A Symmetric Product for Vector Fields and its Geometric Meaning*

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Abstract

We introduce the notion of *geodesic invariance* for distributions on manifolds with a linear connection. This is a natural weakening of the concept of a totally geodesic foliation to allow distributions which are not necessarily integrable. To test a distribution for geodesic invariance, we introduce a symmetric, vector field valued product on the set of vector fields on a manifold with a linear connection. This product serves the same purpose for geodesically invariant distributions as the Lie bracket serves for integrable distributions. The relationship of this product with connections in the bundle of linear frames is also discussed. As an application, we investigate geodesically invariant distributions associated with a left-invariant affine connection on a Lie group.

Keywords. symmetric product, linear connections, affine connections, Riemannian geometry, Lie groups

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1. Introduction

In Riemannian geometry there is a well-known theory of *totally geodesic foliations* where a geodesic which starts tangent to a leaf of a foliation remains on that leaf for all time. Further, the orthogonal complement to a totally geodesic foliation is also totally geodesic. This leads to a local decomposition of a Riemannian manifold into totally geodesic components. In this paper we discuss a notion which is weaker than that of being totally geodesic. Namely, in Section 2 we present the idea of *geodesic invariance* of a distribution D. Loosely speaking, a distribution D is geodesically invariant if the tangent vectors to a geodesic lie in D if the geodesic's initial tangent vector lies in D. It is clear that an integrable distribution which defines a totally geodesic foliation will have the property of geodesic invariance. However, such a distribution may strictly contain another geodesically invariant distribution which is not integrable. We obtain an algebraic test for such distributions which is used in much the same way that the Lie bracket is used to test for integrability. This test involves the use of a product which we define in Section 3 and call the *symmetric product*. Our presentation is, for the most part, in the context of general affine connections and not just Levi-Civita connections. However, the strongest results in Section 4 regarding connections in the bundle of linear frames are for Levi-Civita connections.

A particularly interesting class of affine connections are left-invariant affine connections on Lie groups. In Section 5 we use the techniques of the earlier sections of the paper to

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examine a class of geodesically invariant distributions for these affine connections. We also present a class of non-integrable, geodesically invariant distributions on compact Lie groups which allow a reduction of the structure group of the bundle of orthonormal frames.

We mention that the symmetric product for Levi-Civita connections first arose in the work of Crouch [1981] in the context of gradient systems. The product was rediscovered in the dissertation of Lewis [1995] in the context of nonlinear control theory for mechanical systems. For an abbreviated version of the relevant results of that work, we refer the reader to Lewis and Murray [1996]. For basic background in geometry of the bundle of linear frames, we refer the reader to Kobayashi and Nomizu [1963a,b].

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2. Some Geometry

In this section we review some relevant material from the geometry of the bundle of linear frames. The purpose is two-fold: first to establish our notation, and second to extract from the enormous literature on the geometry of the linear frame bundle, specific information which will be useful to us. Pretty much everything we say in this section can be found in Kobayashi and Nomizu [1963a].

2.1. The Bundle of Linear Frames. Let M be an n-dimensional, paracompact, differentiable manifold. We shall call an ordered basis of T_xM a *linear frame* at x. The collection of linear frames over M is called the *bundle of linear frames* and is denoted $L(M)$. We denote by $\pi\colon L(M) \to M$ the canonical projection. It is well-known that this bundle is a principal fibre bundle with structure group $GL(n;\mathbb{R})$ and that the right action of $GL(n;\mathbb{R})$ on $L(M)$ is given by $R_A\colon (x, \{X_1, \dots, X_n\}) \mapsto (x, \{A^i_1 X_i, \dots, A^i_n X_i\})$. We may also think of a linear frame, $p = \{X_1, \dots, X_n\} \in L_x(M)$, as an isomorphism from $\mathbb{R}^n$ to T_xM. We do this by declaring that p map $\boldsymbol{e}_i$, the ith standard basis element for $\mathbb{R}^n$, to X_i. If we view p in this manner, then the action of $A \in GL(n;\mathbb{R})$ on p is the composition of $A\colon \mathbb{R}^n \to \mathbb{R}^n$ and $p\colon \mathbb{R}^n \to T_xM$.

It will be useful to introduce coordinates for $L(M)$. To do so, let (U, ϕ) be a chart for M with coordinates denoted $(x^1, \dots, x^n)$. Then any linear frame, $\{X_1, \dots, X_n\}$, at $x \in U$ may be written as

$$\left\{ X^i_1 \frac{\partial}{\partial x^i}, \dots, X^i_n \frac{\partial}{\partial x^i} \right\}$$

which defines coordinates (x^i, X^j_k) for $L(M)$. We define $Y^i_j(x)$ by requiring that $Y^i_k(x) X^k_j(x) = \delta^i_j$. There is a natural basis for $\mathfrak{gl}(n;\mathbb{R})$ which we denote by $\{\boldsymbol{E}^i_j \mid i, j = 1, \dots, n\}$. Here $\boldsymbol{E}^i_j$ is the $n \times n$ matrix with a 1 in the (j, i) entry and zeroes elsewhere.

We shall call the subbundle of $T(L(M))$ defined by

$$V(L(M)) = \ker(T\pi)$$

the *vertical subbundle*. A *linear connection* on M is a distribution, $H(L(M))$, with the properties

1. $T(L(M)) = H(L(M)) \oplus V(L(M))$, and
2. $T_pR_A(H_p(L(M))) = H_{R_Ap}(L(M))$ for each $p \in L(M)$ and $A \in GL(n;\mathbb{R})$.

