Decentralized Implementation of Centralized Controllers for Interconnected Systems

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Abstract

Given a centralized controller associated with a linear time-invariant interconnected system, this paper is concerned with designing a parameterized decentralized controller such that the state and input of the system under the obtained decentralized controller can become arbitrarily close to those of the system under the given centralized controller, by tuning the controller's parameters. To this end, a two-level decentralized controller is designed, where the upper level captures the dynamics of the centralized closed-loop system, and the lower level is an observed-based sub-controller designed based on the new notion of *structural initial value observability*. The proposed method can decentralize every generic centralized controller, provided the interconnected system satisfies very mild conditions. The efficacy of this work is elucidated by some numerical examples.

I. INTRODUCTION

Many real-world systems such as communication networks, large-space structures, power systems, and chemical processes can be modeled as interconnected systems with homogeneous or heterogeneous interacting subsystems [1], [2], [3], [4], [5]. The classical control techniques often fail to control such systems, in light of some well-known computation or communication constraints. This has given rise to the emergence of the decentralized control area that aims to design non-classical structurally constrained controllers [6]. A decentralized controller comprises a set of non-interacting local controllers corresponding to disparate subsystems. The analysis and synthesis of a decentralized control system has long been studied by many researchers. In particular, the decentralized control theory has been recently developed for systems with geographically distributed subsystems in the context of distributed control for diverse applications, such as flight formation [7], consensus [8], [9] and Internet congestion control [10].

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To study the decentralized stabilization problem, the notion of decentralized fixed modes (DFM) was introduced in [11] to characterize those modes of a system that cannot be moved using a linear time-invariant (LTI) decentralized controller. Several methods have been proposed in the literature to find the DFMs of a system [12], [13], [14]. The notion of quotient fixed modes (QFM) was subsequently introduced in [15] to investigate the stabilizability of a system with respect to all (nonlinear and time-varying) decentralized controllers.

Although there has been a plethora of research on finding the best achievable decentralized performance, several related problems are still open or partially solved [6], [16], [17], [18], [19]. The main reason is that while many control problems, such as H_2 or H_∞ optimal controller design, have explicit solutions in the centralized case, they are cumbersome and generally nonconvex when restricted to the class of decentralized controllers. The work [20] provides a lower bound on the achievable decentralized H_2 performance for stable discrete-time linear systems with stable finite zeros. The problem of designing a decentralized controller that achieves certain H_∞ requirements on all subsystems as well as the overall system is tackled in [21], where sufficient conditions are derived. The papers [22] and [23] obtain sufficient nonlinear conditions for the existence of a stabilizing decentralized controller with a guaranteed H_∞ performance (the sensors and actuators are also allowed to fail in [22]). To find the best achievable decentralized H_∞ performance, an infinite-dimensional optimization problem is proposed in [24] based on the parameterization of all decentralized stabilizing controllers, and it is then truncated to a nonconvex finite-dimensional optimization problem. The existence of a decentralized controller providing certain closed-loop properties for a stable system is studied in [25]. A closely related decentralized control problem is also tackled in [26], where time-domain performance limitations are obtained for open-loop stable, square, linear systems, which can be used to bound the settling time and undershoot in decentralized architecture control schemes. Sufficient conditions are derived in [18] to make the decentralized H_2 optimal control problem convex.

The above-mentioned decentralized control problems can be asked in a broader, unified context as follows: given a centralized controller associated with a system, what is the best decentralized controller that generates state and input trajectories for the system that are sufficiently close to those generated by the prescribed centralized controller? This question is partially answered in the literature. The paper [27] proposes a technique to design a static decentralized controller in terms of a prescribed centralized one, but the centralized and decentralized closed-loop performances can be very different. The work [28] aims to design a decentralized controller based on a given centralized controller such that the associated sensitivity functions are close to each other. In order to make the problem convex, that work minimizes a weighted H_2 error whose weighting factor is forced to be dependent on the parameters of the unknown decentralized controller, which leads to an undesirable performance measure. The paper [29] derives sufficient conditions for the approximation of a static centralized controller by a decentralized controller, which requires solving a set of nonlinear algebraic equations. The recent work [19] proposes a method to implement a given centralized controller in a decentralized fashion, which is successfully applied to the flight formation problem in [4]. However, it is not guaranteed that the system behaves similarly under both the given centralized controller and its decentralized counterpart. The primary objective of the current paper is to address the aforementioned decentralization question for strongly connected LTI systems, i.e. those LTI systems whose subsystems cannot be renumbered in such a way that the corresponding transfer function matrix becomes upper/lower block triangular.

