \left[ \begin{array}{cc} \tilde{D}_{\overline{c}}^{y} & \tilde{D}_{\overline{c}}^{u} \end{array} \right]$, where $\tilde{D}_{\overline{c}}^{u}$ must be square and invertible.

It now follows that by applying the behavior preserving transformations of Lemma 2, as in Section 4.1, $\Sigma_p$ can now be captured in dual input-output form with $y$ and $e$ as outputs and $d$ and $u$ as inputs, and $\Sigma_{\overline{c}}$ can be captured in dual input-output form with $u$ an output and $y$ an input, as stated in the lemma. Matrices $M_p^{IO}$ and $M_c^{IO}$ can readily be determined from $\tilde{M}_p$ and $\tilde{M}_c$; the details are omitted.

■

Note that in the above lemma, only the existence of an allowable compensator was required to express $\Sigma_p$ in dual input-output form. Furthermore, this form for $\Sigma_p$ is independent of the particular $\Sigma_c$ with which it will be interconnected.

The following Theorem states that with the above change of co-ordinates, a solution to the problem of Section 3 takes the form of an almost standard $\mathcal{H}_\infty$ problem:

**Theorem 2** *Let* $G := \left[ \begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & 0 \end{array} \right]$. *There exists an allowable compensator* $\Sigma_c$ *if and only if there exists* $K := \left[ \begin{array}{c|c} \overline{A} & \overline{B} \\ \hline \overline{C} & \overline{D} \end{array} \right]$ *such that*

**I.** *$K$ internally stabilizes $G$ except for possible modes at* $0$

**II.** $\|G \star K\|_\infty < 1$

*If such a $K$ exists, a dual representation matrix for an allowable $\Sigma_c$ is*

$$\left[ \begin{array}{c} x \\ 0 \end{array} \right] = \left[ \begin{array}{cc} \overline{A} & \overline{B}V_y \\ \overline{C} & (\overline{D}V_y - V_u) \end{array} \right] \left[ \begin{array}{c} \dot{x} \\ c \end{array} \right] \tag{31}$$

**Proof:** Let $\Sigma_c$ be an allowable compensator. By Lemma 5 and equation (29), the following dual input-output representation for $\Sigma$ may be constructed:

$$\left[ \begin{array}{c} x \\ e \\ c \end{array} \right] = \left[ \begin{array}{cc} \hat{A} & \hat{B} \\ \hat{C}^e & \hat{D}^e \\ \hat{C}^c & \hat{D}^c \end{array} \right] \left[ \begin{array}{c} \dot{x} \\ d \end{array} \right] \tag{32}$$

where the above matrices can be determined from the matrices in Lemma 5; the details are omitted. By Corollary 2, the eigenvalues of $\hat{A}$ must be in $\mathbb{C}^- \cup \{0\}$ since $(\hat{A}, \hat{C})$ is an observable pair. There exists, therefore, a state transformation which yields the following equations:

$$\left[ \begin{array}{c} x_s \\ x_{0_1} \\ x_{0_2} \\ x_{0_3} \\ e \end{array} \right] = \left[ \begin{array}{ccccc} \hat{A}_s & 0 & 0 & 0 & \hat{B}_s \\ 0 & \hat{A}_{0_{11}} & 0 & \hat{A}_{0_{13}} & \hat{B}_{0_1} \\ 0 & \hat{A}_{0_{21}} & \hat{A}_{0_{22}} & \hat{A}_{0_{23}} & \hat{B}_{0_2} \\ 0 & 0 & 0 & \hat{A}_{0_{33}} & 0 \\ \hat{C}_s^e & \hat{C}_{0_1}^e & 0 & \hat{C}_{0_3}^e & \hat{D}^e \end{array} \right] \left[ \begin{array}{c} \dot{x}_s \\ \dot{x}_{0_1} \\ \dot{x}_{0_2} \\ \dot{x}_{0_3} \\ d \end{array} \right] \tag{33}$$

where $\hat{A}_s$ is Hurwitz and $\hat{A}_{0_{11}}, \hat{A}_{0_{22}}, \hat{A}_{0_{33}}$ are nilpotent. Equation $x_{0_3} = \hat{A}_{0_{33}} \dot{x}_{0_3}$ implies that $x_{0_3} = 0$; these are the algebraic constraints which result from a singular interconnection. Thus singularity is equivalent to uncontrollable modes at 0. States $x_{0_2}$ are controllable from $d$, but unobservable from $e$; they are, however, observable from $y$ and/or $u$; these states correspond to derivatives of $d$ appearing in the closed loop expressions for $y$ and $u$ (note that this does not violate the stability requirement). The $x_{0_1}$ states are controllable from $d$ and observable from $e$. If any of these states are present, derivatives of $d$ will appear in the closed loop expression for $e$, violating performance requirement $P3$.