Associated with a linear connection is the *connection one-form*, ω, which is a one-form on $L(M)$ taking its values in $\mathfrak{gl}(n;\mathbb{R})$. Intrinsically we define the connection one-form by

$$\omega(X) = \{\xi \in \mathfrak{gl}(n;\mathbb{R}) \mid \quad \xi_{L(M)}(p) = X^{ver}\} \tag{2.1}$$

for $X \in T_p(L(M))$. Here $\xi_{L(M)}$ is the infinitesimal generator corresponding to $\xi \in \mathfrak{gl}(n;\mathbb{R})$ and X^{ver} is the vertical part of X relative to the decomposition given by the connection. In local coordinates for $L(M)$, the connection one-form is given by

$$\omega = Y^i_k \left(dX^k_j + \Gamma^k_{ml} X^l_j dx^m \right) \boldsymbol{E}^j_i, \tag{2.2}$$

which defines the *Christoffel symbols*, Γ^i_{jk}. We say that the connection is *torsion-free* if $\Gamma^i_{jk} = \Gamma^i_{kj}$ for $i, j, k = 1, \ldots, n$. The condition 2 in the definition of a connection gives the condition

$$\omega(T_pR_A(X)) = \mathrm{Ad}_{A^{-1}}\,\omega(X) \tag{2.3}$$

for $p \in L(M)$, $X \in T_p(L(M))$, and $A \in GL(n;\mathbb{R})$. Here $\mathrm{Ad}_A \colon \mathfrak{gl}(n;\mathbb{R}) \to \mathfrak{gl}(n;\mathbb{R})$ is the image of $A \in GL(n;\mathbb{R})$ in $\mathrm{Aut}(\mathfrak{gl}(n;\mathbb{R}))$ under the adjoint representation. In fact, any $\mathfrak{gl}(n;\mathbb{R})$-valued one-form on P satisfying the conditions (2.1) and (2.3) defines a connection on $L(M)$ given by $H(L(M)) = \ker(\omega)$.

2.2. Reduction of Structure Group. It will be important to us to know when it is possible to reduce the structure group of $L(M)$. In this section we will present the theory for a general principal fibre bundle, $\pi \colon P \to M$, with structure group G. A *subbundle* of P consists of a subbundle, Q, of P (regarding P as just a fibre bundle) and a subgroup H of G such that the induced right action of H on P leaves Q invariant, and the associated right action of H on Q makes $\pi_Q \triangleq \pi \mid Q \colon Q \to M$ a principal fibre bundle. Denote by $i_Q \colon Q \to P$ the inclusion. If we have connections, HP on P and HQ on Q, we shall say that HP is *reducible* to HQ if

$$T_p i_Q(H_pQ) = H_pP \tag{2.4}$$

for each $p \in Q$.

2.1 Proposition: *Let $\pi_Q \colon Q \to M$ be a subbundle of $\pi \colon P \to M$ with structure group $H \subset G$ and let HP be a connection. Denote the connection one-form of HP by ω_P. Consider the following propositions:*

(i) HP is reducible to a connection on Q;
(ii) HP is tangent to Q;

(iii) $i_Q^*\omega_P$ *takes its values in* $\mathfrak{h}$*, the Lie algebra of* H*.*
The following implications hold:

$$\text{(i)} \iff \text{(ii)} \implies \text{(iii)}$$

If there is subspace, $\mathfrak{m} \subset \mathfrak{g}$*, complementary to* $\mathfrak{h}$ *which is invariant under the adjoint representation of* H *in* $\mathfrak{g}$*, then (iii) is equivalent to (i) and (ii).*

Proof: (i) $\implies$ (ii) Since HP is reducible to a connection on Q, which we denote by HQ, (2.4) must hold. Since $H_pP = H_pQ \subset T_pQ$ for each $p \in Q$, HP is tangent to Q.

(ii) $\implies$ (i) By hypothesis we have $H_pP \subset T_pQ$ for each $p \in Q$. Therefore we can define a connection on Q by defining $H_pQ = H_pP$ for each $p \in Q$. It is clear that this does, in fact, define a connection on Q and so HP is reducible to a connection on Q.

Denote by $i_H\colon H \to G$ the inclusion homomorphism of H into G and by $i_{\mathfrak{h}}\colon \mathfrak{h} \to \mathfrak{g}$ the induced Lie algebra homomorphism.

We prove a technical lemma.

1 LEMMA: *If* HP *is reducible to a connection on* Q*, then* $\omega_P(T_p i_Q(X)) = i_{\mathfrak{h}}(\omega_Q(X))$ *for* $p \in Q$ *and* $X \in T_pQ$*. Here* ω_Q *is the connection one-form for the connection on* Q*.*

Proof: We will first verify this formula when X is vertical. In this case $X = \xi_Q(p)$ for some $\xi \in \mathfrak{h}$. Since Q is a subbundle of P we have

$$i_Q(R_A p) = R_{i_H(A)} i_Q(p) \tag{2.5}$$

for each $A \in H$ and $p \in Q$. If we set $\xi' = i_{\mathfrak{h}}(\xi)$, differentiating (2.5) along the one-parameter subgroup corresponding to ξ, we get $T_p i_Q(X) = \xi'_P(p)$. Thus

$$\omega_P(T_p i_Q(X)) = \omega_P(\xi'_P(p)) = \xi' = i_{\mathfrak{h}}(\xi) = i_{\mathfrak{h}}(\omega_Q(\xi_Q(p))) = i_{\mathfrak{h}}(\omega_Q(X)).$$

If X is horizontal, since $T_p i_Q$ is an isomorphism between horizontal subspaces, both $\omega(T_p i_Q(X))$ and $i_{\mathfrak{h}}(\omega_Q(X))$ will be zero. This proves the lemma. ▼

The fact that (ii) implies (iii) now follows easily from Lemma 1.

