Robust $\mathcal{H}_2$ Performance: Guaranteeing Margins for LQG Regulators

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Abstract

This paper shows that $\mathcal{H}_2$ (LQG) performance specifications can be combined with structured uncertainty in the system, yielding robustness analysis conditions of the same nature and computational complexity as the corresponding conditions for $\mathcal{H}_\infty$ performance. These conditions are convex feasibility tests in terms of Linear Matrix Inequalities, and can be proven to be necessary and sufficient under the same conditions as in the $\mathcal{H}_\infty$ case.

With these results, the tools of robust control can be viewed as coming full circle to treat the problem where it all began: guaranteeing margins for LQG regulators.

1 Introduction

The advent of modern control in the 60s brought a substantial transformation in control theory, with state-space tools and optimal control offering the promise of tractable, systematic methods for multivariable control design. This era was epitomized by the solution of the LQG control problem (see, for example, [1]) , which provides an elegant, easily computable method for a well-motivated multivariable control design problem: optimizing the rejection of white noise disturbances for a closed loop system. It became increasingly clear in the late 70s that modern control unfortunately provided limited tools to further treat model uncertainty, a fundamental requirement for a practical feedback theory and an issue which was often better addressed by the otherwise more primitive frequency domain techniques of classical control [20].

While LQ state feedback was shown to provide stability margin guarantees [37], further research led to a counterexample showing that full LQG controllers had none [10]. This motivated efforts to reconcile LQG with classical methods [11], with some initial success in providing a robust LQG-based methodology [9]. The most popular development was LQG/LTR [12, 2, 40], a multivariable version of classical loopshaping using LQG machinery. The problem of adding plant uncertainty directly to LQG remained unsolved, however, and ultimately these efforts pointed in other directions [12], particularly toward (structured) singular values and related methods [14, 41].

At about the same time as the critique of LQG robustness was becoming widely accepted, the new performance paradigm of $\mathcal{H}_\infty$ was being put forth [44]. It had close ties to the frequency domain and allowed singular value robustness conditions to be treated directly. More importantly, it allowed for the first time a very natural and relatively transparent analysis of robust performance [13, 26]. While $\mathcal{H}_\infty$ soon replaced LQG (now referred to as $\mathcal{H}_2$) as the centerpiece of modern control, and research on $\mathcal{H}_\infty$ flourished in the 1980s, several developments helped bring $\mathcal{H}_2$ back into the picture.

The main weakness of $\mathcal{H}_\infty$ is that modeling signals as weighted norm balls ignores important structure, typically expressed in terms of spectral or correlation properties, which are often features of more realistic models of physical disturbances. Ignoring this structure makes a worst-case measure like $\mathcal{H}_\infty$ substantially conservative, in much the same way as what happens when uncertainty structure is ignored in singular value robustness conditions; the recognition of these limitations led to a resurgence of interest in $\mathcal{H}_2$ as a performance measure. The desirable design specification, from both the performance and uncertainty points of view, appears to be in most cases Robust $\mathcal{H}_2$ performance: rejection of white signals in the worst-case over a set of plants. (The $\mathcal{L}_1$ theory [7] is another attractive alternative to $\mathcal{H}_\infty$ but still suffers from a pessimistic signal description).

Renewed interest in $\mathcal{H}_2$ performance was also stimulated by the striking fact that the most powerful computational solutions for the $\mathcal{H}_\infty$ control problem [18, 15] relied on the same state-space tools as LQG. This led to a new research direction in mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control (see e.g. [5, 22, 45, 16, 34]), and to various upper bounds for the $\mathcal{H}_2$ cost over a set of plants (e.g., [42, 33, 45, 19, 17]).

In spite of these developments, the robust $\mathcal{H}_2$ problem lagged substantially behind $\mathcal{H}_\infty$ (or $\mathcal{L}_1$), where a sophisticated set of tools is available for the analysis of robust performance under structured uncertainty (see e.g. [26, 43, 4, 7]), including several results that exactly analyze robust performance with structured uncertainty in terms of computationally attractive convex conditions ([21, 38, 25, 35]). No such results have previously been available for robust $\mathcal{H}_2$ performance.

This paper provides the final step in the return of the $\mathcal{H}_2$ performance paradigm, casting it on an equal footing with $\mathcal{H}_\infty$. We present a convex condition for robust $\mathcal{H}_2$ performance analysis under structured uncertainty, of a very similar nature to the the corresponding condition for robust $\mathcal{H}_\infty$ performance, and with analogous properties. Computationally, it reduces to a Linear Matrix Inequality (LMI, see [6]) over frequency which can be handled with analogous tools as in the $\mathcal{H}_\infty$ case. From a theoretical point of view, the condition is shown to be necessary and sufficient under the same assumptions for the uncertainty as in the corresponding $\mathcal{H}_\infty$ conditions. The tools involved in proving the necessity results build on recent work in the Integral Quadratic Constraint (IQC) formulation which has been mainly applied to describe uncertainty [25], but can also be used [24, 27] for signal characterization. In this paper we extend these methods to set characterizations of white signals based on statistical tests on the cumulative spectrum [28], and rely on infinite dimensional convex analysis methods to derive the necessary conditions.

The paper is organized as follows. In Section 2 some material on standard robust control with $\mathcal{H}_\infty$ performance measure is reviewed. Section 3 discusses $\mathcal{H}_2$ norms and set characterizations of white noise signals. In Section 4, the condition for robust $\mathcal{H}_2$ performance is presented and proven to be necessary and sufficient under various uncertainty assumptions. In Section 5 we remark on the computational properties of this test. Section 6 compares these results to the previous work in the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ literature. Some remarks on robust $\mathcal{H}_2$ synthesis are given in Section 7, and the conclusions are presented in Section 8. Some proofs are covered in the Appendix. A partial version of these results was presented in [31].

2 Background and Notation

The results in this paper will be presented for discrete time systems. Analogous conditions hold for the continuous time case, the details of which will be reported elsewhere.

2.1 Uncertain Systems in LFT Form

A standard setup for robustness analysis is depicted in Figure 1, consisting of a nominal map M and a perturbation Δ which enters the system in feedback fashion; the overall uncertain system will be denoted by (M, Δ).

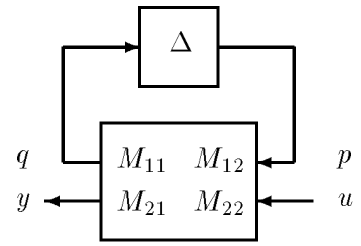

Figure 1: Uncertain system

M will be assumed to be a finite dimensional, linear time invariant (LTI), stable system. Its transfer function is denoted by $M(e^{j\omega})$. The uncertainty Δ is assumed to have spatial structure of block diagonal form

$$\Delta = \mathrm{diag}\,[\delta_1 I_{r_1}, \ldots, \delta_L I_{r_L}, \Delta_{L+1}, \ldots, \Delta_{L+F}] \tag{1}$$

The blocks in Δ can in general represent real parameters or dynamic perturbations; in each case, there is a restricted class $\mathbf{\Delta}$ of allowed perturbations, which are usually assumed normalized to the ball of uncertainty $\mathbf{B}_{\mathbf{\Delta}} = \{\Delta \in \mathbf{\Delta} : \|\Delta\| \leq 1\}$ in some operator norm.

We will consider l_2 signal norms: l_2^n denotes the space of square-summable, $\mathbb{C}^n$-valued sequences over the integers (or positive integers). These are identified via the Fourier transform with square integrable functions on the unit circle $\mathbb{T}$, with respect to the normalized Lebesgue measure $\frac{d\omega}{2\pi}$. The vector dimension will be omitted when clear from context.