Given that there are no $x_{0_1}$ states, and since $\hat{A}_s$ is invertible, we may write the following state space input-output map from $d$ to $e$:

$$\begin{bmatrix} \dot{x}_s \\ e \end{bmatrix} = \begin{bmatrix} \hat{A}_s^{-1} & -\hat{A}_s^{-1}\hat{B}_s \\ \hat{C}_s^e \hat{A}_s^{-1} & \hat{D}^e - \hat{C}_s^e \hat{A}_s^{-1}\hat{B}_s \end{bmatrix} \begin{bmatrix} x_s \\ d \end{bmatrix} \tag{34}$$

Furthermore, $P3$ is satisfied if and only if there are no $x_{0_1}$ states and

$$\begin{aligned} \sup_{w\in\mathbb{R}} \quad & \overline{\sigma}\left(\hat{D}^e - \hat{C}_s^e \hat{A}_s^{-1}\hat{B}_s - \hat{C}_s^e \hat{A}_s^{-1}\left(jwI - \hat{A}_s^{-1}\right)^{-1} \hat{A}_s^{-1}\hat{B}_s\right) = \\ \sup_{\overline{w}\in\mathbb{R}} \quad & \overline{\sigma}\left(\hat{D}^e - \hat{C}_s^e \hat{A}_s^{-1}\hat{B}_s - j\overline{w}\hat{C}_s^e \hat{A}_s^{-1}\left(I - j\overline{w}\hat{A}_s^{-1}\right)^{-1} \hat{A}_s^{-1}\hat{B}_s\right) = \\ \sup_{\overline{w}\in\mathbb{R}} \quad & \overline{\sigma}\left(\hat{D}^e + \hat{C}_s^e \left(j\overline{w}I - \hat{A}\right)^{-1} \hat{B}_s\right) < 1 \end{aligned} \tag{35}$$

This proves the necessity of conditions I and II. The sufficiency of conditions I and II, and the given representation for an allowable $\Sigma_c$, follow directly from the previous arguments and Lemma 5.

■

The reader is referred to Appendix B for a procedure to construct an AR representation given a dual ON representation. Under some mild assumptions, the solution to the $\mathcal{H}_\infty$ synthesis problem of Theorem 2 may be found in Appendix E.

# 6 Example

We return to the example first outlined in Section 3.1. We choose the following values for the system parameters:

$$m = 1,\ b = 20,\ k = 100$$

Furthermore, $e_1$ is scaled by a factor of 100 relative to $e_2$.

Equations (11) are in AR form. Using the procedure of Section A to construct an ON representation, the procedure of Section C to make it minimal, and finally the procedure of

Section D to eliminate latent variables $F_c$, $z$, $r_0$ and $r_1$, results in a representation matrix with the following $D$ matrix

$$D = \begin{bmatrix} 0 & -0.0082 & 0 & 0 & -0.0001 \\ 0 & 0.0100 & 0 & 0.0087 & -0.9998 \\ 0 & 0.0002 & 0 & -0.5773 & -0.0151 \end{bmatrix} \tag{36}$$

and variables $e_1$, $e_2$, $d$, $c_1$ and $c_2$. It follows from Section 4.1 that the only one way to write the above as a proper input-output map is with $e_1$ and $d$ as inputs. Furthermore, since the allowable control strategies must only involve $c_1$ and $c_2$, it follows that all interconnections must be singular. Thus standard $\mathcal{H}_\infty$ tools cannot be applied to this problem.

By next constructing a dual representation matrix as in Section 4.4, applying the procedure of Section 5 to convert it to the form of Theorem 2, using the $\mathcal{H}_\infty$ solution of Section E, and constructing an AR representation from the resulting dual ON representation for $\Sigma_c$ via Section B, the following form for the optimal system $\Sigma_c$ is obtained:

$$1.3379(\ddot{z} - \ddot{r}_0) + 10.1164(\dot{z} - \dot{r}_0) + 17.1342(z - r_0) + 0.1623\dot{F}_c + F_c = 0 \tag{37}$$

with 5.84 the optimal worst case gain from $d$ to $e$. Defining $k_{opt} := 17.1342$, $b_{opt} := 8.2417$, and input-output map $F(s) := \frac{5.5861s}{s+6.1601}$, the following is an equivalent expression for (37):

$$F_c = -k_{opt}(z - r_0) - b_{opt}(\dot{z} - \dot{r}_0) + F(z - r_0) \tag{38}$$

Thus the optimal system may be implemented as a spring with coefficient $k_{opt}$, a damper with coefficient $b_{opt}$, and an active component $F$ (note that this parametrization is not unique. The criterion used to extract $k_{opt}$ and $b_{opt}$ was that the resulting $F(s)$ be proper with a $DC$ gain of 0). For comparison purposes, the resulting worst case gain from $d$ to $e$ without active component $F$ is 6.11.

In order to determine how close the above design comes to predicting what the optimal values of spring and damper coefficients are in the absence of active component $F$, a search was performed by gridding the space of spring and damper coefficients, and determining the $\mathcal{H}_\infty$ norm for the resulting designs. The optimal design was a spring coefficient $k_{search} = 16.70$ and a damper coefficient $b_{search} = 7.55$, with a resulting worst case gain of 6.00 .