Given a centralized controller for a strongly connected LTI system, the objective of this paper is to study the existence of a decentralized controller such that the input and state trajectories of the system under this decentralized controller are arbitrarily close to those of the system under the given centralized controller. To this end, it is shown that under mild conditions, there exists such a decentralized controller composed of high-level and low-level decentralized sub-controllers. The control law of the high-level sub-controller is given explicitly, but the low-level sub-controller is designed based on a new notion of *structural initial value observability*. The developed method is then applied to the optimal LQR decentralized control problem through an example. The problem studied in this work encompasses the ones investigated in [4], [19], [27], [29], and is also related to different problems surveyed earlier on achievable decentralized performance. However, unlike the aforementioned works, the present paper does not derive nonconvex or conservative sufficient conditions. Instead, it shows that every generic centralized controller can be approximated arbitrarily well by a two-level decentralized controller, provided the system satisfies certain mild (easy-to-check) conditions.

The rest of the paper is organized as follows. The problem is formulated in Section II, and the main results are given accordingly in Section III. An illustrative example is provided in Section IV. Concluding remarks are drawn in Section V. Some proofs and further discussions are presented in Appendices 1 and 2.

II. Problem formulation

Consider an interconnected system $\mathcal{S}$ composed of ν interacting LTI subsystems $S_1, S_2, ..., S_\nu$. Let the system $\mathcal{S}$ be governed by the differential equation

$$\begin{aligned} \dot{x}(t) &= Ax(t) + \sum_{i=1}^{\nu} B_i u_i(t), \\ y_j(t) &= C_j x(t), \quad \forall j \in \boldsymbol{\nu} := \{1, 2, ..., \nu\}, \end{aligned} \tag{1}$$

where $x(t) \in \mathbf{R}^n$ represents the state of the system $\mathcal{S}$, and $u_j(t) \in \mathbf{R}^{m_j}$ and $y_j(t) \in \mathbf{R}^{r_j}$ are the input and output of the subsystem S_j, respectively, for every $j \in \boldsymbol{\nu}$. Denote the initial state $x(0)$ with x_0 and assume that x_0 is a random variable. Define

$$\begin{aligned} &B := \begin{bmatrix} B_1 & B_2 & \cdots & B_\nu \end{bmatrix}, \quad C := \begin{bmatrix} C_1^T & C_2^T & \cdots & C_\nu^T \end{bmatrix}^T, \\ &m := m_1 + m_2 + \cdots + m_\nu, \quad r := r_1 + r_2 + \cdots + r_\nu. \end{aligned} \tag{2}$$

Note that although the matrices B and C are block-diagonal for many applications especially when each subsystem has its own sub-state (such as in flight formation), these matrices are considered to be general and unconstrained in the present work. Consider a given stabilizing centralized controller K_c of order n_o with the control law

$$\begin{aligned} \dot{z}(t) &= A_o z(t) + B_o y(t), \\ u(t) &= C_o z(t) + D_o y(t), \end{aligned} \tag{3}$$

where $z(0) = 0$. In the centralized closed-loop system obtained by applying the controller K_c to the system $\mathcal{S}$, represent different signals as follows:

- Let $x_c(t)$, $u_c(t)$ and $y_c(t)$ denote the state, input and output of the system $\mathcal{S}$, respectively.
- Let $z_c(t)$ denote the state of the controller K_c.
- Let $u_{c_i}(t)$ and $y_{c_i}(t)$ denote the input and output of the subsystem S_i, respectively, for every $i \in \boldsymbol{\nu}$.

It is desired to investigate whether there exists a decentralized controller K_d such that the system $\mathcal{S}$ under K_d generates state and input trajectories sufficiently close to the centralized trajectories $x_c(t)$ and $u_c(t)$, respectively. To this end, a new notion will be introduced in the sequel.