Now suppose that ω_P restricted to Q takes its values in $\mathfrak{h}$ and suppose that $\mathfrak{h}$ admits an Ad_H-invariant complementary subspace, $\mathfrak{m}$. We shall show that (iii) $\implies$ (i) in this case. In fact, we claim that the $\mathfrak{h}$-component of ω_P defines a connection one-form, which we denote by ω_Q, on Q. Let $\xi \in \mathfrak{h}$ and let ξ_P denote the corresponding infinitesimal generator. Let ω'_P denote the $\mathfrak{h}$-component of ω_P with respect to the decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$. We then have

$$\omega'_P(\xi_P) = \xi \tag{2.6}$$

for each $\xi \in \mathfrak{h}$. Now let λ denote the $\mathfrak{m}$-component of ω_P restricted to Q and let $X \in T_pQ$ for some $p \in Q$. For $A \in H$ we have

$$\begin{aligned} \omega_P(T_pR_AX) &= \omega'_P(T_pR_AX) + \lambda(T_pR_AX) \\ \mathrm{Ad}_{A^{-1}}(\omega_P(X)) &= \mathrm{Ad}_{A^{-1}}(\omega'_P(X)) + \mathrm{Ad}_{A^{-1}}(\lambda(X)). \end{aligned}$$

Since ω_P is a connection one-form, the two left hand sides must be equal. Since $\lambda(T_pR_AX) \in \mathfrak{m}$ by definition of λ, and since $\mathrm{Ad}_{A^{-1}}(\lambda(X)) \in \mathfrak{m}$ by hypothesis, we must have

$$\omega'_P(T_pR_AX) = \mathrm{Ad}_{A^{-1}}(\omega'_P(X))$$

for each $p \in Q$, $A \in H$, and $X \in T_pQ$. This, combined with (2.6), shows that $\omega_Q \triangleq i_Q^*\omega'_P$ defines a connection one-form on Q, and hence shows that the connection HP is reducible to a connection on Q. ■

2.3. Affine Connections on TQ. As is well-known, a connection in the bundle of linear frames induces an *affine connection* on M. Covariant differentiation with respect to this affine connection is defined in coordinates by

$$\nabla_X Y = \left(\frac{\partial Y^i}{\partial x^j}X^j + \Gamma^i_{jk}X^jY^k\right)\frac{\partial}{\partial x^i}. \tag{2.7}$$

Associated with an affine connection is the notion of *parallel translation*. Given a curve $c\colon [0,1] \to M$, parallel translation defines an isomorphism from $T_{c(0)}M$ to $T_{c(1)}M$. A curve $c\colon [a,b] \to M$ satisfying the relation $\nabla_{c'(t)}c'(t) = 0$ is called a *geodesic*. In coordinates, c must satisfy the second-order differential equation

$$\ddot{x}^i + \Gamma^i_{jk}\dot{x}^j\dot{x}^k = 0$$

which in turn defines a vector field on TM which we call the *geodesic spray* and denote by Z_g. In natural coordinates (x^i, v^j) for TM we have

$$Z_g = v^i\frac{\partial}{\partial x^i} - \Gamma^i_{jk}v^jv^k\frac{\partial}{\partial v^i}. \tag{2.8}$$

Given an affine connection on M, we may define the *torsion tensor field* as the tensor field on M of type $(1,2)$ given by

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y].$$

It is easy to verify that this does, in fact, define a tensor field, and that the components of T in a local chart are $T^i_{jk} = \Gamma^i_{jk} - \Gamma^i_{kj}$. From this expression it is clear that if the linear connection defining ∇ is torsion-free, then $T = 0$.

2.4. The Bundle of Orthonormal Frames. Some of the results we state will only be valid for connections on the so-called orthonormal frame bundle. In this setup we consider a Riemannian metric, $\langle\cdot,\cdot\rangle$, on M and we call an ordered, orthonormal basis for T_xM an *orthonormal frame* at x. The collection of orthonormal frames over M will be called the *bundle of orthonormal frames* and will be denoted $O(M)$. This is a principal fibre bundle whose structure group is $O(n)$. An orthonormal frame, $p = \{X_1, \ldots, X_n\} \in O_x(M)$, can be thought of as a linear isometry from $\mathbb{R}^n$, with its standard inner product, to T_xM. As with the bundle of linear frames, we may talk about connections on $O(M)$. The connection one-form for a connection on $O(M)$ will take its values in $\mathfrak{o}(n)$, the Lie algebra of skew-symmetric $n \times n$ matrices.

On a Riemannian manifold there is a distinguished connection in $O(M)$ with the properties that

1. the connection is torsion-free, and
2. parallel translation with respect to the induced affine connection is an isometry.

We shall call this connection the *Levi-Civita* connection and will also use this name to refer to the induced affine connection on M. The Christoffel symbols for the Levi-Civita connection are given by

$$\Gamma^i_{jk} = \frac{1}{2} g^{il} \left(\frac{\partial g_{lj}}{\partial x^k} + \frac{\partial g_{lk}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^l} \right).$$

Here g_{ij} are the components of the Riemannian metric.

2.5. The Tangent Bundle as an Associated Bundle of $L(M)$. To properly relate the linear frame bundle to TM, we need to realise TM as an associated bundle of $L(M)$. This is done, for example, in Kobayashi and Nomizu [1963a].

Note that $GL(n;\mathbb{R})$ acts on $\mathbb{R}^n$ on the left by $(A, \boldsymbol{u}) \mapsto A\boldsymbol{u}$. Thus we can define a *right* action of $GL(n;\mathbb{R})$ on $L(M) \times \mathbb{R}^n$ by $(A, (p, \boldsymbol{u})) \mapsto (R_A p, A^{-1}\boldsymbol{u})$.

2.2 LEMMA: *There is a natural diffeomorphism from $(L(M) \times \mathbb{R}^n)/GL(n;\mathbb{R})$ onto TM.*

Proof: Let $(p, \boldsymbol{u}) \in L(M) \times \mathbb{R}^n$ and denote by $[(p, \boldsymbol{u})]$ the orbit through $(p, \boldsymbol{u})$. If $p \in \pi^{-1}(x)$, we may define $v \in T_x M$ by $v = p\boldsymbol{u}$. We claim that this definition does not depend on which element of $[(p, \boldsymbol{u})]$ we choose. Indeed suppose that $(p', \boldsymbol{u}') \in [(p, \boldsymbol{u})]$. Then $(p', \boldsymbol{u}') = (R_A p, A^{-1}\boldsymbol{u})$ for some $A \in GL(n;\mathbb{R})$. It follows from the definition of the right action that $p'\boldsymbol{u}' = v$.