$\mathcal{L}_c(l_2)$ denotes the set of causal, linear, bounded operators in l_2. The largest class of uncertainty considered here is the set of structured linear time-varying (LTV) perturbations

$$\mathbf{\Delta}^{\mathbf{LTV}} = \{\Delta \in \mathcal{L}_c(l_2) : \Delta = \mathrm{diag}\,[\delta_1 I_{r_1}, \ldots, \delta_L I_{r_L}, \Delta_{L+1}, \ldots, \Delta_{L+F}]\} \tag{2}$$

The uncertainty can also be restricted to be linear time-invariant, which means it commutes with the unit delay operator λ. This gives the structured set $\mathbf{\Delta}^{\mathbf{LTI}} = \{\Delta \in \mathbf{\Delta}^{\mathbf{LTV}} : \lambda\Delta = \Delta\lambda\}$. Some recent work [35] has shown it is useful to introduce the mildly larger class of slowly varying operators, by defining for $\nu > 0$ the class

$$\mathbf{\Delta}^{\nu} = \{\Delta \in \mathbf{\Delta}^{\mathbf{LTV}} : \|\lambda\Delta - \Delta\lambda\| \leq \nu\} \tag{3}$$

of operators with "variation slower than ν". For $\nu = 0$ we recover $\mathbf{\Delta}^{\mathbf{LTI}}$, but some of the necessary conditions will be proven for an arbitrarily small $\nu > 0$. The unit balls of uncertainty for each class are denoted, correspondingly, $\mathbf{B}_{\mathbf{\Delta}^{\mathbf{LTV}}}$, $\mathbf{B}_{\mathbf{\Delta}^{\mathbf{LTI}}}$, and $\mathbf{B}_{\mathbf{\Delta}^{\nu}}$.

The system of Figure 1 is said to be robustly stable if M is stable, and if $I - \Delta M_{11}$ has an inverse in $\mathcal{L}_c(l_2)$ for every $\Delta \in \mathbf{B}_{\mathbf{\Delta}}$. When this holds, the closed loop map from u to y is well defined for all $\Delta \in \mathbf{B}_{\mathbf{\Delta}}$ and given by the Linear Fractional Transformation (LFT)

$$\Delta \star M := M_{22} + M_{21}\Delta(I - M_{11}\Delta)^{-1}M_{12} \tag{4}$$

A performance specification can then be imposed on the map $\Delta \star M$. In our case of l_2 signal norms, the standard choice is the l_2-induced norm (which we call the $\mathcal{H}_\infty$ norm, although this is an abuse of notation for non-LTI systems). The system is said to have robust $\mathcal{H}_\infty$ performance if it is robustly stable, and

$$\sup_{\Delta \in \mathbf{B}_{\mathbf{\Delta}}} \|\Delta \star M\|_{l_2 \to l_2} < 1 \tag{5}$$

2.2 Robust $\mathcal{H}_\infty$ Performance Tests

The main method for obtaining tractable robust $\mathcal{H}_\infty$ performance tests is to add scalings to a small gain condition. For this purpose we introduce scaling matrices of the form

$$X = \mathrm{diag}\,[X_1, \ldots, X_L,\, x_{L+1} I_{m_1}, \ldots, x_{L+F} I_{m_F}] \tag{6}$$

which commute with the elements in $\boldsymbol{\Delta}$. We will denote by $\boldsymbol{X}$ the set of positive definite, continuous scaling functions $X(\omega)$ with the structure (6). The tests for robust $\mathcal{H}_\infty$ performance can be summarized as follows:

Condition 1 *There exists a function $X(\omega) \in \boldsymbol{X}$ such that for all $\omega \in [0, 2\pi]$,*

$$M(e^{j\omega})^* \begin{bmatrix} X(\omega) & 0 \\ 0 & I \end{bmatrix} M(e^{j\omega}) - \begin{bmatrix} X(\omega) & 0 \\ 0 & I \end{bmatrix} < 0 \tag{7}$$

Since M is finite dimensional, $M(e^{j\omega})$ is continuous and the continuity assumption of $X(\omega)$ entails no loss of generality. For this case of frequency dependent scales, we can state the following:

Proposition 1 *If Condition 1 holds for a function $X(\omega) \in \boldsymbol{X}$, then the system (M, Δ) has robust $\mathcal{H}_\infty$ performance for $\Delta \in \mathbf{B}_{\boldsymbol{\Delta}_{\mathbf{LTI}}}$.*

The previous result follows by showing that this condition provides a bound for the structured singular value μ [14, 26], which is the exact robustness test for LTI uncertainty. Although Condition 1 is in general conservative for this case, it remains as an attractive condition since exact computation of μ is not tractable. Also, computational experience with bounds such as those used in [4] shows evidence that the two tests are not far apart, at least for full block structures.

Another argument to support the claim that Condition 1 has small conservatism is given by the following result from Poolla and Tikku [35]:

Proposition 2 *There exists $\nu > 0$ such that the system (M, Δ) has robust $\mathcal{H}_\infty$ performance for $\Delta \in \mathbf{B}_{\mathbf{\Delta}^\nu}$ if and only if there exists a function $X(\omega) \in \boldsymbol{X}$ satisfying Condition 1.*

Consequently, if one is willing to include an arbitrarily small amount of time variation, in the sense of (3), in the uncertainty, Condition 1 characterizes exactly the robust performance problem.

Finally, if we allow an unrestricted time variation, robustness analysis is obtained from Condition 1 by imposing $X(\omega)$ to be a constant function. The following result was shown independently by Shamma [38] and Megretski [25]:

Proposition 3 *The system (M, Δ) has robust $\mathcal{H}_\infty$ performance with $\Delta \in \mathbf{B}_{\mathbf{\Delta}^{\mathbf{LTV}}}$ if and only if there exists a constant matrix $X(\omega) \equiv X \in \boldsymbol{X}$ satisfying Condition 1.*

The tests provided by Condition 1 amount to an infinite dimensional convex feasibility problem, in terms of a parametrized (by frequency) family of Linear Matrix Inequalities (LMIs). Conditions of this type allow for tractable computation, as will be discussed in Section 5.

2.3 Mathematical Preliminaries

The following mathematical facts are collected here for ease of reference (see, e.g., [23] and [39]).

First, we introduce the space $BV[a, b]$ of real-valued functions of bounded variation in the interval $[a, b] \subset \mathbb{R}$. A function $\Psi(t)$ is of bounded variation if

$$TV(\Psi) := \sup \sum_{i=1}^{N} |\Psi(t_i) - \Psi(t_{i-1})| < \infty \tag{8}$$

where the supremum is taken over partitions of $[a, b]$. $TV(\Psi)$ is called the total variation of Ψ.

We will also use the space $C_R[a, b]$ of continuous, real-valued functions on $[a, b]$, with the norm $||g||_\infty := \sup_{t \in [a,b]} |g(t)|$.

Given $\Psi \in BV[a, b]$ and $g \in C_R[a, b]$, we introduce the Stieltjes integral (see [39])

$$\int_a^b g(t) d\Psi(t) \tag{9}$$

Since

$$\left|\int_a^b g(t)d\Psi(t)\right| \leq \|g\|_\infty TV(\Psi) \tag{10}$$

the map $\Gamma_\Psi : C_R[a,b] \rightarrow \mathbb{R}$ given by $\Gamma_\Psi(g) = \int_a^b g(t)d\Psi(t)$ defines a bounded linear functional on $\mathbb{C}_R[a,b]$. In fact, the Riesz representation theorem states that every functional in the dual space $C_R[a,b]^*$ is of this form.

We will also use the formula of integration by parts for the Stieltjes integral,

$$\int_a^b g(t)d\Psi(t) = g(b)\Psi(b) - g(a)\Psi(a) - \int_a^b \Psi(t)dg(t) \tag{11}$$

which holds, for example, for $\Psi \in BV[a,b]$, $g \in C_R[a,b]$. Furthermore, if g has an integrable derivative $g'(t)$, the integral on the right can be written as $\int_a^b \Psi(t)g'(t)dt$.

Finally, a key element in the proofs of this paper is the following geometric version of the Hahn-Banach theorem, taken from [23]:

Theorem 4 *Let $\mathcal{K}_1$, $\mathcal{K}_2$ be convex sets in a real normed space V, such that $\mathcal{K}_2$ has non-empty interior, and $\mathcal{K}_1$ contains no interior points of $\mathcal{K}_2$. Then there exists a bounded functional $\Gamma \in V^*$, $\Gamma \neq 0$, and a real number α such that*

$$\Gamma(k_1) \leq \alpha \leq \Gamma(k_2), \quad \textit{for all } k_1 \in \mathcal{K}_1, k_2 \in \mathcal{K}_2 \tag{12}$$

3 White Signals and $\mathcal{H}_2$ Norms

As argued in the introduction, $\mathcal{H}_\infty$ performance takes a conservative view of disturbances; in many situations a more useful performance measure is given by the $\mathcal{H}_2$ norm of a system, which characterizes the response to white signals. For an LTI system $H(e^{j\omega})$, this norm is defined by

$$\|H\|_2 := \left(\int_0^{2\pi} \mathrm{trace}(H(e^{j\omega})^* H(e^{j\omega}))\frac{d\omega}{2\pi}\right)^{\frac{1}{2}} \tag{13}$$

White signals typically arise in two situations. One is as chaotic, high dimensional fluctuations known as white noise, which are usually modeled as a stationary, uncorrelated random process of unit covariance matrix; if such a signal is input to an LTI system, the variance of the output is given by $\|H\|_2^2$. Another source of white signals are impulsive disturbances, or impulsive signals used to test the response of a system to fixed reference signals; the output energy for an impulsive (scalar, or vector-valued with random direction) input is $\|H\|_2^2$. For more motivation see [46].