Definition 1: The controller K_c is said to be *decentrally implementable* if there exists a natural number μ and a stabilizing linear decentralized controller $K_d(\xi)$ parameterized in terms of a multivariate parameter $\xi \in \mathbf{R}^{\mu}$ such that for every given positive reals ε and Δ, there exists a vector $\xi_0 \in \mathbf{R}^{\mu}$ for which the relation

$$\int_{\varepsilon}^{\infty} \left(\|x_c(t) - x_d(t)\|^2 + \|u_c(t) - u_d(t)\|^2\right) dt < \Delta \tag{4}$$

holds if $x_d(t)$ and $u_d(t)$ denote the state and the input of the system $\mathcal{S}$ under the controller $K_d(\xi_0)$, respectively, and $\|\cdot\|$ represents an arbitrary vector norm.

Regarding the parameterized controller $K_d(\xi)$ in Definition 1, a switching-type nonlinear controller $K_d(\xi)$ can be designed using the technique proposed in [30] to make the inequality (4) hold; however, this definition requires $K_d(\xi)$ to be linear due to the linearity of the original controller K_c. It is noteworthy that the notion of *decentralized implementation* is instrumental in understating the gap between the achievable centralized and decentralized performances.

The objective of this paper is twofold. First, it is desired to prove that the given controller K_c is decentrally implementable under mild conditions. Second, it is aimed to construct a parameterized controller $K_d(\xi)$ associated with K_c.

III. MAIN RESULTS

Denote the modes of the system $\mathcal{S}$ with $\lambda_1, \lambda_2, ..., \lambda_n$. Define the structural graph of the system $\mathcal{S}$ to be a directed graph with ν vertices such that for every $i, j \in \boldsymbol{\nu},\ i \neq j$, vertex i is connected to vertex j by a directed edge if the transfer function $C_j(sI - A)^{-1}B_i$ is nonzero. The system $\mathcal{S}$ is said to be *strongly connected* if its associated structural graph is strongly connected, meaning that there exist directed paths from every vertex to all remaining vertices of the graph [15]. A few technical assumptions are required for the development of this paper, as provided below.

Assumption 1: The system $\mathcal{S}$ has no decentralized fixed mode, i.e., it is controllable, observable and the inequality

$$\text{rank}\begin{bmatrix} A - \lambda_i I & B_{j_1} & \dots & B_{j_p} \\ C_{j_{p+1}} & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ C_{j_\nu} & 0 & \dots & 0 \end{bmatrix} \geq n, \qquad \forall i \in \mathbf{n} := \{1, 2, ..., n\} \tag{5}$$

holds for every permutation $(j_1, j_2, ..., j_\nu)$ of the set $\{1, 2, ..., \nu\}$ and $p \in \{1, ..., \nu - 1\}$ [13].

Assumption 2: The system $\mathcal{S}$ is strongly connected.

Assumption 3: The inequality $m_j \leq r$ holds for all $j \in \boldsymbol{\nu}$ and, in addition, there exists a matrix $M \in \mathbf{R}^{(\max_{i\in\nu} m_i)\times r}$ such that

$$\begin{bmatrix} A-\lambda_i I & B \\ MC & 0 \end{bmatrix} = \text{full rank}, \quad \forall i \in \mathbf{n}. \tag{6}$$

Consider a decentralized controller K_d with ν local controllers $K_{d_1}, K_{d_2}, ..., K_{d_\nu}$, where the local controller K_{d_i} ($\forall i \in \boldsymbol{\nu}$) receives $y_i(t)$ as its input to generate $u_i(t)$ as its output. Let K_d be an interconnection of two <u>linear</u> decentralized sub-controllers K_d^1 and K_d^2 with the sets of local controllers $\{K_{d_1}^1, K_{d_2}^1, ..., K_{d_\nu}^1\}$ and $\{K_{d_1}^2, K_{d_2}^2, ..., K_{d_\nu}^2\}$, respectively, such that $K_{d_i}^1$ generates the input signal $u_i(t)$ and another signal $y_{d_i}(t)$ in terms of $y_i(t)$ and $u_{d_i}(t)$, where $u_{d_i}(t)$ and $y_{d_i}(t)$ are the respective output and input of $K_{d_i}^2$, for every $i \in \boldsymbol{\nu}$. This implies that K_d can be regarded as a two-level decentralized controller with the high-level sub-controller K_d^1 and the low-level sub-controller K_d^2. The topology of the controller K_d is illustrated in Figure 1 for the particular case $\nu = 2$, which shows that the high-level controller K_d^1 interacts directly with the system $\mathcal{S}$ while the low-level controller K_d^2 can only communicate with the high-level controller K_d^1. Let the local controller $K_{d_i}^1$, $\forall i \in \boldsymbol{\nu}$, be as follows