Now suppose that $v \in T_x M$. Then there exists $p \in \pi^{-1}(x)$ and $\boldsymbol{u} \in \mathbb{R}^n$ so that $p\boldsymbol{u} = v$. It is clear that this choice of $(p, \boldsymbol{u})$ depends only upon the equivalence class $[(p, \boldsymbol{u})]$ thus proving the lemma. ■

Given this lemma, it seems reasonable that the geodesic spray might be induced from a $GL(n;\mathbb{R})$-invariant vector field on $L(M) \times \mathbb{R}^n$. This is in fact the case. For $\boldsymbol{u} \in \mathbb{R}^n$, define a horizontal vector field, $\Sigma(\boldsymbol{u})$, on $L(M)$ by defining $\Sigma(\boldsymbol{u})(p) \in T_p L(M)$ to be the unique horizontal vector with the property $T_p\pi(\Sigma(\boldsymbol{u})(p)) = p\boldsymbol{u}$. The vector field $\Sigma(\boldsymbol{u})$ is called the *standard horizontal vector field* corresponding to $\boldsymbol{u}$. Clearly we may regard $\Sigma\colon (p, \boldsymbol{u}) \mapsto (\Sigma(\boldsymbol{u})(p), 0)$ as a vector field on $L(M) \times \mathbb{R}^n$. The following result characterises the vector field Σ in terms of the geodesic spray. It is a consequence of Proposition 6.3 in Chapter III of Kobayashi and Nomizu [1963a].

2.3 PROPOSITION: *Let $F_t\colon L(M) \times \mathbb{R}^n \to L(M) \times \mathbb{R}^n$ denote the flow of Σ and let $f_t\colon TM \to TM$ denote the flow of Z_g. Then the following diagram commutes for each t for which the flows are defined.*

$$\begin{tikzcd} L(M) \times \mathbb{R}^n \arrow[r, "F_t"] \arrow[d, "\tau"'] & L(M) \times \mathbb{R}^n \arrow[d, "\tau"] \\ TM \arrow[r, "f_t"'] & TM \end{tikzcd}$$

Here $\tau\colon L(M) \times \mathbb{R}^n \to TM \simeq L(M) \times \mathbb{R}^n/GL(n;\mathbb{R})$ is the canonical projection.

2.6. Distributions on Manifolds with a Linear Connection. In this section we present some definitions for distributions on manifolds with a linear connection, some of which are new and some of which are well-known. Of particular interest is the introduction of the notion of a *geodesically invariant* distribution. We also present some constructions on $L(M)$ associated with a distribution on M.

We shall denote by $\mathscr{D}$ the set of sections of a distribution D.

2.4 DEFINITION: Let D be a distribution on M. We say that D is *geodesically invariant* if, for every geodesic $c\colon [a, b] \to M$ such that $c'(a) \in D_{c(a)}$, $c'(t) \in D_{c(t)}$ for each $t \in (a, b]$. If D is integrable and geodesically invariant, we say that it is *totally geodesic*. □

2.5 REMARK: Suppose that D is totally geodesic and denote by $\mathscr{F}_D$ the foliation defined by D (since, by definition, D is integrable). It is clear in this case that each geodesic whose initial velocity is tangent to a leaf $\Lambda \in \mathscr{F}_D$ will remain on Λ for all time for which the geodesic is defined. It is well-known (see Kobayashi and Nomizu [1963b], (pages 53–57)) that a distribution is totally geodesic if $\nabla_X Y \in \mathscr{D}$ for every $X, Y \in \mathscr{D}$. The converse is true when the linear connection is torsion-free. □

Note that a distribution which is totally geodesic is geodesically invariant, but that the converse is not necessarily true. Thus the notion of a distribution being geodesically invariant is *weaker* than that of being totally geodesic. We wish to characterise geodesically invariant distributions.

Now we say a few things about distributions and frame bundles. Let D be a distribution on M of rank σ. We shall say that a linear frame, $\{X_1, \ldots, X_n\}$, at $x \in M$ is *D-adapted* if $\{X_1, \ldots, X_\sigma\}$ is a basis for D_x. We shall denote by $L(M, D)$ the set of D-adapted frames on M. If we denote by $\mathbb{R}^\sigma$ the subset of $\mathbb{R}^n$ consisting of those vectors whose last $n - \sigma$ entries are zero, we denote by $GL(n, \sigma; \mathbb{R})$ the subgroup of $GL(n; \mathbb{R})$ which leaves $\mathbb{R}^\sigma$ invariant. Thus an element of $GL(n, \sigma; \mathbb{R})$ has the form

$$\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$$

where $A \in GL(\sigma; \mathbb{R})$, $C \in GL(n - \sigma; \mathbb{R})$, and B is a $\sigma \times (n - \sigma)$ matrix. It is then clear that $L(M, D)$ is a principal fibre bundle over M with structure group $GL(n, \sigma; \mathbb{R})$. We may think of $p \in L(M, D)$ as a linear map from $\mathbb{R}^n$ to $T_x M$ which maps $\mathbb{R}^\sigma$ to D_x.

When the linear connection is the Levi-Civita connection associated with a Riemannian metric, $\langle \cdot, \cdot \rangle$, on M, we may make a stronger statement about the structure of the bundle of D-adapted *orthonormal* frames, which we denote by $O(M, D)$. Let $O(n, \sigma)$ denote the subgroup of $O(n)$ which leaves $\mathbb{R}^\sigma$ invariant. Since the elements of $O(n)$ are orthogonal, we see that $O(n, \sigma) \simeq O(\sigma) \times O(n - \sigma)$. That is to say, a typical element of $O(n, \sigma)$ has the form

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

where $A \in O(\sigma)$ and $B \in O(n - \sigma)$. We denote by $\mathbb{R}^{n-\sigma}$ the subset of $\mathbb{R}^n$ whose first σ entries are zero. Note that $p \in O_x(M, D)$ will have the property that it maps $\mathbb{R}^{n-\sigma}$ to $D_x^\perp$, the orthogonal complement of D_x.

3. The Symmetric Product

In this section we introduce the symmetric product and use it to define a property of distributions on manifolds with an affine connection. In Section 4 we will relate the symmetric product to the distributions discussed in Section 2.6.