The objective of this paper is to analyze the effect of white signals, in the worst-case for the uncertain system given in Figure 1. For the case of LTI uncertainty, the closed loop $\Delta \star M$ is LTI and we will simply analyze for the worst-case $\mathcal{H}_2$ norm as defined in (13).

We will also want to consider, however, the classes $\mathbf{\Delta^{LTV}}$ and $\mathbf{\Delta}^{\nu}$ involving time-varying uncertainty, which come in naturally to characterize the necessity of the robust performance conditions, as was shown in Section 2 for the $\mathcal{H}_\infty$ case. For this purpose, it will be convenient to describe white signals in terms of a set, rather than a random process, which will allow the natural formulation of worst-case analysis problems over the uncertainty and over this set.

A non-stochastic treatment of $\mathcal{H}_2$-performance was in fact given in Zhou et al. [45], where the classes of *bounded power* and *bounded spectrum* signals are employed to motivate both the $\mathcal{H}_2$ and the $\mathcal{H}_\infty$ norms. This formulation is conceptually appealing, but poses a number of mathematical difficulties. First of all, as noted in [45], the formalization of such classes would require limiting arguments which raise a number of technical issues. More importantly, these classes do not have a rich mathematical structure, which greatly restricts the applicability of functional analytic tools.

As a counterpart, the class of *bounded energy* (l_2) signals offers the rich mathematical structure of a Hilbert space. This structure plays a key role in the most powerful results on control with an $\mathcal{H}_\infty$ performance measure, and is equally satisfactory from a conceptual point of view, since bounded power and bounded energy signals differ only in their asymptotic behavior. For this reason our treatment of white signals sets will be based on l_2-space; this paper will consider the discrete time version, which is more straightforward. The same methodology can be applied to the continuous time case, as will be reported elsewhere, and could also be used to provide a more complete foundation to the material in [45].

We begin with the case of scalar signals in l_2. Ideally, a white signal has a flat spectrum, i.e. lies in the set

$$W_0 = \left\{ f \in l_2 : |f(\omega)|^2 = \|f\|_2^2, \ \omega \text{ a.e. in } [0, 2\pi] \right\} \tag{14}$$

A key technical requirement for the results of this paper is to introduce a set of signals which are approximately (up to a small accuracy) white. For this purpose, we take the standpoint (developed extensively in [28]) that such a notion should be based on standard statistical tests for stochastic white noise.

More specifically, if one is given a time series $f_0, \ldots, f_{N-1}$, deciding whether it is a sample of white noise is usually done based on the values of a chosen statistic; one such choice is the sample autocorrelation, and leads to a definition for white noise sets which was exploited in [28, 29] for robust $\mathcal{H}_2$ analysis. This paper is based on a frequency domain definition for white noise, which corresponds to the so-called Bartlett cumulative periodogram test for time series. This test consists of accumulating the periodogram (squared magnitude of the Discrete Fourier Transform of the series $f_0, \ldots, f_{N-1}$), and comparing the result uniformly with a linear function. For more details see [28]. Inspired by this, we consider here the difference

$$F_f(s) := \int_0^s |f(\omega)|^2 \frac{dw}{2\pi} - \frac{s}{2\pi}\|f\|_2^2 \tag{15}$$

between the cumulative spectrum and a linear function, and bound it in a uniform sense. Define the set of "white up to accuracy η" signals

$$W_\eta := \{ f \in l_2 : \sup_{s \in [0, 2\pi]} |F_f(s)| < \eta \} \tag{16}$$

The gain of an LTI single input system $H(e^{j\omega})$ under signals in W_0 is easily seen to be $\|H\|_2$. We now consider the worst-case gain under signals in W_η,

$$\|H\|_{W_\eta} := \sup\{ \|Hf\|_2 : f \in W_\eta, \|f\|_2 \leq 1 \} \tag{17}$$

Lemma 5 *Let $Y(\omega) \in BV[0, 2\pi]$. If $f \in W_\eta$, then*

$$\left| \int_0^{2\pi} Y(\omega)|f(\omega)|^2 \frac{d\omega}{2\pi} - \|f\|_2^2 \int_0^{2\pi} Y(\omega) \frac{d\omega}{2\pi} \right| \leq \eta \, TV(Y) \tag{18}$$

Proof: Defining $F_f(s)$ as in (15) ($F_f(0) = F_f(2\pi) = 0$), an integration by parts yields

$$\int_0^{2\pi} Y(\omega)(|f(\omega)|^2 - \|f\|_2^2) \frac{d\omega}{2\pi} = - \int_0^{2\pi} F_f(\omega) dY(\omega) \tag{19}$$

Since $f \in W_\eta$, then $\|F_f(\omega)\|_\infty = \sup_\omega |F_f(\omega)| < \eta$, so (10) implies that the right hand side of (19) can be bounded by $\eta \, TV(Y)$.

□

A consequence of this Lemma (picking $Y(\omega) = |H(e^{j\omega})|^2$) is that for an LTI system H,

$$\|H\|_2^2 \leq \|H\|_{W_\eta}^2 \leq \|H\|_2^2 + \eta \, TV(|H|^2) \tag{20}$$

which implies that under the mild assumption that $|H(e^{j\omega})|^2 \in BV[0, 2\pi]$,

$$\|H\|_{W_\eta} \stackrel{\eta \to 0+}{\longrightarrow} \|H\|_2 \tag{21}$$

We now analyze the multivariable case; vector-valued white noise is characterized by having a matrix spectrum equal to a constant times the identity matrix across frequency. In the l_2 setting, ideally white signals do not appear since the spectrum $f(\omega)f(\omega)^*$ is always a rank one matrix, but for $\eta > 0$ we can define the set of approximately white signals in l_2^m

$$W_\eta^m = \left\{ f : \left\| \int_0^s f(\omega) f(\omega)^* \frac{dw}{2\pi} - \frac{s}{2\pi} \frac{\|f\|^2}{m} I_m \right\|_\infty < \eta \right\} \tag{22}$$

where the infinity norm of a continuous matrix function is taken to be the maximum across the coordinates of the supremum norm. It is easy to show that for every $\eta > 0$, W_η^m is non-trivial.

With this definition, an extension of Lemma 5 can be written, which leads for LTI systems to

$$\|H\|_{W_\eta^m} := \sup\{\|Hf\| : f \in W_\eta^m, \frac{1}{m}\|f\|_2^2 \leq 1\} \stackrel{\eta \to 0+}{\longrightarrow} \|H\|_2 \tag{23}$$

The normalization by a factor $\frac{1}{\sqrt{m}}$ in the input norm is done for convenience, since if $f(\omega)f(\omega)^*$ approximates the ideal unit white spectrum I_m, $\|f\|_2$ is approximately $\sqrt{m}$.

If one is interested in analyzing white noise rejection for systems which are not LTI, the $\mathcal{H}_2$ norm (13) is no longer meaningful, but it is natural give a definition based on $\|H\|_{W_\eta^m}$ as in (23):

$$\|H\|_2 := \lim_{\eta \to 0+} \|H\|_{W_\eta^m} \tag{24}$$

This system measure (a seminorm) captures the response to signals of flat spectrum, the interesting object from the point of view of applications, and extends the LTI definition.

4 Main Results

In this section we provide conditions for Robust $\mathcal{H}_2$ performance for the uncertain system of Figure 1, which are analogous to those presented in Section 2.2 for Robust $\mathcal{H}_\infty$ performance. We now state an analysis test, which is a convex feasibility condition on the unknowns X, Y

Condition 2 *There exists $X(\omega) \in \boldsymbol{X}$, and a matrix function $Y(\omega) = Y^*(\omega) \in \mathbb{C}^{m \times m}$, such that*

$$M(e^{j\omega})^* \begin{bmatrix} X(\omega) & 0 \\ 0 & I \end{bmatrix} M(e^{j\omega}) - \begin{bmatrix} X(\omega) & 0 \\ 0 & Y(\omega) \end{bmatrix} < 0 \tag{25}$$

holds for all $\omega \in [0, 2\pi]$, and

$$\int_0^{2\pi} \mathrm{trace}(Y(\omega)) \frac{d\omega}{2\pi} < 1 \tag{26}$$

This condition is in fact very similar to Condition 1 for Robust $\mathcal{H}_\infty$ performance. The only addition is the incorporation of the function $Y(\omega)$, which can be assumed continuous. Heuristically, for $m = 1$, $Y(\omega)$ allows for the gain to be larger than 1 at some frequencies, provided that it is compensated at other frequencies by keeping the total effect $\int Y(\omega) d\omega$ less than 1; this imposes an "average over frequency" performance which corresponds to the $\mathcal{H}_2$ norm.