For this simple example, the optimal $\mathcal{H}_\infty$ design essentially gives the parallel interconnection of a spring and a damper as the optimal compensator $\Sigma_c$, the suspension. The difference between this design and that obtained by searching for optimal spring and damper values was less than 3%. This is mainly due to the simple performance specifications and road disturbance profile, which result in a relation between $F_c$ and $z - r_0$ which can be approximated very well by a spring and a damper. In fact, by residualizing the tire dynamics (equivalently, by setting $r_0 = r_1$), the optimal $\mathcal{H}_\infty$ design results in a value of 6.23 for the worst case gain from $d$ to $e$, a value of 16.06 for $k_{opt}$, a value of 6.23 for $b_{opt}$, and a value of 0 for $F$.

# 7 Conclusion

In this paper, a general $\mathcal{H}_\infty$ problem for continuous time, linear time invariant systems was formulated and solved. One of the main benefits of this more general problem formulation is that singular interconnections are not precluded from the design process. These types of interconnections occur naturally when interconnecting first principle models; for example, the simple suspension design presented in Section 6 resulted in three algebraic constraints on the states.

A desirable feature of this design methodology is also apparent in the simple example; the optimal design consists of a part which can be implemented with passive components, and an active part. Typically, these designs are not performed simultaneously; ie., the choice of which spring and damper values to use would not typically be made at the same time that an active suspension design was being performed. The problem formulation in this paper makes no distinction between these two phases, and views the design process as determining the optimal relation between a given set of variables.

There are several logical continuations to this work. On the technical side, assumptions $A_2$, $A_3$, and $A_4$ in Theorem 2 need to be relaxed to provide a purely general solution. To that end, a linear matrix inequality (LMI) formulation is currently being investigated.

While we have explored optimality in this paper, the important issue of implementability has not been addressed. In many cases, the optimal relation between a system's variables may not be physically realizable; for example, how would one implement relation $F_c = \ddot{z} - \ddot{r}_1$ in the suspension design of Section 6 ? More generally, designs of real systems must take into account numerous other types of constraints, such as mass and size limitations, and other properties of a model, such as non-linearities, distributed effects, and model uncertainty, which make the design techniques presented in this paper not directly applicable. The results in this paper should thus be seen as providing bounds on the best achievable performance, and provide guidelines on how to proceed with the design of the overall system. The next step towards a more feasible design methodology is to expand the results in this paper to include the implicit descriptions of systems with uncertainty as per [5] and [4], and the latest results in the analysis and synthesis of systems described by implicit equations in [16] and [6].


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## A Constructing ON representations from AR representations

The following procedure yields an observable, but not necessarily minimal, ON representation given an AR representation. The reduction procedure of Section C can be used to make the resulting representation minimal.

Given the following set of AR equations,

$$R_L \frac{d^L w}{dt^L} + \cdots + R_0 w = 0 \tag{39}$$

consider the following set of equations:

$$\begin{aligned} \dot{x}_1 &= R_0 w && (40)\\ &\vdots \\ \dot{x}_L &= x_{L-1} + R_{L-1} w \\ 0 &= x_L + R_L w && (41) \end{aligned}$$

which can be captured in ON form by setting $M$ equal to

$$M := \begin{bmatrix} 0 & R_0 \\ I & \hat{R} \end{bmatrix}, \text{ with } \hat{R} := \begin{bmatrix} R_1 \\ \vdots \\ R_L \end{bmatrix} \tag{42}$$

Note that the above partition for $M$ does not correspond to the partition of equation (12). By repeated differentiation of equation (41) and substituting for the $x_l$, it immediately follows that any $w$ which satisfies the ON equations must also satisfy the AR equations. To show the converse, let $w$ satisfy the AR equations. By integrating (39) $L$ times, it follows that

$$R_L w(t) + x_L(t) = 0 \tag{43}$$

where

$$x_l(t) := \int_0^t (x_{l-1}(\tau) + R_{L-1} w(\tau))\, d\tau + c_l, \quad c_l \in \mathbb{R} \tag{44}$$

for $1 \leq l \leq L$. These $x_l$ and $w$ also satisfy the ON equations.

Note that $M$ in (42) is not necessarily minimal, since $R_L$ need not be full row rank. The total number of states in this representation are $rL$. By building an ON representation for each AR equation, and interconnecting them as described in Section 4.3, a lower state dimension representation can be constructed, with the total number of states equal to $\sum_{l=1}^{L} d_l(R(\xi))$, with $d_l(R(\xi)) :=$ the degree of the l-th row of $R(\xi)$. The resulting representation is minimal if and only if the *leading coefficient matrix* of $R(\xi)$ is of full row rank; equivalently, if $R(\xi)$ is *row proper* [23].