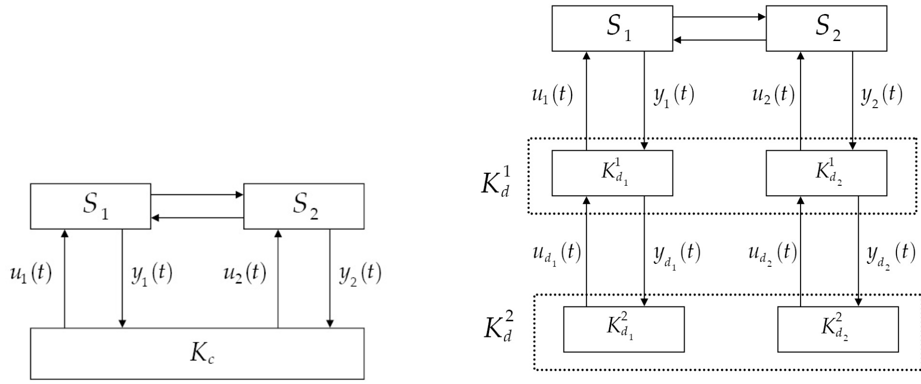
$$\begin{bmatrix} \dot{x}_{d_i}(t) \\ \dot{z}_{d_i}(t) \end{bmatrix} = \begin{bmatrix} A+BD_oC & BC_o \\ B_oC & A_o \end{bmatrix} \begin{bmatrix} x_{d_i}(t) \\ z_{d_i}(t) \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u_{d_i}(t), \tag{7a}$$

$$y_{d_i}(t) = y_i(t) - \begin{bmatrix} C_i & 0 \end{bmatrix} \begin{bmatrix} x_{d_i}(t) \\ z_{d_i}(t) \end{bmatrix}, \tag{7b}$$

$$\begin{aligned} u_i(t) = & \begin{bmatrix} 0_{m_i\times m_1} & \cdots & 0_{m_i\times m_{i-1}} & I_{m_i} & 0_{m_i\times m_{i+1}} & \cdots & 0_{m_i\times m_\nu} \end{bmatrix} \\ & \times \begin{bmatrix} D_oC & C_o \end{bmatrix} \begin{bmatrix} x_{d_i}(t) \\ z_{d_i}(t) \end{bmatrix}, \end{aligned} \tag{7c}$$

where $x_{d_i}(t) \in \mathbf{R}^n$ and $z_{d_i}(t) \in \mathbf{R}^{n_o}$ together form the state vector of $K_{d_i}^1$ (note that $0_{\mu_1\times\mu_2}$ denotes a zero matrix of dimension $\mu_1 \times \mu_2$ in this paper, for every natural numbers μ_1 and μ_2). To comprehend the basic idea behind the above control law, the configuration of the system $\mathcal{S}$ under the introduced decentralized controller K_d is sketched in Figure 2 for the particular case $\nu = 2$. Before designing the low-level decentralized sub-controller K_d^2, it is essential to understand why K_d^1 is defined as such.

Theorem 1: Consider the decentralized controller K_d with the high-level sub-controller K_d^1 given in (7) and an arbitrary low-level linear sub-controller K_d^2. If the initial state of the high-level



(a) (b)

Fig. 1. (a): The block diagram of the system $\mathcal{S}$ under the centralized controller K_c (assuming $\nu = 2$); (b): The block diagram of the system $\mathcal{S}$ under the two-level decentralized controller K_d (assuming $\nu = 2$).

decentralized sub-controller K_d^1 is taken as

$$Z_0 := \Big[\underbrace{\left[\begin{array}{cc} x_0^T & 0_{1\times n_0}\end{array}\right]\left[\begin{array}{cc} x_0^T & 0_{1\times n_0}\end{array}\right]\cdots\left[\begin{array}{cc} x_0^T & 0_{1\times n_0}\end{array}\right]}_{\nu \text{ times}}\Big]^T \tag{8}$$

and the state of K_d^2 is initialized as zero, then the state and the input of the system $\mathcal{S}$ under the decentralized controller K_d will be identical to $x_c(t)$ and $u_c(t)$, respectively.