The main result of this paper is that this test answers the robust $\mathcal{H}_2$ performance problem. As in the case of Condition 1, different results can be stated in accordance with the nature of the perturbations in Δ, which exactly parallel Propositions 1, 2 and 3.

For the first one involving LTI perturbations, the LFT $\Delta \star M$ is an LTI system so the result can be stated using the standard definition (13) of the $\mathcal{H}_2$ norm, and proved with elementary frequency domain tools.

Theorem 6 *Suppose Condition 2 holds for matrix functions $X(\omega)$,$Y(\omega)$. If $\Delta \in \mathbf{B}_{\mathbf{\Delta}\mathrm{LTI}}$, then the system is robustly stable and*

$$\sup_{\Delta \in \mathbf{B}_{\mathbf{\Delta}\mathrm{LTI}}} \|\Delta \star M\|_2 < 1 \tag{27}$$

Proof: The first block of the inequality (25) gives $\|X(\omega)^{\frac{1}{2}} M_{11}(e^{j\omega}) X(\omega)^{-\frac{1}{2}}\|_\infty < 1$, which implies (see [26]) robust stability of the system under LTI perturbations. Furthermore, defining

$$\hat{M} = \begin{bmatrix} X^{\frac{1}{2}} & 0 \\ 0 & I \end{bmatrix} M \begin{bmatrix} X^{-\frac{1}{2}} & 0 \\ 0 & I \end{bmatrix} \tag{28}$$

we conclude that for some $\epsilon > 0$, and all ω,

$$(\hat{M}^* \hat{M})(\omega) - \begin{bmatrix} I & 0 \\ 0 & Y(\omega) \end{bmatrix} < -\epsilon I \tag{29}$$

Fix $\Delta \in \mathbf{B}_{\mathbf{\Delta}}$, LTI. For any fixed frequency, since $\Delta(e^{j\omega})$, $X^{\frac{1}{2}}(\omega)$ commute, we can replace M by $\hat{M}$ in Figure 1, giving $\Delta(e^{j\omega}) \star M(e^{j\omega}) = \Delta(e^{j\omega}) \star \hat{M}(e^{j\omega})$. Using (29), we have

$$|y(\omega)|^2 + |q(\omega)|^2 \leq (1-\epsilon)|p(\omega)|^2 + u(\omega)^* Y(\omega) u(\omega) - \epsilon |u(\omega)|^2 \tag{30}$$

where we use the signal denominations of Figure 1.

Since Δ is LTI, contractive we have $|p(\omega)|^2 \leq |q(\omega)|^2$, which leads to

$$u(\omega)^* (\Delta \star M)(\omega)^* (\Delta \star M)(\omega) u(\omega) = |y(\omega)|^2 \leq u(\omega)^* Y(\omega) u(\omega) - \epsilon |u(\omega)|^2 \tag{31}$$

Since this holds for any $u(\omega)$, we have

$$(\Delta \star M)(\omega)^*(\Delta \star M)(\omega) \leq Y(\omega) - \epsilon I \tag{32}$$

across frequency. Computing the trace and integrating gives, using (26),

$$\|\Delta \star M\|_2^2 \leq \int_0^{2\pi} \mathrm{trace}(Y(\omega))\frac{d\omega}{2\pi} - m\epsilon < 1 \tag{33}$$

□

We have obtained a convex sufficient condition for Robust $\mathcal{H}_2$ performance under LTI uncertainty; we now show that it has the same necessity properties as Condition 1 for the $\mathcal{H}_\infty$ case. To state the following results for which include non-LTI systems, we adopt the approach of (24) and give the following definition:

Definition 1 *The uncertain system (M, Δ) with input $u \in l_2^m$ has robust $\mathcal{H}_2$ performance if it is robustly stable, and there exists $\eta > 0$ such that*

$$\sup_{\Delta \in \mathbf{B}_{\mathbf{\Delta}}} \|\Delta \star M\|_{W_\eta^m} < 1 \tag{34}$$

The first result concerns the case of frequency-dependent X-scales. For brevity the proof is not included here, although we will later remark on the modifications to the proof of Theorem 8 needed to obtain it. A full account is given in [32].

Theorem 7 *There exists $\nu > 0$ such that the system (M, Δ) has robust $\mathcal{H}_2$ performance for $\Delta \in \mathbf{B}_{\mathbf{\Delta}^\nu}$ if and only if there exist bounded variation functions $X(\omega) \in \boldsymbol{X}$, $Y(\omega)$ satisfying Condition 2.*

Theorem 7 gives indication that there is mild conservatism involved in using Condition 2 for LTI uncertainty, in a totally analogous way to Proposition 2 for $\mathcal{H}_\infty$ performance.

In the $\mathcal{H}_\infty$ case there was also supporting empirical evidence with computation of lower bounds for the LTI case based on μ [26], which is not available for $\mathcal{H}_2$. In fact, the restriction on *causality* of the LTI perturbations will provide an additional gap for $\mathcal{H}_2$ performance. To see this, consider the case of unstructured uncertainty, where the $\mathcal{H}_\infty$ conditions are known to be exact [26]. In the $\mathcal{H}_2$ case, with scalar inputs, it is easy to show that Condition 2 is exact for non-causal LTI perturbations, by simply choosing $\Delta(e^{j\omega})$ to produce the worst gain at every frequency; this interpolation is in general only possible with non-causal Δ. The gap due to causality has not been quantified in general, but the results of [45] (see Section 6) suggest that it is not significant.

In any case, the only necessary conditions available for the $\mathcal{H}_\infty$ and the $\mathcal{H}_2$ frequency dependent scales tests are Proposition 2 and Theorem 7, both indicating that these gaps are a modest price to pay for a convex characterization.

We now turn to the constant X-scales condition:

Theorem 8 *The system (M, Δ) has robust $\mathcal{H}_2$ performance for $\Delta \in \mathbf{B}_{\mathbf{\Delta}\mathrm{LTV}}$, if and only if there exists a constant matrix $X \in \boldsymbol{X}$, and a bounded variation function $Y(\omega)$, satisfying Condition 2.*

Proof: For simplicity, the proof will be described in detail for the case of scalar inputs $u \in l_2^1$, and for uncertainty $\Delta = \mathrm{diag}[\Delta_1, \ldots, \Delta_n]$ consisting only of full blocks. For the general case see the remarks at the end of the section.

[Sufficiency]: The first block of (25) gives $\|X^{\frac{1}{2}} M_{11}(e^{j\omega}) X^{-\frac{1}{2}}\|_\infty < 1$, which implies [26, 38] robust stability of the system under LTV perturbations. Also, $X^{\frac{1}{2}}$ and Δ commute, so define $\hat{M}$ as in (28), which verifies (29), and leads to (30), which can be integrated across frequency to give

$$\|y\|^2 + \|q\|^2 \le (1-\epsilon)\|p\|^2 + \int_0^{2\pi} u(\omega)^* Y(\omega) u(\omega) \frac{d\omega}{2\pi} - \epsilon \|u\|^2 \tag{35}$$

Since $\|\Delta\| \le 1$, then $\|p\| \le \|q\|$, leading (for scalar u) to

$$\|(\Delta \star M) u\|^2 \le \int_0^{2\pi} Y(\omega) |u(\omega)|^2 \frac{d\omega}{2\pi} - \epsilon \|u\|^2 \tag{36}$$

Fix $\eta > 0$; for $u \in W_\eta$, we invoke Lemma 5 to bound

$$\int_0^{2\pi} Y(\omega)|u(\omega)|^2 \frac{d\omega}{2\pi} \leq \eta TV(Y) + \|u\|^2 \int_0^{2\pi} Y(\omega) \frac{d\omega}{2\pi} \tag{37}$$

Substituting (37) into (36) and using (26) leads for $\|u\| \leq 1$, to

$$\|(\Delta \star M)u\|^2 \leq \eta TV(Y) + 1 - \epsilon \tag{38}$$

By choosing small enough η, this implies

$$\sup_{\Delta \in \mathbf{B_\Delta}} \|(\Delta \star M)\|_{W_\eta} < 1$$

[Necessity]: The converse implication is based on an extended "S-procedure losslessness" theorem on quadratic functions on l_2. This type of result was first obtained by Megretski and Treil in [25] for a finite number of scalar quadratic forms. The nature of the constraints defining the white noise sets W_η requires the extension of this procedure to quadratic functions on l_2 which take values on the function space $C_R[0, 2\pi]$.