# B Constructing AR representations from dual ON representations

Let $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ be an observable ON representation matrix for $\Sigma$. $\mathcal{B}$ is captured by the following equations:

$$\begin{aligned} x &= A\dot{x} + Bw && (45)\\ 0 &= C\dot{x} + Dw \end{aligned}$$

Let matrix $F$ be such that $A+FC$ is nilpotent. The following equations capture the same behavior as (45):

$$
\begin{aligned}
x &= \overline{A}\dot{x} + \overline{B}w = (A+FC)\dot{x} + (B+FD)w && (46)\\
0 &= C\dot{x} + Dw && (47)
\end{aligned}
$$

Let $L$ be the dimension of the largest Jordan block of $\overline{A}$. Define the following:

$$
\begin{aligned}
R_0 &:= D && (48)\\
R_l &:= C\overline{A}^{l-1}\overline{B}, \ \ 1 \le l \le L
\end{aligned}
$$

We claim that $R(\frac{d}{dt})w = 0$ is an equivalent AR representation for $\Sigma$. If $w$ satisfies (46) and (47), repeated differentiation of (46), substitution into (47), and the fact that $\overline{A}^L = 0$ implies that $w$ satisfies $R(\frac{d}{dt})w = 0$. Now assume that $w$ satisfies $R(\frac{d}{dt})w = 0$. Define

$$x := \overline{A}^{L-1}\overline{B}\frac{dw^{L-1}}{dt^{L-1}} + \cdots + \overline{B}w \tag{49}$$

It follows that (47) is satisfied. Furthermore, since $\overline{A}^L = 0$, (46) is satisfied as well. Note that the above construction allows one to construct a map from variables $w$ to states $x$, similar to the results of Rapisarda and Willems in [17].

One could use the above procedure to construct an AR representation from an ON representation by first converting the latter to the form in (45); for this case, however, a more direct approach exists.

## C Constructing minimal ON representations

Let $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ be an observable representation matrix. Let $P_1^{\mathrm{T}}$ be a basis for Im $(\begin{bmatrix} C & D \end{bmatrix})$, $P_2^{\mathrm{T}}$ a basis for Im $(P_1 D)$, and $T_1$ a basis for Ker $(P_2^{\perp} P_1 C)$. Define $P := \begin{bmatrix} P_2 P_1 \\ P_2^{\perp} P_1 \\ P_1^{\perp} \end{bmatrix}$ and $T := \begin{bmatrix} T_1 & T_1^{\perp} \end{bmatrix}$. Applying the behavior preserving transformations of Lemma 2 results in the following equivalent representation:

$$
\begin{bmatrix}
A_{11} & A_{12} & B_1 \\
A_{21} & A_{22} & B_2 \\
C_1 & C_2 & D_1 \\
0 & C_4 & 0 \\
0 & 0 & 0
\end{bmatrix} \tag{50}
$$

where $C_4$ is an invertible, square matrix. Partitioning $x$ into $x_1$ and $x_2$, consistent with the partition in (50), the second to last equation implies that $x_2 = 0$, and hence $\dot{x}_2 = 0$. It follows that that following is an equivalent, observable representation:

$$\begin{bmatrix} A_{11} & B_1 \\ C_1 & D_1 \\ A_{21} & B_2 \end{bmatrix} \tag{51}$$

The above procedure can be repeated until the resulting $D$ matrix has full row rank. Note that the above procedure yields an equivalent representation with less number of states $n$, or less number of equations $r$ (or both), if the original representation was not minimal.

# D Latent variable elimination

Given a system $\Sigma$, with behavior $\mathcal{B}$ and variables $w = (\overline{w}, l)$, it is required to construct a representation for system $\overline{\Sigma}$ with behavior $\overline{\mathcal{B}}$ defined by:

$$\overline{\mathcal{B}} := \{\overline{w} \in C^\infty | (\overline{w}, l) \in \mathcal{B} \text{ for some } l \in C^\infty\} \tag{52}$$

In the terminology of [22], variables $l$ are the latent variables, while variables $\overline{w}$ are the *manifest* variables. Let $M$ be a representation matrix for $\Sigma$:

$$M = \begin{bmatrix} A & B_w & B_l \\ C & D_w & D_l \end{bmatrix} \tag{53}$$

Let $V_3$ be a basis for Ker $\left(\begin{bmatrix} B_l \\ D_l \end{bmatrix}\right)$, and $V_2$ a basis for $\left(\text{Ker}\begin{bmatrix} B_l \\ D_l \end{bmatrix}\right)^\perp \cap \text{Ker}(D_l)$. Let $V_1 = \begin{bmatrix} V_2 & V_3 \end{bmatrix}^\perp$. Since $D_l V_1$ is full column rank, there exists $L$ such that $(B_l + LD_l)V_1 = 0$. Define $P := \begin{bmatrix} V_1^{\mathrm{T}} D_l^{\mathrm{T}} \\ (V_1^{\mathrm{T}} D_l^{\mathrm{T}})^\perp \end{bmatrix}$ and $T := \begin{bmatrix} V_2^{\mathrm{T}} B_l^{\mathrm{T}} \\ (V_2^{\mathrm{T}} B_l^{\mathrm{T}})^\perp \end{bmatrix}^{-1}$. Defining $l =: V_1 l_1 + V_2 l_2 + V_3 l_3 =: V\overline{l}$ and applying the behavior preserving transformations of Lemma 2 results in the following representation which does not change behavior $\overline{\mathcal{B}}$ (since $V$ is square and invertible):