Proof: This theorem can be proved in line with the state-space approach proposed in the proof of Theorem 1 in [19]. However, an alternative method will be pursued here based on a well-known technique for circuit analysis. First, consider the system $\mathcal{S}$ under the high-level sub-controller K_d^1 and assume that the low-level sub-controller K_d^2 does not exist (i.e. $u_{d_1}(t) = \cdots = u_{d_\nu}(t) = 0$). Since $u_{d_1}(t), ..., u_{d_\nu}(t)$ are equal to zero for all $t \geq 0$, it follows from (7a), (7c) and the assumption

$$\left[\begin{array}{cc} x_{d_i}(0)^T & z_{d_i}(0)^T \end{array}\right] = \left[\begin{array}{cc} x_0^T & 0_{1\times n_o}\end{array}\right], \quad \forall i \in \boldsymbol{\nu} \tag{9}$$

that

$$x_{d_i}(t) = x_c(t), \quad z_{d_i}(t) = z_c(t), \quad u_i(t) = u_{c_i}(t), \quad \forall t \geq 0,\ i \in \boldsymbol{\nu}. \tag{10}$$

This means that the input $u(t)$ of the system $\mathcal{S}$ is equal to $u_c(t)$, which makes the state of the system be equal to $x_c(t)$, i.e.,

$$x(t) = x_c(t), \quad u(t) = u_c(t), \quad \forall t \geq 0. \tag{11}$$

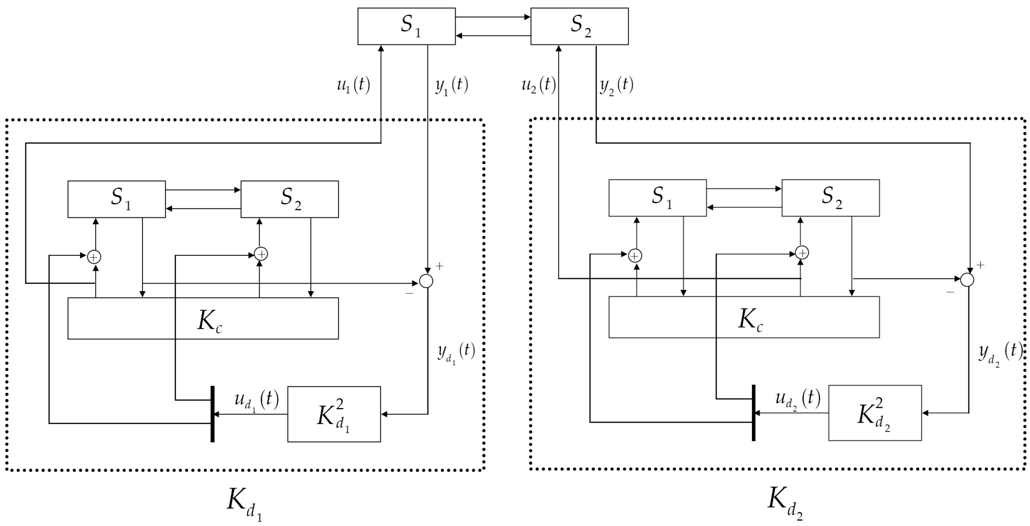


Fig. 2. The detailed block-diagram of the system under the decentralized controller K_d (assuming $\nu = 2$).

The above relations yield that $y(t) = y_c(t)$, which can be combined with (7b) to obtain $y_{d_i}(t) = 0$ for every $i \in \boldsymbol{\nu}$. Now, notice that the signals $u_{d_1}(t), ..., u_{d_\nu}(t)$ and $y_{d_1}(t), ..., y_{d_\nu}(t)$ are all equal to zero. As a result, if a <u>linear</u> controller K_d^2 with zero initial state is augmented to the system $\mathcal{S}$ under K_d^1 from the inputs $u_{d_1}(t), ..., u_{d_\nu}(t)$ to the outputs $y_{d_1}(t), ..., y_{d_\nu}(t)$, it will not change the zero values of these signals. In other words, the linear controller K_d^2 has no effect on the signals of the system $\mathcal{S}$ and the controller K_d^1. The proof now follows immediately from (11). ■