Let $z = \mathrm{col}(z_1, \ldots, z_{n+1})$ be the vector of all inputs to the M system, where $z_1 \ldots z_n$ partition p in correspondence with the blocks $\Delta_1, \ldots, \Delta_n$, and $z_{n+1} = u$. Analogously $(Mz)_i, i = 1 \ldots n+1$ denotes the partition of the output of M.

Now define the following scalar valued quadratic functions of $z \in l_2$,

$$\sigma_i(z) = \|(Mz)_i\|^2 - \|z_i\|^2, \quad i = 1 \ldots n+1 \tag{39}$$

Roughly, the motivation behind these functions is the following: if the σ_i are all non-negative at a certain $z \neq 0$, then M expands this signal in all the channels, and therefore a contractive, structured LTV operator Δ can be constructed where the $\mathcal{H}_\infty$ performance is violated. Now for this signal z to violate $\mathcal{H}_2$ performance, the u portion must be white (belong to W_η as defined in Section 3); this requirement is imposed by an additional quadratic constraint. Consider the function $\rho : l_2 \mapsto C_R[0, 2\pi]$,

$$\rho(z) = F_{z_{n+1}}(s) \tag{40}$$

where $F_{z_{n+1}}$ is defined as in (15). With this definition, $z_{n+1} \in W_\eta$ if and only if $\rho(z) \in B(0, \eta)$, the ball of radius η in $C_R[0, 2\pi]$.

By considering the real Banach space $V = \mathbb{R}^{n+1} \oplus C_R[0, 2\pi]$, we can collect all these functions together in a quadratic map $\Lambda : l_2 \mapsto V$, given by

$$\Lambda(z) = (\sigma_1(z), \ldots, \sigma_{n+1}(z), \rho(z)) \tag{41}$$

Define a set in V,

$$\nabla = \{\Lambda(z) : z \in l_2, \|z\| = 1\} \tag{42}$$

The previous discussion suggests that robust $\mathcal{H}_2$ performance will be violated if for all $\eta > 0$, ∇ intersects the set $\mathcal{K} := \{(r_1, \ldots, r_{n+1}, g) : \ r_i \geq 0, \ \|g(s)\|_\infty < \eta\}$. For technical reasons, we introduce instead the set

$$\mathcal{K}_\epsilon := \{(r_1, \ldots, r_{n+1}, g) : \ r_i > -\epsilon^2, \ \|g(s)\|_\infty < \eta\} \tag{43}$$

We formalize these ideas in the following Proposition, which reduces robust performance to geometric separation condition in the space V.

Proposition 9 *Suppose (M, Δ) has robust $\mathcal{H}_2$ performance for $\Delta \in \mathbf{B}_{\mathbf{\Delta}\mathrm{LTV}}$. Define ∇ as in (42), $\mathcal{K}_\epsilon$ as in (43) (for η given in Definition 1). Then there exists $\epsilon > 0$ such that $\nabla \cap \mathcal{K}_\epsilon = \emptyset$.*

To bring in the Hahn-Banach Theorem, we note that $\mathcal{K}_\epsilon$ is open and convex in V, and that

Proposition 10 *The closure $\bar{\nabla}$ of ∇ is convex in V.*

Propositions 9 and 10 are proved in the Appendix. By choosing $\mathcal{K}_1 = \bar{\nabla}$, $\mathcal{K}_2 = \mathcal{K}_\epsilon$, we are in a position to apply Theorem 4, and obtain the corresponding $\Gamma \in V^*$, $\Gamma \neq 0$, $\alpha \in \mathbb{R}$. The structure of V and the Riesz representation theorem imply that Γ can be represented by $(x_1, \ldots, x_{n+1}, \Psi)$, where $x_i \in \mathbb{R}$, $\Psi \in BV[0, 2\pi]$. Then (12) yields

$$\sum_{i=1}^{n+1} x_i \sigma_i(z) + \int_0^{2\pi} \rho(z) d\Psi \ \leq \ \alpha, \quad \forall z \in l_2, \ \ \|z\| = 1 \tag{44}$$

$$\sum_{i=1}^{n+1} x_i r_i + \int_0^{2\pi} g d\Psi \ \geq \ \alpha, \quad \text{for} \begin{cases} r_i > -\epsilon^2 \\ \|g\|_\infty < \eta \end{cases} \tag{45}$$

Concentrating on (45), we conclude that $x_i \geq 0,\ i = 1 \ldots n+1$; also, since $\mathcal{K}_\epsilon$ contains a ball of 0, and $\Gamma \neq 0$, then $\alpha < 0$. Now turning to (44), it is possible to perturb x_i to make them strictly positive, and since $\sigma_i(z)$ are bounded functions, (44) will still hold for a new value $\alpha < 0$. Similarly, x_{n+1} can be normalized to 1.

It only remains to rewrite (44) using the definitions of σ_i, ρ. In the first place, simple manipulations show that

$$\sum_{i=1}^{n+1} x_i \sigma_i(z) = \left\langle \left(M^* \begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix} M - \begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix} \right) z, z \right\rangle \tag{46}$$

where $X = \mathrm{diag}[x_1 I, \ldots, x_n I] > 0$ is a scaling which commutes with Δ. Secondly, an integration by parts (note $F_{z_{n+1}}(0) = F_{z_{n+1}}(2\pi) = 0$) gives

$$\int_0^{2\pi} F_{z_{n+1}}(s) d\Psi(s) = -\int_0^{2\pi} \Psi(s) F'_{z_{n+1}}(s) ds = -\int_0^{2\pi} \Psi(s)(|z_{n+1}(s)|^2 - \|z_{n+1}\|^2)\frac{ds}{2\pi} \tag{47}$$

Defining $Y(s) = 1 + \Psi(s) - \int_0^{2\pi} \Psi(s)\frac{ds}{2\pi}$, the right hand side of (47) is equal to

$$-\int_0^{2\pi} (Y(s) - 1)|z_{n+1}(s)|^2 \frac{ds}{2\pi} = \left\langle \begin{bmatrix} 0 & 0 \\ 0 & I - Y \end{bmatrix} z, z \right\rangle \tag{48}$$

Combining everything into (44), we obtain

$$\left\langle \left(M^* \begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix} M - \begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix} \right) z, z \right\rangle \leq \alpha < 0, \quad \forall z \in l_2,\ \|z\| = 1 \tag{49}$$

This implies (25) holds. Finally, from the definition of $Y(s)$, we know that $\int_0^{2\pi} Y(s)\frac{ds}{2\pi} = 1$, and a small perturbation in Y will preserve (25) and yield

$$\int_0^{2\pi} Y(s)\frac{ds}{2\pi} < 1 \tag{50}$$

which is (26) for this scalar case.

□

We now comment briefly on the various extensions to the above proof; these are developed in detail in [32].

Multivariable noise.

If $u \in l_2^m$, then from (23) the performance quadratic constraint is changed to

$$\sigma_{n+1}(z) = \|(Mz)_{n+1}\|^2 - \frac{1}{m}\|z_{n+1}\|^2 \tag{51}$$

and (22) indicates the natural definition for $\rho(z) = F_{z_{n+1}}(s)$, given by

$$F_{z_{n+1}}(s) = \int_0^s z_{n+1}(\omega) z_{n+1}(\omega)^* \frac{d\omega}{2\pi} - \frac{s}{2\pi m}\|z_{n+1}\|^2 I_m$$

$F_{z_{n+1}}(s)$ now takes values in the space of continuous, hermitian matrix-valued functions; the dual of this space can be identified with the space of hermitian, bounded variation matrix functions Ψ on $[0, 2\pi]$ (up to a constant matrix), with the convention

$$\Gamma_\Psi(g) = \int_0^{2\pi} \mathrm{trace}(g(s) d\Psi(s)) = \sum_{i,j} \int_0^{2\pi} g_{i,j}(s) d\Psi_{i,j}(s)$$

The proof then follows in a similar way, giving $Y(s) = \Psi(s) + \left(1 - \int_0^{2\pi} \mathrm{trace}\Psi(s)\frac{ds}{2\pi}\right)\frac{I}{m}$ which satisfies (25) and is perturbed to satisfy (26).