$$\begin{bmatrix} A_{11} & A_{12} & B_{w,1} & 0 & B_{l_2} & 0 \\ A_{21} & A_{22} & B_{w,2} & 0 & 0 & 0 \\ C_{11} & C_{12} & D_{w,1} & D_{l_1} & 0 & 0 \\ C_{21} & C_{22} & D_{w,2} & 0 & 0 & 0 \end{bmatrix} \tag{54}$$

where $B_{l_2}$ and $D_{l_1}$ are square and invertible. Let state $x$ be partitioned into $x_1$ and $x_2$, consistent with the above partition. We claim that

$$\begin{bmatrix} A_{22} & B_{w,2} & A_{21} \\ C_{22} & D_{w,2} & C_{21} \end{bmatrix} \tag{55}$$

captures behavior $\overline{\mathcal{B}}$, where $x_1$ is now a latent variable: If $\overline{w}$ satisfies (54), it must also satisfy (55); if $\overline{w}$ satisfies (55) for some $x_1$, it also satisfies (54) by appropriately defining $l_1$ and $l_2$.

The above procedure can be repeated until there are no more latent variables left. Note that the minimality of the representation is not necessarily preserved, since $(A_{22}, C_{22})$ need not be an observable pair; in that case, however, the unobservable modes can be truncated to yield a minimal representation.

# E $\mathcal{H}_\infty$ solution

We present a Ricatti based solution to the synthesis problem of Theorem 2, in the style of Doyle et al [8]. The development will closely parallel that of Glover and Doyle [11], and specific references to this work will be made to streamline the proofs and arguments.

The techniques used to solve this problem are very similar to those of Mita et al [14] where an $\mathcal{H}_\infty$ control problem with unstable weighting functions is solved. In fact, the derived conditions in [14] are equivalent to those presented in this section, even though extra assumptions in their problem data are made. Similarly, in Copeland and Safonov [2], a general synthesis procedure is outlined where pre-compensators are used to cancel zeros on the $j\omega$ axis; this approach, however, cannot be applied in general to the problem of Theorem 2.

In the setup of Theorem 2, a controller which internally stabilizes the system with the exception of possible modes at 0 will be termed an *admissible* controller. Note that the state space equations for the closed loop system are:

$$G \star K = \left[\begin{array}{cc|c} A + B_2\overline{D}C_2 & B_2\overline{C} & B_1 + B_2\overline{D}D_{21} \\ \overline{B}C_2 & \overline{A} & \overline{B}D_{21} \\ \hline C_1 + D_{12}\overline{D}C_2 & D_{12}\overline{C} & D_{11} + D_{12}\hat{D}D_{21} \end{array}\right] \tag{56}$$

For $K$ to be admissible, any modes at 0 must be either unobservable or uncontrollable (or both). As is shown in [24], the unobservable 0 modes of (56) must correspond to the 0 invariant zeros of $G_{12} := \left[\begin{array}{c|c} A & B_2 \\ \hline C_1 & D_{12} \end{array}\right]$, and the uncontrollable 0 modes must correspond to the 0 invariant zeros of $G_{21} := \left[\begin{array}{c|c} A & B_1 \\ \hline C_2 & D_{21} \end{array}\right]$. These violate the invariant zero assumptions of [11], as will be discussed next.

## E.1 Assumptions

The following assumptions are made on the problem data:

(A2) $D_{12}$ full column rank with $\left[\begin{array}{cc} D_{12} & D_\perp \end{array}\right]$ unitary, $D_{21}$ full row rank with $\left[\begin{array}{c} D_{21} \\ \tilde{D}_\perp \end{array}\right]$ unitary.

(A3) $\left[\begin{array}{cc} A - j\omega & B_2 \\ C_1 & D_{12} \end{array}\right]$ full column rank $\forall \omega \neq 0$ .

(A4) $\begin{bmatrix} A - j\omega & B_1 \\ C_2 & D_{21} \end{bmatrix}$ full row rank $\forall \omega \neq 0$.

Condition $(A3)$ is equivalent to $(D_\perp^{\mathrm{T}} C_1, A - B_2 D_{12}^{\mathrm{T}} C_1)$ having no purely imaginary unobservable modes, except possibly at 0. A Kalman decomposition induces the following state transformation $S$:

$$\begin{aligned} A^{FI} &= \begin{bmatrix} A_{11}^{FI} & A_{12}^{FI} \\ A_{21}^{FI} & A_{22}^{FI} \end{bmatrix} := S^{-1} A S \\ \begin{bmatrix} B_1^{FI} & B_2^{FI} \end{bmatrix} &= \begin{bmatrix} B_{11}^{FI} & B_{21}^{FI} \\ B_{12}^{FI} & B_{22}^{FI} \end{bmatrix} := S^{-1} \begin{bmatrix} B_1 & B_2 \end{bmatrix} \\ \begin{bmatrix} C_1^{FI} \\ C_2^{FI} \end{bmatrix} &= \begin{bmatrix} C_{11}^{FI} & C_{12}^{FI} \\ C_{21}^{FI} & C_{22}^{FI} \end{bmatrix} := \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} S \end{aligned} \tag{57}$$

In this co-ordinate system, $D_\perp^{\mathrm{T}} C_1^{FI} = \begin{bmatrix} \overline{C}^{FI} & 0 \end{bmatrix}$ and $(A^{FI} - B_2^{FI} D_{12}^{\mathrm{T}} C_1^{FI}) = \begin{bmatrix} \overline{A}^{FI} & 0 \\ * & \Sigma^{FI} \end{bmatrix}$, where $\Sigma^{FI}$ is nilpotent, and $(\overline{C}^{FI}, \overline{A}^{FI})$ has no purely imaginary unobservable modes.