Given arbitrary vectors $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}$ of appropriate dimensions, let $x_d(t; \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$ denote the state of the system $\mathcal{S}$ under the decentralized controller K_d provided the initial states of $\mathcal{S}$, K_d^1 and K_d^2 are taken as $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$, respectively. Likewise, define the input signal $u_d(t; \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$. Theorem 1 states that

$$x_d(t; x_0, Z_0, 0) = x_c(t), \quad u_d(t; x_0, Z_0, 0) = u_c(t), \quad \forall t \geq 0, \tag{12}$$

for every arbitrary low-level linear sub-controller K_d^2. This implies that if the initial state of K_d^1 is taken as Z_0, then the left side of the inequality (4) is equal to zero for the above-defined controller K_d, which would make the centralized controller K_c decentrally implementable. Nevertheless, since the initial state x_0 is unknown by assumption and Z_0 is based on x_0, this initialization of K_d^1 is infeasible. Instead, the high-level controller K_d^1 can be initialized as zero, which leads to the state $x_d(t; x_0, 0, 0)$ and the input $u_d(t; x_0, 0, 0)$ for the system $\mathcal{S}$. Now, it follows from the linearity of the controller K_d and the relations given in (12) that

$$\begin{aligned}&\int_{\varepsilon}^{\infty}\left(\|x_c(t)-x_d(t;x_0,0,0)\|^2+\|u_c(t)-u_d(t;x_0,0,0)\|^2\right)dt=\\&=\int_{\varepsilon}^{\infty}\left(\|x_d(t;x_0,Z_0,0)-x_d(t;x_0,0,0)\|^2+\|u_d(t;x_0,Z_0,0)-u_d(t;x_0,0,0)\|^2\right)dt\\&=\int_{\varepsilon}^{\infty}\left(\|x_d(t;0,Z_0,0)\|^2+\|u_d(t;0,Z_0,0)\|^2\right)dt.\end{aligned} \tag{13}$$

Hence, to prove that the centralized controller K_c is decentrally implementable for the system $\mathcal{S}$ with an unknown initial state x_0, it is enough to solve the next problem.

Problem 1: For every given positive numbers ε and Δ, design a low-level decentralized sub-controller K_d^2 such that when the initial states of $\mathcal{S}$, K_d^1 and K_d^2 are taken as 0, Z_0 and 0, respectively, then the state and input of the system $\mathcal{S}$ under the decentralized controller K_d satisfy the relation

$$\int_{\varepsilon}^{\infty}\left(\|x_d(t;0,Z_0,0)\|^2+\|u_d(t;0,Z_0,0)\|^2\right)dt<\Delta. \tag{14}$$

Augment the system $\mathcal{S}$ with the high-level decentralized sub-controller K_d^1 to obtain an interconnected system $\tilde{\mathcal{S}}$ with ν subsystems, where the input and output of its i^{th} subsystem are $u_{d_i}(t)$ and $y_{d_i}(t)$, respectively, for every $i \in \boldsymbol{\nu}$ (this augmentation may be observed in Figure 1). Problem 1 stated above amounts to designing a linear decentralized controller K_d^2 for the system $\tilde{\mathcal{S}}$ with the initial state $\begin{bmatrix}0_{1\times n} & Z_0^T\end{bmatrix}^T$ such that the state of the system is regulated arbitrarily fast. Although a state regulation problem is normally easy-to-solve for a controllable and observable system, it will be shown in the sequel that the system $\tilde{\mathcal{S}}$ is not observable.

Denote the eigenvalues of the centralized closed-loop system (i.e. the system $\mathcal{S}$ under the controller K_c) with $\bar{\lambda}_1, \bar{\lambda}_2, ..., \bar{\lambda}_{n+n_o}$. Besides, let Φ be an $(n+n_o)$-dimensional subspace defined as

$$\Phi := \left\{\left[\boldsymbol{\eta}\ \underbrace{\left[\begin{array}{cc}\boldsymbol{\eta} & \boldsymbol{\gamma}\end{array}\right]\ \left[\begin{array}{cc}\boldsymbol{\eta} & \boldsymbol{\gamma}\end{array}\right]\ \cdots\ \left[\begin{array}{cc}\boldsymbol{\eta} & \boldsymbol{\gamma}\end{array}\right]}_{\nu \text{ times}}\right]^T \,\middle|\, \boldsymbol{\eta}\in\mathbf{R}^{1\times n},\ \boldsymbol{\gamma}\in\mathbf{R}^{1\times n_o}\right\} \tag{15}$$