δI perturbations in Δ.

If the i-th block of Δ is δI_{r_i}, then the scalar quadratic function σ_i must be replaced (see [32]) by a matrix-valued function

$$\Sigma_i(z) = \int_0^{2\pi} \left[(Mz)_i(\omega)(Mz)_i^*(\omega) - z_i(\omega) z_i^*(\omega)\right] \frac{d\omega}{2\pi}$$

which takes values in the space of hermitian $r_i \times r_i$ matrices. The functionals in this space are of the form $\Gamma_{X_i}(A) = \mathrm{trace}(X_i A)$, where X_i is a full, hermitian matrix. The argument then proceeds in a similar fashion, X_i becoming a sub-block of the scaling matrix X.

Slowly varying perturbations

To modify this proof to obtain Theorem 7, we need a quadratic constraint for slowly varying perturbations. It can be shown (see [35, 30, 32]) that given $p, q \in l_2$, if

$$\int_s^{s+h} (|q|^2 - |p|^2)\frac{d\omega}{2\pi} \geq 0$$

for every interval of length h, then there exists a contractive operator Δ, with $\|\lambda\Delta - \Delta\lambda\| \leq \nu = 2\sin(\frac{h}{2})$, such that $\Delta q = p$. Hence it is natural to replace the scalar σ_i with a quadratic function $\varphi_i : l_2 \mapsto C_R[0, 2\pi]$,

$$[\varphi_i(z)](s) = \int_s^{s+h} (|(Mz)_i|^2 - |z_i|^2)\frac{d\omega}{2\pi}$$

The corresponding functional will be a bounded variation function $x_i(s)$, which will give the frequency varying portion of the X scale. In this way one can also handle a combination of slowly varying and LTV perturbations for robust $\mathcal{H}_2$ performance, as was done in [30] for $\mathcal{H}_\infty$ performance.

5 Computational Issues

A test has been developed in Section 4 which characterizes robust $\mathcal{H}_2$ performance analysis of an uncertain system. This test is an infinite dimensional convex feasibility condition on the unknowns X and Y, specified as a Linear Matrix Inequality (LMI) across the frequency axis, of a similar complexity as Condition 1 for robust $\mathcal{H}_\infty$ performance.

There are two standard approaches for handling the infinite dimensionality of these conditions, and turn them into finite dimensional LMIs, for which efficient algorithms are available [6]: one used in [4] is to grid the frequency axis, the other is to select a finite set of basis functions and search for a scaling in the span of these functions, which reduces to a single LMI via a state-space approach (see, e.g., [3]).

Both approaches can indeed be applied to Condition 2, and involve minor modifications to their counterparts for Condition 1. We demonstrate this by commenting on the gridding approach for this problem: Condition 2 is approximated by considering frequency points $0 = w_0 \dots w_N = 2\pi$.

Although this approximation offers no hard guarantees, since it is based on the frequency domain it allows for engineering judgement to be used in choosing the number and location of the grid-points. The finite dimensional approximation to Condition 2 is the LMI problem

$$M(\omega_i)^* \begin{bmatrix} X_i & 0 \\ 0 & I \end{bmatrix} M(\omega_i) - \begin{bmatrix} X_i & 0 \\ 0 & Y_i \end{bmatrix} \; < \; 0 \quad i = 1 \dots N \tag{52}$$

$$X_i \; > \; 0 \quad i = 1 \dots N \tag{53}$$

$$\sum_{i=1}^{N} \mathrm{trace}(Y_i)(\omega_i - \omega_{i-1}) \; < \; 1 \tag{54}$$

where the unknowns Y_i are hermitian matrices and the X_i structured matrices. For the LTV test, $X_i \equiv X$ is constant across the ω_i, which makes conditions (52-54) intrinsically coupled across frequency. For the LTI/slowly varying test, we use different variables X_i, $i = 1 \dots N$. Although (54) still involves all frequency points, the following strategy can be used to decouple the problem across frequency:

- For each fixed frequency point ω_i, pose the problem:

  Minimize $\mathrm{trace}(Y_i)$, subject to

$$M(\omega_i)^* \begin{bmatrix} X_i & 0 \\ 0 & I \end{bmatrix} M(\omega_i) - \begin{bmatrix} X_i & 0 \\ 0 & Y_i \end{bmatrix} \; < \; 0 \tag{55}$$

$$X_i \; > \; 0 \tag{56}$$

  The problem of minimizing a linear function of the unknowns, subject to an LMI constraint, falls in the class of eigenvalue problems (EVPs) considered in [6], and can be computed efficiently.

- Given all the solutions $Y_1 \dots Y_N$, compute $\sum_{i=1}^{N} \mathrm{trace}(Y_i)(\omega_i - \omega_{i-1})$, and compare the answer to 1. More directly, this sum will provide an approximation to the square of the worst-case $\mathcal{H}_2$ norm of the system; this follows from the fact that to test if the worst-case $\mathcal{H}_2$ norm is less than γ, it suffices to change 1 for γ^2 in (26) or (54).

6 Connections to Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ Performance

In this section we relate these results to earlier work in the so-called mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problem. There are many versions of this problem in the literature (a few are [5, 22, 45, 34]), all of which attempt to get a handle on robust $\mathcal{H}_2$ performance by studying first the situation where there is no uncertainty, but the performance specification is a combination of the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms.

A mixed $\mathcal{H}_2/\mathcal{H}_\infty$ performance problem can in fact be cast naturally in our setting, and leads to an analysis test which looks exactly like Condition 2, except that the scaling matrix $X(\omega)$ does not appear and is fixed to be the identity.

Proposition 11 *Consider a system $M = [M_1 \; M_2]$ where the input z is partitioned in the vectors $z_1, \; z_2 \in \mathbb{C}^m$. The following are equivalent:*

$$\text{(i)}: \; \exists \eta > 0: \; \sup \left\{ \|Mz\|^2 : \|z_1\|^2 + \frac{1}{m}\|z_2\|^2 \leq 1, \; z_2 \in \hat{W}_\eta \right\} < 1 \tag{57}$$

(ii) : *There exists* $Y(\omega) = Y(\omega)^*$:

$$M(e^{j\omega})^* M(e^{j\omega}) - \begin{bmatrix} I & 0 \\ 0 & Y(\omega) \end{bmatrix} < 0 \qquad \forall\, \omega \in [0, 2\pi] \tag{58}$$

$$\int_0^{2\pi} \operatorname{trace}(Y(\omega)) \frac{d\omega}{2\pi} < 1 \tag{59}$$

This result is proved along the same lines as Theorem 8, only that since there is no uncertainty, one only considers the whiteness constraint $\rho(z) = F_{z_2}$, and the performance quadratic constraint $\sigma(z) = \|Mz\|^2 - \|z_1\|^2 - \frac{1}{m}\|z_2\|^2$, with $\|z_2\|^2$ weighted by $\frac{1}{m}$ for the reasons explained in Section 3.

Conditions (58) and (59) are therefore interpreted as a mixed performance problem where a portion of the input signal is constrained to be white. Various problems like this were considered in [45] for continuous time, with different assumptions on the relationship between z_1 and z_2. It is shown in [45] that the performance costs are not substantially different for these alternatives, and subsequently the attention is concentrated on the case where z_1 is restricted to be causally dependent on z_2. State-space methods for both analysis and synthesis for this alternative are given in [45] and the sequel paper [16] (a stochastic version appears in [34]).

Our condition, in contrast, corresponds to the case where there is no such causality restriction, which is only treated summarily in [45]. While at the level of the mixed performance problem it is not obvious which alternative to prefer, the version considered here has advantages from the point of the robust $\mathcal{H}_2$ problem, since it is allows for the inclusion of X scales in a convex condition with the strong interpretation given in Theorems 7 and 8.

Remarks:

- These causality issues are strongly connected to the remarks of Section 4 regarding the causality of LTI perturbations. A possible conclusion from the results in [45] is that gaps due to this are not very significant, although this issue warrants further investigation.