Similarly, condition $(A4)$ is equivalent to $\left(A - B_1 D_{21}^{\mathrm{T}} C_2, B_1 \tilde{D}_\perp^{\mathrm{T}}\right)$ having no purely imaginary uncontrollable modes, except possibly at 0. State transformation $T$ is defined analogously to $S$.

The final assumption on the problem data follows, presented last since it is more natural to do so in the appropriate co-ordinate system:

(A1) $(A_{11}^{FI}, B_{21}^{FI})$ stabilizable, $(C_{21}^{FC}, A_{11}^{FC})$ detectable.

Assumption $(A2)$ is equivalent to requiring that $D_{12}$ and $D_{21}$ be full column rank and full row rank, respectively, by the freedom in the change of co-ordinates of equation (30). In general, assumptions $(A2)$, $(A3)$, and $(A4)$ are not necessary for a solution to exist, but they allow the Ricatti based approach of [11] to be used. $(A1)$, however, is necessary for a solution to exist. If not satisfied, it can be shown that all closed loop maps will have unstable modes, with any modes at 0 appearing in the map from $d$ to $e$. The simplest method of relaxing assumptions $(A2)$, $(A3)$, and $(A4)$ is to modify the various $\mathcal{H}_\infty$ LMI solutions of Gahinet and Apkarian [10] and Packard [15]. This is a topic of future research.

The only differences between the above assumptions and those in [11] are the relaxation on the 0 invariant zeros of $G_{12}$ and $G_{21}$; this is done to allow the closed loop system to have 0 unobservable and/or uncontrollable modes, as previously discussed.

## E.2 Equivalence of Special Problems

Most of the complications which arise from relaxing the 0 invariant zero assumption can be eliminated by showing the equivalence between two full information problems:

(FI) Given $G(s) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ \begin{bmatrix} I \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ I \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{array}\right]$, find an admissible $K(s)$ such that $\|G \star K\|_\infty < 1$.

($\overline{\text{FI}}$) Given $\overline{G}(s) = \left[\begin{array}{c|cc} A_{11}^{FI} & B_{11}^{FI} & B_{21}^{FI} \\ \hline C_{11}^{FI} & D_{11} & D_{12} \\ \begin{bmatrix} I \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ I \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{array}\right]$, find an internally stabilizing $\overline{K}(s)$ such that $\|\overline{G} \star \overline{K}\|_\infty < 1$.

Associated with full information problems FI and $\overline{\text{FI}}$ are Hamiltonians $H_\infty$ and $\overline{H}_\infty$, respectively; their definitions may be found in [11], equation (3.1). If $\overline{H}_\infty \in dom(Ric)$, we will denote $\overline{X}_\infty := Ric(\overline{H}_\infty)$, and $X_\infty := S^{-\mathrm{T}} \begin{bmatrix} \overline{X}_\infty & 0 \\ 0 & 0 \end{bmatrix} S^{-1}$. Note that $X_\infty$ is not defined in terms of $Ric(H_\infty)$; when $H_\infty \in dom(Ric)$, however, it can be shown that $X_\infty = Ric(H_\infty)$.

The following Lemma outlines the equivalence of the two problems:

**Lemma 6** FI *has a solution if and only if* $\overline{\text{FI}}$ *has a solution. Furthermore, all admissible* $K$ *such that* $\|G \star K\| < 1$ *are given by the formulas in [11], Theorem 3.1 (c).*

**Proof:** Assume, without loss of generality, that the state space data for FI is in the same co-ordinates as $\overline{\text{FI}}$, ie., $S = I$. We will thus drop the superscript $FI$ to simplify the notation. Let $K = \left[\begin{array}{c|c} \hat{A} & \hat{B} \\ \hline \hat{C} & \hat{D} \end{array}\right]$ be a candidate controller for FI, resulting in the following equations:

$$\begin{aligned} \dot{x}_1 &= A_{11}x_1 + B_{11}d + B_{21}(D_{12}^{\mathrm{T}}C_{12}x_2 + u) \\ e &= C_{11}x_1 + D_{11}^d + D_{12}(D_{12}^{\mathrm{T}}C_{12}x_2 + u) \\ \dot{x}_2 &= A_{21}x_1 + A_{22}x_2 + B_{12}d + B_{22}u \\ \dot{\hat{x}} &= \hat{A}\hat{x} + \hat{B}_1x_1 + \hat{B}_2x_2 + \hat{B}_3d \\ u + D_{12}^{\mathrm{T}}C_{12}x_2 &= \hat{C}\hat{x} + \hat{D}_1x_1 + \hat{D}_2x_2 + \hat{D}_3d + D_{12}^{\mathrm{T}}C_{12} \end{aligned} \tag{58}$$