Theorem 2: There exists a nonzero multivariate polynomial $F : \mathbf{R}^{n_o\times n_o}\times\mathbf{R}^{n_o\times r}\times\mathbf{R}^{m\times n_o}\times\mathbf{R}^{m\times r}\to\mathbf{R}$ (whose coefficients are dependent only on the parameters of the system $\mathcal{S}$ rather than K_c) such that the following statements hold if $F(A_o, B_o, C_o, D_o)\neq 0$:

i) The system $\tilde{\mathcal{S}}$ is strongly connected.

ii) The modes of the system $\tilde{\mathcal{S}}$ are

$$\left(\bigcup_{j=1}^{n+n_o}\{\underbrace{\bar{\lambda}_j, ..., \bar{\lambda}_j}_{\nu \text{ times}}\}\right)\bigcup\{\lambda_1, ..., \lambda_n\} \tag{16}$$

(the multiplicities of λ_i and $\bar{\lambda}_j$ in the above set are 1 and ν, respectively, for every $i \in \mathbf{n}$ and $j \in \{1, 2, ..., n + n_o\}$).

iii) The system $\tilde{\mathcal{S}}$ is controllable.

iv) The system $\tilde{\mathcal{S}}$ is unobservable with the unobservable subspace Φ corresponding to the $n + n_0$ unobservable modes $\bar{\lambda}_1, ..., \bar{\lambda}_{n+n_o}$ (each of these repeated modes is only one time unobservable).

v) The mode λ_j is not a DFM of the system $\tilde{\mathcal{S}}$, for every $j \in \mathbf{n}$.

vi) The repeated mode $\bar{\lambda}_j$ with multiplicity ν is a single DFM of the system $\tilde{\mathcal{S}}$, for every $j \in \{1, 2, ..., n + n_o\}$.

Proof: The proof of this theorem is provided in Appendix 1. ■

Several properties of the system $\tilde{\mathcal{S}}$ have been derived in Theorem 2 under the condition $F(A_o, B_o, C_o, D_o) \neq 0$. A question arises: how can the validity of this condition be checked? A method is proposed in Appendix 2 to find the multivariate polynomial $F(A_o, B_o, C_o, D_o)$. However, this polynomial has a complicated structure with several monomials, in general. Hence, it may not be useful to find this polynomial explicitly. In contrast, one can argue that given a fixed number n_o, almost all (generic) finite-dimensional LTI controllers K_c satisfy the property $F(A_o, B_o, C_o, D_o) \neq 0$; more precisely, the set of controllers K_c for which this condition is violated forms a set of measure zero. The reason is that the set of real zeros of a (nonzero) polynomial is a hypersurface (real algebraic variety) with a positive codimension. Therefore, instead of checking the condition $F(A_o, B_o, C_o, D_o) \neq 0$ directly for a given controller K_c, it is easier to verify whether properties (i)-(vi) stated in Theorem 2 hold. Based on the aforementioned discussion, if any of these properties is violated, then one can randomly perturb the parameters A_0, B_0, C_0, D_o arbitrarily small to obtain a perturbed controller for which the properties given in Theorem 2 hold with probability 1.

Assume henceforth that the condition $F(A_o, B_o, C_o, D_o) \neq 0$ is satisfied for the centralized controller K_c. To design the low-level decentralized sub-controller K_d^2, let its first $\nu - 1$ local controllers be simply static feedbacks, while its last local controller may possibly be a dynamic controller. For this purpose, given arbitrary gains $G_i \in \mathbf{R}^{m \times r_i}$, $\forall i \in \{1, 2, ..., \nu - 1\}$, let $\tilde{\mathcal{S}}(G_1, ..., G_{\nu-1})$ denote the system $\tilde{\mathcal{S}}$ under the following local controllers $K_{d_1}^2, ..., K_{d_{\nu-1}}^2$:

$$u_{d_i}(t) = G_i y_{d_i}(t), \quad \forall i \in \{1, 2, ..., \nu - 1\}, \tag{17}$$

where the input and output of $\tilde{\mathcal{S}}(G_1, ..., G_{\nu-1})$ are $u_{d_\nu}(t)$ and $y_{d_\nu}(t)$, respectively.