We first present some algebraic preliminaries. We do this in an effort to emphasise as much as possible the similarities between the symmetric product and the Lie bracket for vector fields, recalling that the Lie bracket makes the set of vector fields on a manifold a real Lie algebra.

Let R be a commutative ring with unit. An *R-symmetric algebra* is a module, A, over R with a product, which we denote $(u, v) \mapsto \langle u : v\rangle \in A$, which has the properties

1. $\langle u : v\rangle = \langle v : u\rangle$ for each $u, v \in A$, and
2. $\langle au + bv : w\rangle = a\,\langle u : w\rangle + b\,\langle v : w\rangle$ for each $u, v, w \in A$ and for each $a, b \in R$.

We shall call $\langle \cdot : \cdot\rangle$ the *symmetric product* on A. In Lewis [1995] the notion of a free symmetric algebra is presented.

Now suppose that we have a manifold M with an affine connection ∇. We make the set of vector fields on M a real symmetric algebra by defining a symmetric product by

$$\langle X : Y\rangle = \nabla_X Y + \nabla_Y X.$$

Clearly this product satisfies the conditions required of a symmetric product. The following lemma gives some sense of what the symmetric product means, at least in local coordinates.

3.1 Lemma: *Let ∇ be an analytic affine connection on M with $\langle \cdot : \cdot\rangle$ the associated symmetric product. Let X and Y be vector fields on M with flows F^X_t and F^Y_t, respectively. Fix $x \in M$ and consider the following construction:*

(i) parallel transport $X(x)$ along the integral curve for Y through x for a time ϵ to arrive at a vector X_ϵ;

(ii) parallel transport X_ϵ along the integral curve of X through $F^Y_\epsilon(x)$ for a time ϵ to arrive at a vector $X_{2\epsilon}$.

In a coordinate chart, (U, ϕ), for M with coordinates $(x^1, \dots, x^n)$ the following relation is true:

$$\langle X : Y\rangle^i (x) = \lim_{\epsilon\to 0} \frac{X^i_{2\epsilon} - X^i(x)}{\epsilon}, \quad i = 1, \dots, n$$

where X^i is the principal part of X in the chart (U, ϕ).

Proof: Denote by $\tau^Y_\epsilon : T_xM \to T_{F^Y_\epsilon}M$ parallel translation along Y. If we Taylor expand $\tau^Y_\epsilon(X^i(x))$ with respect to ϵ we get

$$X^i_\epsilon = \tau^Y_\epsilon(X^i(x)) = X^i(x) + \epsilon(\nabla_Y X)^i(x) + \mathscr{O}(\epsilon^2).$$

Similarly, if we denote by $\tau^X_\epsilon : T_{F^Y_\epsilon(x)}M \to T_{F^X_\epsilon(F^Y_\epsilon(x))}M$ parallel translation along X, we get

$$X^i_{2\epsilon} = \tau^X_\epsilon(\tau^Y_\epsilon(X^i(x))) = X^i(x) + \epsilon(\nabla_Y X)^i(x) + \epsilon(\nabla_Y X)^i(x) + \mathscr{O}(\epsilon^2).$$

Taking the limit and using the definition of the symmetric product gives the lemma. ■

Now we give a property for distributions on manifolds with an affine connection.

3.2 Definition: Let M be a manifold with an affine connection ∇ with $\langle \cdot : \cdot \rangle$ the associated symmetric product. We shall say that a distribution D is *symmetric* if $\langle X : Y \rangle \in \mathscr{D}$ for every $X, Y \in \mathscr{D}$. □

The following result leads up to the results of Section 4.

3.3 Proposition: *Let M be a manifold with a torsion-free linear connection with associated affine connection ∇, and let D be a distribution on M. Then D is totally geodesic if and only if it is symmetric and integrable.*

Proof: First suppose that D is totally geodesic. By definition, D is integrable. Also, by Remark 2.5, $\nabla_X Y$ is a section of D for sections X and Y of D since the connection is torsion-free. Therefore, $\langle X : Y \rangle$ is a section of D for each pair of sections X and Y of D. Thus D is also symmetric.

Now suppose that D is symmetric and integrable. This means that $\langle X : Y \rangle$ and $[X, Y]$ are sections of D for sections X and Y of D. Thus, since the connection is torsion free, we have

$$\langle X : Y \rangle = \nabla_X Y + \nabla_Y X \quad \text{and} \quad [X, Y] = \nabla_X Y - \nabla_Y X.$$

Therefore, $\nabla_X Y \in \mathscr{D}$ for every $X, Y \in \mathscr{D}$. This proves that D is totally geodesic by Remark 2.5. ■

4. Characterising Geodesically Invariant Distributions

In this section we tie together the constructions and definitions of Section 2.6 with the definitions of Section 3.

First we recall the notion of the *vertical lift* of a vector field on M. Let X be a vector field on M. Its vertical lift is the vector field on TM defined by

$$X^{lift}(v_x) = \left.\frac{d}{dt}\right|_{t=0} (v_x + tX(x)).$$

4.1 Theorem: *Suppose that $L(M)$ is equipped with a linear connection, $H(L(M))$. Let D be a distribution on M of rank σ. Then D is symmetric if and only if it is geodesically invariant.*

Proof: First we gather three easy technical lemmas for the purpose of reference later in the proof. The first is obvious and so we give no proof.