Assume $K$ solves FI. Then defining $\overline{K}$ by the last three equations of (58) results in $\|\overline{G} \star \overline{K}\|_\infty < 1$. Note that it isn't clear whether $\overline{K}$ is an internally stabilizing controller, since there could be closed loop modes at 0. Since $\begin{bmatrix} A_{11} & B_{21} \\ C_{11} & D_{12} \end{bmatrix}$ is full column rank by construction, however, $(u + D_{12}^{\mathrm{T}}C_{12}x_2) \in \mathcal{L}_{2+}$ $\forall d \in \mathcal{L}_{2+}$. Thus a minimal realization for $\overline{K}$ will internally stabilize $\overline{G}$.

Conversely, if $\overline{K} = \begin{bmatrix} K_1 & K_d \end{bmatrix}$ solves $\overline{\text{FI}}$, it follows that

$$K := \begin{bmatrix} K_1 & -D_{12}^{\mathrm{T}}C_{12} & K_d \end{bmatrix} \tag{59}$$

is admissible since $(A_{22} - B_{22}D_{12}^{\mathrm{T}}C_{12})$ is nilpotent, and the $x_2$ states are unobservable. Furthermore, $\|G \star K\|_\infty < 1$.

By equations (58) and the above arguments, it is clear that any $K$ which solves FI can be decomposed into the form of (59), where $\overline{K} = \begin{bmatrix} K_1 & K_d \end{bmatrix}$ solves $\overline{\text{FI}}$. It thus follows that if one could generate all $\overline{K}$ which solve $\overline{\text{FI}}$, all $K$ which solve FI could be generated as well. It can readily be verified that the equations in [11] Theorem 3.1 (c) give this parametrization.

■

Problems FC and $\overline{\text{FC}}$ can be defined analogously, along with $J_\infty$, $\overline{J}_\infty$, and $\overline{Y}_\infty$. It can be shown that if $\overline{J}_\infty \in dom(Ric)$ and $\overline{Y}_\infty = Ric(\overline{J}_\infty)$, the corresponding definition for $Y_\infty$ is $Y_\infty := T \begin{bmatrix} \overline{Y}_\infty & 0 \\ 0 & 0 \end{bmatrix} T^{\mathrm{T}}$. The equivalence of the FC and $\overline{\text{FC}}$ problems follow by duality.

## E.3 Output Feedback

The formulas for all admissible controllers and the conditions for their existence are virtually identical to those of [11], Theorem 4.1. In the interest of brevity, we will present the formulas and conditions for $D_{11} = 0$, although the proofs presented hold for $D_{11} \neq 0$. The formulas for the $D_{11} \neq 0$ case may be generalized as in [11], Theorem 4.1.

**Theorem 3** *Suppose $G$ satisfies assumptions (A1) through (A4).*

1. *There exists an admissible controller $K$ such that $\|G \star K\|_\infty < 1$ if and only if*

   (a) $\overline{H}_\infty \in dom(Ric)$, *with* $\overline{X}_\infty = Ric(\overline{H}_\infty) \geq 0$

   (b) $\overline{J}_\infty \in dom(Ric)$, *with* $\overline{Y}_\infty = Ric(\overline{J}_\infty) \geq 0$

   (c) $\rho(X_\infty Y_\infty) < 1$

2. *Given that the conditions of part 1 are satisfied, then all rational admissible controllers $K$ satisfying $\|G \star K\|_\infty < 1$ are given by $K = (K_a \star \Phi)$ for arbitrary $\Phi \in \mathcal{RH}_\infty$ such that $\|\Phi\|_\infty < 1$ where*

$$\begin{aligned}
K_a &= \left[\begin{array}{c|cc} \hat{A} & \hat{B}_1 & \hat{B}_2 \\ \hline \hat{C}_1 & 0 & I \\ \hat{C}_2 & I & 0 \end{array}\right] := \left[\begin{array}{c|cc} A + BF + \hat{B}_1\hat{C}_2 & -Z_\infty^{-1}L_2 & Z_\infty^{-1}(B_2 + L_{12}) \\ \hline F_2 & 0 & I \\ -(C_2 + F_{12}) & I & 0 \end{array}\right] \qquad (60) \\
Z_\infty &:= (I - Y_\infty X_\infty) \\
F &:= \begin{bmatrix} B_1^{\mathrm{T}} X_\infty \\ -(D_{12}^{\mathrm{T}}C_1 + B_2^{\mathrm{T}}X_\infty) \end{bmatrix} \\
L &:= \begin{bmatrix} Y_\infty C_1^{\mathrm{T}} & -(B_1D_{21}^{\mathrm{T}} + Y_\infty C_2^{\mathrm{T}}) \end{bmatrix}
\end{aligned}$$

Note, in fact, that the only difference between Theorem 3 and Theorem 4.1 in [11] is that only hamiltonians $\overline{H}_\infty$ and $\overline{Y}_\infty$ are required to be in $dom(Ric)$, not $H_\infty$ and $Y_\infty$. Also note that the coupling condition is on $X_\infty$ and $Y_\infty$, not on $\overline{X}_\infty$ and $\overline{Y}_\infty$. $X_\infty$ and $Y_\infty$ can be constructed from $\overline{X}_\infty$ and $\overline{Y}_\infty$, as outlined earlier.