- Analogously to the case treated in detail in [45], the mixed conditions (58-59) reduce to a finite dimensional test if $M(e^{j\omega})$ is rational. In fact, a Schur complement operation and some algebra shows that (58) is equivalent to

$$\|M_1\|_\infty < 1 \tag{60}$$

$$M_2^*(I - M_1 M_1^*)^{-1} M_2 < Y \quad \forall\, \omega \in [0, 2\pi] \tag{61}$$

  This implies that (58-59) can be tested by first checking (60), and then imposing that $\|N^{-1}M_2\|_2 < 1$, where N is the inversely stable spectral factor (see, e.g., [46]) satisfying $I - M_1 M_1^* = NN^*$. Both these operations can be computed efficiently by the same state-space techniques used in the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ literature.

7 Robust $\mathcal{H}_2$ Synthesis

Having obtained conditions for robust $\mathcal{H}_2$ performance analysis under structured uncertainty, it is natural to consider the problem of controller synthesis. If the nominal system M is obtained as the closed loop in a feedback configuration, the problem is to design the controller such that M satisfies the robust $\mathcal{H}_2$ performance conditions.

It is unlikely that a tractable global solution will appear for this problem, since none is known for the case of $\mathcal{H}_\infty$ performance. Except for very special configurations, the only general method for robust $\mathcal{H}_\infty$ synthesis is the so-called "D-K" iteration, where an an analysis step (Condition 1) is alternated with $\mathcal{H}_\infty$ synthesis.

Such iteration schemes can easily be extended to robust $\mathcal{H}_2$ synthesis, as is now described. Assume that the functions $X(\omega)$ and $Y(\omega)$ are rational and satisfy Condition 2 (they could be obtained by fitting frequency points, or with basis functions). They are both positive definite across frequency, so by a spectral factorization (see [46]) they can be expressed as

$$X(\omega) = D(e^{j\omega})^* D(e^{j\omega}), \qquad Y(\omega) = E(e^{j\omega})^* E(e^{j\omega}), \tag{62}$$

where D, E are rational, stable, stably invertible transfer functions, and D has the structure of X. Using these factorizations, it follows that (25) can be rewritten as

$$\left\| \begin{bmatrix} D & 0 \\ 0 & I \end{bmatrix} M \begin{bmatrix} D^{-1} & 0 \\ 0 & E^{-1} \end{bmatrix} \right\|_\infty < 1 \tag{63}$$

which leads to the following iteration procedure:

- For fixed D, E, reduce the norm in (63) by $\mathcal{H}_\infty$ synthesis.
- For a fixed controller, solve the analysis problem for D, E.

As for the standard D-K iteration, each step in this "(D,E)-K" iteration can be shown to improve the robust performance cost, but there is no reason to expect convergence to the global optimum.

The previous iteration was based in $\mathcal{H}_\infty$ synthesis. An alternative is suggested by the discussion in Section 6, where at the synthesis step one only includes the D scales with the plant, and employs the design schemes for the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problem. As remarked before, the techniques in [16] correspond to a slightly different mixed problem, and it is not clear whether they extend to the situation of Proposition 11. The approximate method presented in [8] could also be used.

Therefore a number of issues remain open for future research, regarding this second iteration and the comparison between the two alternatives for practical problems.

8 Conclusion

The results in this paper restore the $\mathcal{H}_2$ performance paradigm to the level of mainstream robust control: there is no longer a significant advantage from the point of view of robustness analysis in the consideration of $\mathcal{H}_\infty$ norms, since it is essentially no harder to test for the corresponding $\mathcal{H}_2$-performance problem.

Condition 2 appears in fact as a summary of tractable exact conditions for robustness analysis. Setting the blocks in X to be either constant or frequency-varying selects between LTV or LTI (slowly-varying) uncertainty, or a combination thereof. Selecting either a constant or frequency varying Y chooses between maximum over frequency or average over frequency performance (roughly, $\mathcal{H}_\infty$ or $\mathcal{H}_2$ performance; in rigor, for $\mathcal{H}_\infty$ the trace must be removed from (26)). Also, combinations of $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performance can be studied in the style of Section 6, by including Y terms only for the signals which are assumed white. For all these choices, we have an exact characterization of the robust performance problem for which Condition 2 is necessary and sufficient.

A number of research questions are raised by these results. One of these is the issue of computation of Condition 2; we demonstrated its tractability in Section 5, but further research and practical experience with algorithms are in order. Another important question which we will investigate in future papers is the applicability of state-space methods for this problem, which would reduce Condition 2 to a finite dimensional LMI in the case of constant scales. Finally, the problem of controller synthesis for robust $\mathcal{H}_2$ performance is now reduced to the complexity level of robust $\mathcal{H}_\infty$ synthesis; further investigation is required, in particular on the different iteration schemes which where proposed in Section 7.

To some extent, this paper closes a cycle of research which originated in the 70s. The methods developed during this period have reached the maturity to address one the main problems which motivated the appearance of robust control theory. The impact of this theory goes, however, far beyond the robustness of LQG regulators, constituting a fundamental addition to the understanding of models, uncertainty, and feedback control.

Appendix

We will give proofs of Propositions 9 and 10.

Lemma 12 *Let $\Lambda : l_2 \rightarrow V$ be defined as in (41). If $z, f \in l_2$, and λ is the delay operator, then*

$$\|z + \lambda^k f\|^2 \stackrel{k\rightarrow\infty}{\longrightarrow} \|z\|^2 + \|f\|^2 \tag{64}$$

$$\Lambda(z + \lambda^k f) \stackrel{k\rightarrow\infty}{\longrightarrow} \Lambda(z) + \Lambda(f) \tag{65}$$

where (65) means convergence in the topology of $V = \mathbb{R}^{n+1} \oplus C_R[0, 2\pi]$.

Proof: The main observation (which is easily seen by looking at the shift operator in the time domain) is that $\langle z, \lambda^k f \rangle \stackrel{k\rightarrow\infty}{\longrightarrow} 0$ for any functions $z, f \in l_2$. From this it follows that

$$\|z_i + \lambda^k f_i\|^2 \stackrel{k\rightarrow\infty}{\longrightarrow} \|z_i\|^2 + \|f_i\|^2 \tag{66}$$

$$\|(Mz)_i + (M\lambda^k f)_i\|^2 \stackrel{k\rightarrow\infty}{\longrightarrow} \|(Mz)_i\|^2 + \|(Mf)_i\|^2 \tag{67}$$

where in (67) we use $M\lambda^k = \lambda^k M$ from the time invariance of M. This implies (64), and also

$$\sigma_i(z + \lambda^k f) \stackrel{k\rightarrow\infty}{\longrightarrow} \sigma_i(z) + \sigma_i(f) \quad i = 1 \ldots n+1 \tag{68}$$

We now show that

$$\rho(z + \lambda^k f) \stackrel{k\rightarrow\infty}{\longrightarrow} \rho(z) + \rho(f) \tag{69}$$

with convergence in the sense of $C_R[0, 2\pi]$. Starting from (40) and (15), some algebra gives

$$\left[\rho(z + \lambda^k f) - \rho(z) - \rho(f)\right](s) = \left(\int_0^s 2Re[z_{n+1}^*(\omega)\, e^{j\omega k} f_{n+1}(\omega)]\frac{d\omega}{2\pi} - \frac{s}{\pi} Re\left\langle z_{n+1}, \lambda^k f_{n+1}\right\rangle\right) \tag{70}$$

It therefore suffices to show that the sequence of functions

$$\varphi_k(s) := \int_0^s z_{n+1}^*(\omega)\, e^{j\omega k} f_{n+1}(\omega) \frac{d\omega}{2\pi} \tag{71}$$

converges uniformly to 0 as $k \to \infty$. Pointwise convergence follows from $\varphi_k(s) = \langle 1_{[0,s]} z_{n+1}, \lambda^k f_{n+1} \rangle$. If convergence were not uniform, we could find $\epsilon > 0$, a subsequence k_j and points s_{k_j} with $|\varphi_{k_j}(s_{k_j})| \geq \epsilon$. By compactness, taking a partial subsequence we can assume $s_{k_j} \stackrel{j\to\infty}{\longrightarrow} s_0$. Now

$$\varphi_{k_j}(s_{k_j}) = \varphi_{k_j}(s_0) + \int_{s_0}^{s_{k_j}} z_{n+1}^*(\omega)\, e^{j\omega k_j} f_{n+1}(\omega) \frac{d\omega}{2\pi} \tag{72}$$

so

$$0 < \epsilon \leq |\varphi_{k_j}(s_{k_j})| \leq |\varphi_{k_j}(s_0)| + \left| \int_{s_0}^{s_{k_j}} |z_{n+1}||f_{n+1}| \frac{d\omega}{2\pi} \right| \tag{73}$$

The right hand side of (73) converges to 0 from the pointwise convergence of φ_{k_j}, and the fact that $|z_{n+1}||f_{n+1}| \in \mathcal{L}_1[0, 2\pi]$. This is a contradiction, so we have shown (69), which together with (68) implies that $\Lambda(z + \lambda^k f) \stackrel{k\to\infty}{\longrightarrow} \Lambda(z) + \Lambda(f)$ in the topology of V.