1 Lemma: *Regard D as a submanifold of TM. Then D is geodesically invariant if and only if Z_g is tangent to $D \subset TM$.*

2 Lemma: *Let X and Y be vector fields on M. Then $[X^{lift}, [Z_g, Y^{lift}]] = \langle X : Y \rangle^{lift}$.*

Proof: We use local coordinates $(x^1, \dots, x^n)$ for M with $(x^1, \dots, x^n, v^1, \dots, v^n)$ the associated natural coordinates for TM. We have

$$X^{lift} = X^i(x)\frac{\partial}{\partial v^i}, \quad Y^{lift} = Y^i(x)\frac{\partial}{\partial v^i}, \quad \text{and} \quad Z_g = v^i\frac{\partial}{\partial x^i} - \Gamma^i_{jk}(x)v^j v^k\frac{\partial}{\partial v^i}.$$

It is an easy matter to compute

$$[Z_g, Y^{lift}] = -Y^i\frac{\partial}{\partial x^i} + \left(\frac{\partial Y^i}{\partial x^j}v^j + +\Gamma^i_{jk}Y^j v^k + \Gamma^i_{jk}v^k Y^j\right)\frac{\partial}{\partial v^i}.$$

We then compute

$$[X^{lift}, [Z_g, Y^{lift}]] = \left(\frac{\partial X^i}{\partial x^j}Y^j + \frac{\partial Y^i}{\partial x^j}X^j + \Gamma^i_{jk}X^jY^k + \Gamma^i_{jk}Y^jX^k\right)\frac{\partial}{\partial v^i}$$

which we recognise as the vertical lift of $\langle X : Y\rangle$ by (2.7). ▼

3 LEMMA: *A vector field X on M is a section of D if and only if X^{lift} is tangent to D thought of as a submanifold of TM.*

Proof: Note that if X is a vector field on M, then X^{lift} is tangent to the fibres of TM. Therefore, X^{lift} is tangent to D if and only if $X^{lift} \mid T_xM$ is tangent to D_x for each $x \in M$. It is clear from the definition of the vertical lift that $X^{lift} \mid T_xM$ is tangent to D_x if and only if $X(x) \in D_x$. This proves the lemma. ▼

Suppose that D is geodesically invariant. Thus Z_g is tangent to $D \subset TM$ by Lemma 1. Now let X and Y be sections of D. By Lemma 3, X^{lift} and Y^{lift} are tangent to D. Therefore, any iterated Lie bracket of X^{lift}, Y^{lift}, and Z_g will also be tangent to D. In particular, $[X^{lift}, [Z_g, Y^{lift}]] = \langle X : Y\rangle^{lift}$ is tangent to D. Therefore, by Lemma 3, $\langle X : Y\rangle$ is a section of D and so D is symmetric.

Now suppose that D is symmetric so that $\langle X : Y\rangle \in \mathscr{D}$ for sections X and Y of D. Thus $[X^{lift}, [Z_g, Y^{lift}]]$ is tangent to $D \subset TM$ for each pair of sections X and Y of D by Lemmas 2 and 3. Now we introduce special coordinates on TM to complete this part of the proof. Choose a coordinate chart (U, ϕ) for M and choose a basis, $\{V_1, \dots, V_n\}$, for vector fields on U with the property that $\{V_1(x), \dots, V_n(x)\}$ is D-adapted for each $x \in U$. We write

$$V_i(x) = V_i^j(x)\frac{\partial}{\partial x^j}, \quad i = 1, \dots, n.$$

Therefore

$$v^i\frac{\partial}{\partial x^i} = v^iW_i^jV_j$$

where W_i^j are defined by $W_k^iV_j^k = \delta_j^i$. Thus, as coordinates for TQ we may use $(x^i, w^j = W_k^jv^k)$. We shall be performing summations over three different ranges of indices. To limit confusion, we will denote by a, b, c, d indices which lie in $\{1, \dots, \sigma\}$, by α, β, γ indices which lie in $\{\sigma + 1, \dots, n\}$, and by i, j, k, l, m, r indices which lie in $\{1, \dots, n\}$.

A straightforward calculation using (2.8) yields Z_g in the coordinates (x^i, w^j) to be

$$Z_g = V^i_j w^j \frac{\partial}{\partial x^i} + \left(\frac{\partial W^i_l}{\partial x^m} V^m_j V^l_k w^j w^k - \Gamma^r_{lm} V^l_j V^m_k W^i_r w^j w^k \right) \frac{\partial}{\partial w^i}.$$

For brevity we define

$$S^i_{jk} = \Gamma^r_{lm} V^l_j V^m_k W^i_r - \frac{\partial W^i_l}{\partial x^m} V^m_j V^l_k. \tag{4.1}$$

Also note that

$$\frac{\partial}{\partial w^i} = V^j_i \frac{\partial}{\partial v^j}$$

for $i = 1, \ldots, n$. That is to say, in the coordinates (x^i, w^j), V^{lift}_i is simply given by $\frac{\partial}{\partial w^i}$. This makes it straightforward to compute

$$[Z_g, V^{lift}_a] = -V^i_a \frac{\partial}{\partial x^i} + (S^i_{aj} w^j + S^i_{ja} w^j) \frac{\partial}{\partial w^i}$$

and

$$[V^{lift}_b, [Z_g, V^{lift}_a]] = (S^i_{ab} + S^i_{ba}) \frac{\partial}{\partial w^i}$$

for $a, b = 1, \ldots, \sigma$. Since V_a and V_b are sections of D and since D is symmetric, by Lemmas 2 and 3 we must have

$$S^\alpha_{ab} + S^\alpha_{ba} = 0, \quad a, b = 1, \ldots, \sigma, \;\; \alpha = \sigma + 1, \ldots, n.$$

We now write

$$Z_g = V^i_j w^j \frac{\partial}{\partial x^i} - S^a_{jk} w^j w^k \frac{\partial}{\partial w^a} - \left(S^\gamma_{a\alpha} w^a w^\alpha + S^\gamma_{\alpha a} w^\alpha w^a + S^\gamma_{\alpha\beta} w^\alpha w^\beta \right) \frac{\partial}{\partial w^\gamma}.$$

Note that in the coordinates (x^i, w^j), the inclusion of D in TM is given by

$$(x^1, \ldots, x^n, w^1, \ldots, w^\sigma) \mapsto (x^1, \ldots, x^n, w^1, \ldots, w^\sigma, 0, \ldots, 0).$$

Thus the restriction of Z_g to D is tangent to D implying that D is geodesically invariant. ■

Now we present some results which relate to reductions of the structure group for linear connections. The first result we state is implicit in Proposition 8.2 and Theorem 8.4 in Chapter VII of Kobayashi and Nomizu [1963b].