**Proof:** The main idea in [11] is to convert the output feedback problem to an output estimation problem, given that the full information problem has a solution. The solution to the output estimation problem, in turn, can be derived from the full control problem. This approach can also be used to prove Theorem 3; the only technical difficulty is allowing modes at 0, which must be uncontrollable and/or unobservable throughout the development. Thus one needs to ensure that the arguments used throughout the proofs in [11] carry over when internal stability is relaxed to allowing modes at 0. This has already been done for the full information problem in Lemma 6, and the full control problem by duality.

It can be verified that the results on the disturbance feedforward and output estimation problems carry through once the results for the full information and full control problems are established; the main observation is that Lemma 3.4 in [11] is still valid, since the 0 modes of $A_F$ are unobservable ([11], equation (3.22)). The conversion from an output feedback problem to an output estimation problem follows immediately as well. The last step is then to solve the resulting output estimation problem, and establish the coupling condition and formulas.

The generalized plant for the derived output estimation problem is the following:

$$G_{tmp} := \left[\begin{array}{c|cc} A+B_1F_1 & B_1 & B_2 \\ \hline -D_{12}F_2 & D_{11} & D_{12} \\ C_2+D_{21}F_1 & D_{21} & 0 \end{array}\right] \tag{61}$$

As shown in [11], it is required to solve the corresponding full control problem. Assume, without loss of generality, that we are in the $\overline{\mathrm{FC}}$ co-ordinate system. $G_{tmp}$ then has the following form:

$$G_{tmp} := \left[\begin{array}{cc|cc} A_{11}+B_{11}F_{11} & A_{12}+B_{11}F_{12} & B_{11} & B_{21} \\ A_{21}+B_{12}F_{11} & A_{22}+B_{12}F_{12} & B_{12} & B_{22} \\ \hline -D_{12}F_{21} & -D_{12}F_{22} & D_{11} & D_{12} \\ C_{21}+D_{21}F_{11} & C_{22}+D_{21}F_{12} & D_{21} & 0 \end{array}\right] \tag{62}$$

It follows that $(G_{tmp})_{21}$ inherits the same invariant zeros of $G_{21}$. The corresponding full control problem to be solved (by Lemma 6 and duality) is therefore

$$\overline{G}_{tmp}^{FC} := \left[\begin{array}{c|c|c} A_{11}+B_{11}F_{11} & B_{11} & \begin{bmatrix} I & 0 \end{bmatrix} \\ \hline -D_{12}F_{21} & D_{11} & \begin{bmatrix} 0 & I \end{bmatrix} \\ C_{21}+D_{21}F_{11} & D_{21} & \begin{bmatrix} 0 & 0 \end{bmatrix} \end{array}\right] \tag{63}$$

with corresponding hamiltonian $\overline{J}_{tmp}$. We thus need to establish that $\overline{J}_{tmp} \in dom(Ric)$ and that $\overline{Y}_{tmp} = Ric(\overline{J}_{tmp}) \geq 0$. In [11], the following condition is derived:

$$J_{tmp} \begin{bmatrix} I - X_\infty Y_\infty \\ Y_\infty \end{bmatrix} = \begin{bmatrix} I - X_\infty Y_\infty \\ Y_\infty \end{bmatrix} (A+LC)^{\mathrm{T}} \tag{64}$$

where $J_{tmp}$ is the hamiltonian associated with the full control problem of (62). Because of the co-ordinate system chosen, this condition implies that

$$\overline{J}_{tmp}\begin{bmatrix} I-(\overline{X}_{\infty})_{11}\overline{Y}_{\infty} \\ \overline{Y}_{\infty} \end{bmatrix} = \begin{bmatrix} I-(\overline{X}_{\infty})_{11}\overline{Y}_{\infty} \\ \overline{Y}_{\infty} \end{bmatrix}\Lambda^{\mathrm{T}} \tag{65}$$

where $\Lambda$ is associated with the $\overline{\mathrm{FC}}$ problem, and is Hurwitz. It then follows that $\overline{Y}_{tmp} := \overline{Y}_{\infty}\left(I-(\overline{X}_{\infty})_{11}\overline{Y}_{\infty}\right)^{-1} \geq 0 \iff \rho\left((\overline{X}_{\infty})_{11}\overline{Y}_{\infty}\right) < 1 \iff \rho(X_{\infty}Y_{\infty}) < 1$. Furthermore, defining $Y_{tmp} := \begin{bmatrix} \overline{Y}_{tmp} & 0 \\ 0 & 0 \end{bmatrix}$, yields the required formulas, as per [11].

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