$\square$

Remark: The previous Lemma can easily be extended to the sum of shifted versions of N signals $z^{(0)}, \ldots, z^{(N-1)} \in l_2$, giving

$$\| \sum_{r=0}^{N-1} \lambda^{kr} z^{(r)} \|^2 \stackrel{k\to\infty}{\longrightarrow} \sum_{r=0}^{N-1} \|z^{(r)}\|^2 \tag{74}$$

$$\Lambda\left(\sum_{r=0}^{N-1} \lambda^{kr} z^{(r)} \right) \stackrel{k\to\infty}{\longrightarrow} \sum_{r=0}^{N-1} \Lambda(z^{(r)}) \tag{75}$$

Proof of Proposition 10

Let $co(\nabla)$ denote the convex hull of ∇; an element $\Lambda_0 \in co(\nabla)$ is a convex combination of the form

$$\Lambda_0 = \sum_{r=0}^{N-1} \alpha_r \Lambda(z^{(r)}), \;\; \alpha_r \geq 0, \;\; \sum_{r=0}^{N-1} \alpha_r = 1, \;\; \|z^{(r)}\| = 1 \tag{76}$$

Define $f^k = \sum_{r=0}^{N-1} \sqrt{\alpha_r} \lambda^{kr} z^{(r)}$. From (75) and the quadratic nature of Λ, it follows that

$$\Lambda(f^k) \stackrel{k\to\infty}{\longrightarrow} \sum_{r=0}^{N-1} \Lambda(\sqrt{\alpha_r} z^{(r)}) = \Lambda_0 \tag{77}$$

From (74), we find $\|f^k\|^2 \stackrel{k\to\infty}{\longrightarrow} \sum_{r=0}^{N-1} \alpha_r = 1$. Therefore $\Lambda(\frac{f^k}{\|f^k\|}) \stackrel{k\to\infty}{\longrightarrow} \Lambda_0$, so $\Lambda_0 \in \bar{\nabla}$.

We have shown $co(\nabla) \subset \bar{\nabla}$. This implies that $co(\bar{\nabla}) \subset \overline{co(\nabla)} \subset \bar{\nabla}$, so $\bar{\nabla}$ is convex.

□

For the next Lemma we consider the configuration of Figure 2, where a disturbance signal d is injected at the output of Δ.

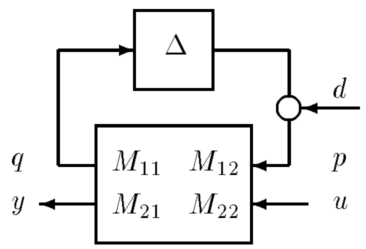

Figure 2: Uncertain system with injected d

Lemma 13 *Given $\epsilon > 0$, suppose $\nabla \cap \mathcal{K}_\epsilon \neq \emptyset$. Then there exists an operator $\Delta \in \mathbf{B}_{\mathbf{\Delta}_{\mathrm{LTV}}}$, and l_2 signals $z = col(p, u)$, $Mz = col(q, y)$, d, satisfying the equations of Figure 2, with $u \in W_\eta$, $\|z\| = 1$, $\|d\| = O(\epsilon)$, and $\|y\| \geq \|u\| - O(\epsilon)$.*

Proof: By hypothesis, we can find $z = col(v, u)$, $\|z\| = 1$, such that $\Lambda(z) \in \mathcal{K}_\epsilon$. This means that $\rho(z) \in B(0, \eta)$, so $u \in W_\eta$, and

$$\sigma_i(z) = \|(Mz)_i\|^2 - \|z_i\|^2 > -\epsilon^2, \quad i = 1 \ldots n+1 \tag{78}$$

From (78) we can find a contractive operator $\Delta_i : l_2 \to l_2$, and a disturbance d_i, $\|d_i\| = O(\epsilon)$, such that

$$z_i = \Delta_i (Mz)_i + d_i \quad i = 1 \ldots n \tag{79}$$

Also, $\|y\| \geq \|u\| - \epsilon$ follows from the constraint on σ_{n+1}. This would complete the proof, except that the given operator $\Delta = \mathrm{diag}[\Delta_1, \ldots, \Delta_n]$ need not be causal. For (79) to hold with causal Δ_i

would require (see [36]) the stronger condition

$$\|P^T(Mz)_i\|^2 - \|P^T z_i\|^2 > -O(\epsilon) \quad \forall T \in \mathbb{Z}_+ \tag{80}$$

(without loss of generality, assume z supported in $\mathbb{Z}_+$) where P^T denotes the truncation operator:

$$(P^T z)(t) = \begin{cases} z(t) & \text{for } t < T \\ 0 & \text{for } t \geq T \end{cases} \tag{81}$$

In fact, (80) can indeed be satisfied by means of a construction due to Shamma [38], where z is replaced by a signal $\hat{z}$ obtained from repetition of shifted versions of z. Consider the signal

$$\hat{z}^k = \frac{1}{\sqrt{N}} \sum_{r=0}^{N-1} \lambda^{kr} z \tag{82}$$

Choosing $\frac{1}{N} = O(\epsilon)$, and sufficiently large k, it can be shown that this signal satisfies (80) as required. The details of this argument are quite involved and are omitted (refer to [32] for a complete treatment of causal perturbations). Note also that $\rho(\hat{z}^k) \stackrel{k\to\infty}{\longrightarrow} \rho(z)$ from (75), and $\|\hat{z}^k\| \stackrel{k\to\infty}{\longrightarrow} 1$ from (74), so $\hat{z}^k$ will satisfy all the required conditions for large k.

□

Proof of Proposition 9

From robust stability, $(I - \Delta M_{11})^{-1}$ exists for each $\Delta \in \mathbf{B}_{\mathbf{\Delta}\mathrm{LTV}}$. We will use the uniform bound

$$\sup_{\Delta \in \mathbf{B}_{\mathbf{\Delta}\mathrm{LTV}}} \|(I - \Delta M_{11})^{-1}\| = \beta < \infty \tag{83}$$

which appears to be slightly stronger but can be shown (see [32]) to be equivalent to robust stability. We also know by hypothesis that

$$\sup_{\Delta \in \mathbf{B}_{\mathbf{\Delta}\mathrm{LTV}}} \|\Delta \star M\|_{\hat{W}_\eta} = \gamma < 1. \tag{84}$$

Given $\epsilon > 0$, suppose $\mathcal{K}_\epsilon \cap \nabla \neq \emptyset$. We apply Lemma 13 and construct the corresponding $\Delta \in \mathbf{B}_{\mathbf{\Delta}\mathrm{LTV}}$, $z = col(p, u)$, and d. We now state the following bounds:

$$-O(\epsilon) + \|u\| \leq \|y\| \leq \beta \|M_{21}\| \|d\|_2 + \gamma \|u\|_2 \tag{85}$$

The lower bound is a direct consequence of Lemma 13. The upper bound is obtained by writing y as the superposition of the contributions of the inputs $\hat{d}$ and $u \in \hat{W}_\eta$ in Figure 2, and using the bounds (83) and (84).

If $\mathcal{K}_\epsilon \cap \nabla \neq \emptyset$ holds for every ϵ, we can choose sequences Δ^k, $z^k = \mathrm{col}(p^k, u^k)$, $\|z^k\| = 1$, d^k corresponding to $\epsilon_k \stackrel{k\to\infty}{\longrightarrow} 0$. From the bounds (85) and $\gamma < 1$ it follows that we must have $\|u^k\| \stackrel{k\to\infty}{\longrightarrow} 0$; also, $\|d^k\| = O(\epsilon_k) \stackrel{k\to\infty}{\longrightarrow} 0$. Now from (83) the gain from (d, u) to z can be uniformly bounded across Δ, which implies that $\|z^k\| \stackrel{k\to\infty}{\longrightarrow} 0$, which is a contradiction. Therefore there must exist $\epsilon > 0$ such that $\mathcal{K}_\epsilon \cap \nabla = \emptyset$.

□

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