4.2 Proposition: *Let $H(L(M))$ be a linear connection on $L(M)$, and let D be an integrable distribution on M. If $H(L(M))$ is reducible to a connection in $L(M, D)$ then D is totally geodesic. Conversely, if the connection is torsion-free, then $H(L(M))$ is reducible to a connection in $L(M, D)$ if D is totally geodesic.*

If the linear connection is the Levi-Civita connection on a Riemannian manifold $(M, \langle\cdot,\cdot\rangle)$, then we can make a stronger statement about the structure of the bundle of orthonormal frames in some cases.

4.3 THEOREM: *Let $(M, \langle\cdot,\cdot\rangle)$ be a Riemannian manifold and let D be a distribution of rank σ on M with $D^\perp$ its orthogonal complement. The following statements are equivalent:*

(i) D and $D^\perp$ are geodesically invariant;

(ii) the Levi-Civita connection in $O(M)$ is reducible to a connection in $O(M, D)$;

(iii) if ω is the connection one-form for the Levi-Civita connection, then ω restricted to $O(M, D)$ takes its values in $\mathfrak{o}(n,\sigma)$.

Proof: (i) $\Longrightarrow$ (ii) We will show that the integral curve of every horizontal vector field starting on $O(M, D)$ remains on $O(M, D)$. We do this by showing that a basis of standard horizontal vector fields has this property. Let $\Sigma(\boldsymbol{e}_a)$ be the standard horizontal vector field associated with the basis vector $\boldsymbol{e}_a$ for $a = 1, \ldots, \sigma$. Since $\boldsymbol{e}_a \in \mathbb{R}^\sigma \subset \mathbb{R}^n$, for $p \in O(M, D)$ we have $v = p\boldsymbol{e}_a \in D$. Since D is geodesically invariant, the integral curve of Z_g through v will remain on D. Let p' be a point along the integral curve of $\Sigma(\boldsymbol{e}_a)$ through p. By Proposition 2.3 and geodesic invariance of D we must have $p'\boldsymbol{e}_a \in D$. Thus $p' \in O(M, D)$ and so $\Sigma(\boldsymbol{e}_a)$ is tangent to $O(M, D)$. Now consider the basis element $\boldsymbol{e}_\alpha$ for $\alpha = \sigma+1, \ldots, n$. For $p \in O_x(M, D)$, $v = p\boldsymbol{e}_\alpha \in D_x^\perp$. An argument similar to the one above will show that $p' \in O(M, D)$ for every point p' along the integral curve of $\Sigma(\boldsymbol{e}_\alpha)$ through p. Thus $\Sigma(\boldsymbol{e}_\alpha)$ is tangent to $O(M, D)$ which proves this part of the theorem.

(ii) $\Longrightarrow$ (i) If we suppose that $H_p(O(M)) \subset T_p(O(M, D))$, then it is true that the integral curve of every horizontal vector whose initial condition lies on $O(M, D)$ will lie on $O(M, D)$. Let $v \in D_x$, let $p \in \pi^{-1}(x) \cap O(M, D)$, and define $\boldsymbol{u} = p^{-1}v$. Since $p \in O(M, D)$, $\boldsymbol{u} \in \mathbb{R}^\sigma$. By hypothesis, the integral curve of $\Sigma(\boldsymbol{u})$ through p will lie on $O(M, D)$. Therefore, if $p' \in O_x(M, D)$ then $p'\boldsymbol{u} \in D_{\pi(p')}$. By Proposition 2.3 this means that the geodesic with initial condition $v \in D_x$ will remain in D. Thus D is geodesically invariant. A similar argument shows that $D^\perp$ is geodesically invariant.

(ii) $\Longrightarrow$ (iii) This follows from Proposition 2.1.

(iii) $\Longrightarrow$ (ii) By Proposition 2.1 it is sufficient to find an $\mathrm{Ad}_{O(n,\sigma)}$-invariant complement of $\mathfrak{o}(n,\sigma)$ in $\mathfrak{o}(n)$. Consider the subspace, $\mathfrak{m}$, of $\mathfrak{o}(n)$ consisting of matrices of the form

$$\begin{pmatrix} 0 & a \\ -a^t & 0 \end{pmatrix}$$

where a is a $\sigma \times (n-\sigma)$ matrix and a^t is its transpose. We claim that $\mathfrak{m}$ is $\mathrm{Ad}_{O(n,\sigma)}$-invariant. Consider an element of $O(n,\sigma)$ of the form

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}.$$

Then, recalling that the inverse of an orthogonal matrix is its transpose, we compute

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} 0 & a \\ -a^t & 0 \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}^{-1} = \begin{pmatrix} 0 & AaB^t \\ -Ba^tA^t & 0 \end{pmatrix}.$$

The matrix on the right hand side is an element of $\mathfrak{m}$ which shows that $\mathfrak{m}$ is $\mathrm{Ad}_{O(n,\sigma)}$-invariant. It is also clear that $\mathfrak{m}$ is a complement to $\mathfrak{o}(n,\sigma)$ in $\mathfrak{o}(n)$. This completes the proof of the theorem. ∎

4.4 REMARKS: 1. We remark that it is not necessary that the orthogonal complement of a geodesically invariant distribution on a Riemannian manifold be geodesically invariant. Counterexamples are readily constructed.

2. It is easy to construct examples of totally geodesic distributions whose orthogonal complement is also totally geodesic. One may ask if this is the only situation in which Theorem 4.3 applies. We shall see in Section 5 that it is not. □

5. Geodesically Invariant Distributions on Lie Groups

An affine connection on a Lie group G is said to be *left-invariant* if $\nabla_X Y(g) = T_e L_g(\nabla_X Y)(e)$ for every pair of left-invariant vector fields X and Y. Here $L_g h = gh$. We refer to Helgason [1978] for a discussion of left-invariant affine connections. If ∇ is a left-invariant affine connection on G, we denote by $\overline{\nabla}$ the restriction of ∇ to $\mathfrak{g}$, the Lie algebra of G. Thus, for $\xi, \eta \in \mathfrak{g}$, $\overline{\nabla}_\xi \eta = T_g L_{g^{-1}}(\nabla_{T_e L_g \xi} T_e L_g \eta)$ for any $g \in G$. Since ∇ is left-invariant, this definition makes sense. Observe that $\overline{\nabla}$ is simply a bilinear product on $\mathfrak{g}$. We may define the *symmetric product on* $\mathfrak{g}$ by $\langle \xi : \eta \rangle_{\mathfrak{g}} = \overline{\nabla}_\xi \eta + \overline{\nabla}_\eta \xi$.

Now we characterise geodesically invariant distributions on G. We shall say that a subspace $\mathfrak{m}$ of $\mathfrak{g}$ is *symmetric* if $\langle \xi : \eta \rangle_{\mathfrak{g}} \in \mathfrak{m}$ for each $\xi, \eta \in \mathfrak{m}$. We may state the form of left-invariant geodesically invariant distributions on Lie groups.

5.1 PROPOSITION: *Let ∇ be a left-invariant affine connection on a Lie group G, and denote by $\overline{\nabla}$ the associated bilinear product on $\mathfrak{g}$. A left-invariant distribution D on G is geodesically invariant if and only if $D_e \subset \mathfrak{g}$ is symmetric.*

Proof: Let $\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n\}$ be a basis for $\mathfrak{g}$ so that $\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_\sigma\}$ is a basis for D_e. We denote by E_a, $a = 1, \ldots, \sigma$, the left-invariant extension of $\boldsymbol{e}_a$. If X and Y are sections of D, since D is left-invariant we may write

$$X(g) = X^a(g) E_a(g), \qquad Y(g) = Y^a(g) E_a(g)$$

for functions $X^a(g)$ and $Y^a(g)$, $a = 1, \ldots, \sigma$.

Now suppose that D_e is symmetric. Then we have

$$\langle \boldsymbol{e}_a : \boldsymbol{e}_b \rangle = \gamma^c_{ab} \boldsymbol{e}_c$$

for some constants γ^c_{ab}, $a, b, c = 1, \ldots, \sigma$. Using the derivation properties of affine connections we obtain

$$\begin{aligned}\langle X : Y \rangle(g) =& \nabla_{X^a E_a} Y^b E_b(g) + \nabla_{Y^a E_a} X^b E_b(g) \\ =& X^a(g) Y^b(g) \nabla_{E_a} E_b(g) + (\mathscr{L}_X Y^b(g)) E_b(g) + \\ & \quad Y^a(g) X^b(g) \nabla_{E_a} E_b(g) + (\mathscr{L}_Y X^b(g)) E_b(g) \\ =& \gamma^c_{ab} X^a(g) Y^b(g) T_e L_g \boldsymbol{e}_c + (\mathscr{L}_X Y^b(g)) E_b(g) + (\mathscr{L}_Y X^b(g)) E_b(g).\end{aligned}$$

Thus $\langle X : Y \rangle(g) \in D_g$ and so D is geodesically invariant by Theorem 4.1.

Now suppose that D is geodesically invariant. Then, for $a, b = 1, \ldots, \sigma$,

$$\langle E_a : E_b \rangle (g) = Z^c(g) E_c(g)$$

for some functions $Z^c(g)$, $c = 1, \ldots, \sigma$, on G. Now define γ^i_{ab}, $i = 1, \ldots, n$, by

$$\langle \boldsymbol{e}_a : \boldsymbol{e}_b \rangle = \gamma^i_{ab} \boldsymbol{e}_i.$$

We then have

$$T_e L_g(\gamma^i_{ab} \boldsymbol{e}_i) = T_e L_g(Z^c(g) \boldsymbol{e}_c).$$

This implies that $\gamma^\alpha_{ab} = 0$ for $\alpha = \sigma + 1, \ldots, n$ and so D_e is symmetric. ■

From this proposition it is clear that we may generate a class of left-invariant, geodesically invariant distributions on G by left translating symmetric subspaces of $\mathfrak{g}$. It turns out that there are geodesically invariant distributions on Lie groups which are *not* left-invariant. However, we shall not discuss this here.

We will, however, resolve a question that was raised in Remark 4.4.2. Towit, we will show that there exists a non-integrable distribution, D, on a Riemannian manifold, $(M, \langle \cdot, \cdot \rangle)$, so that both D and $D^\perp$ are geodesically invariant. This turns out to be easy to do for Lie groups and relies on the following result which follows easily from Proposition 5.1.

5.2 Lemma: *Let $\langle \cdot, \cdot \rangle$ be a left-invariant Riemannian metric on a Lie group G and consider the corresponding left-invariant affine connection on G. A left-invariant distribution D on G has the property that D and $D^\perp$ are geodesically invariant if and only if $D_e, D_e^\perp \subset \mathfrak{g}$ are symmetric.*

Now we suppose that G is a compact Lie group with $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ the inner product on $\mathfrak{g}$ which is the negative of the Killing form. Since G is compact, $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ *is* an inner product and may be extended to a left-invariant Riemannian metric $\langle \cdot, \cdot \rangle$ on G. It is well-known that for this metric the bilinear product $\overline{\nabla}$ is given by

$$\overline{\nabla}_\xi \eta = \frac{1}{2}[\xi, \eta]_{\mathfrak{g}}$$

where $[\cdot, \cdot]_{\mathfrak{g}}$ is the Lie algebra bracket on $\mathfrak{g}$. We refer the reader to Helgason [1978] for results of this type. It is clear that the induced symmetric product on $\mathfrak{g}$ for this metric is zero. Therefore, every subspace of $\mathfrak{g}$ is symmetric. Thus, if $\mathfrak{m} \subset \mathfrak{g}$ is not a subalgebra, then the distribution D, which is the left-invariant extension of $\mathfrak{m}$, has the following properties:

1. both D and $D^\perp$ are geodesically invariant;
2. D is not integrable.

This then provides a proof of the following answer to the question of Remark 4.4.2.

5.3 Proposition: *There exists a Riemannian manifold $(M, \langle \cdot, \cdot \rangle)$ and a non-integrable distribution D on M such that the Levi-Civita connection in $O(M)$ is reducible to a connection in $O(M, D)$.*

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