Theorem 3: There exists a multivariate polynomial $\bar{F} : \mathbf{R}^{m\times r_1} \times \cdots \times \mathbf{R}^{m\times r_{\nu-1}} \to \mathbf{R}$ such that for every given gains $G_i \in \mathbf{R}^{m\times r_i}$, $\forall i \in \{1, 2, ..., \nu-1\}$, if $\bar{F}(G_1, G_2, ..., G_{\nu-1}) \neq 0$, then the system $\tilde{\mathcal{S}}(G_1, ..., G_{\nu-1})$ satisfies the following properties:

i) $\tilde{\mathcal{S}}(G_1, ..., G_{\nu-1})$ is controllable.

ii) $\tilde{\mathcal{S}}(G_1, ..., G_{\nu-1})$ is unobservable with the unobservable subspace Φ.

Proof: Since the system $\tilde{\mathcal{S}}$ is unobservable (due to Theorem 2), consider a minimal realization of this system and denote it with $\bar{\mathcal{S}}$. Define the system $\bar{\mathcal{S}}(G_1, ..., G_{\nu-1})$ in the same way that $\tilde{\mathcal{S}}(G_1, ..., G_{\nu-1})$ was defined. It follows from Theorem 2 that the interconnected system $\bar{\mathcal{S}}$ is strongly connected, controllable, observable, and has no DFMs. Hence, it can be concluded from [31] and [32] that there exists a polynomial $\bar{F} : \mathbf{R}^{m\times r_1} \times \cdots \times \mathbf{R}^{m\times r_{\nu-1}} \to \mathbf{R}$ such that the system $\bar{\mathcal{S}}(G_1, ..., G_{\nu-1})$ is both controllable and observable for every set of gains $\{G_1, G_2, ..., G_{\nu-1}\}$ satisfying the relation $\bar{F}(G_1, G_2, ..., G_{\nu-1}) \neq 0$. The proof of this theorem is a consequence of this property of the system $\bar{\mathcal{S}}(G_1, ..., G_{\nu-1})$ and the following facts:

- $\bar{\mathcal{S}}$ is a reduced-order observable realization of the system $\tilde{\mathcal{S}}$.
- The unobservable subspace of the system $\tilde{\mathcal{S}}$ is shown in Theorem 2 to be Φ. ■

The discussion given right after Theorem 2 concerning the polynomial F is applicable to the polynomial $\bar{F}$ as well. In other words, even though the polynomial $\bar{F}$ can be found explicitly, one can argue that the condition $\bar{F}(G_1, G_2, ..., G_{\nu-1}) \neq 0$ holds for a generic choice of gains $G_1, G_2, ..., G_{\nu-1}$. From now on, let $\{G_1, G_2, ..., G_{\nu-1}\}$ denote a specific set of gains for which properties (i) and (ii) mentioned in Theorem 3 hold for the system $\tilde{\mathcal{S}}(G_1, ..., G_{\nu-1})$.

Recall from Problem 1 and its subsequent discussion that the decentralized implementability of the centralized controller K_c is guaranteed by designing a local controller $K_{d_\nu}^2$ for the system $\tilde{\mathcal{S}}(G_1, ..., G_{\nu-1})$ with the initial state $\begin{bmatrix} 0_{1\times n} & Z_0^T \end{bmatrix}^T$ so that the state of the closed-loop system is regulated arbitrarily fast. To address the latter problem, note that although the system $\tilde{\mathcal{S}}(G_1, ..., G_{\nu-1})$ is not observable (due to Theorem 3), the unobservable modes of the system are all stable. This implies that one can design an observer-based controller $K_{d_\nu}^2$ for the system $\tilde{\mathcal{S}}(G_1, ..., G_{\nu-1})$ to make its state attenuate to zero. However, the question of interest is to design an *arbitrarily fast* regulating controller for this unobservable system. This question is addressed in the sequel through the introduction of a new notion.

A. Structural initial value observer

Consider an LTI continuous-time system $\mathbf{S}$ with the state-space representation

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t), \end{aligned} \tag{18}$$

where $\mathbf{x}(t) \in \mathbf{R}^{\bar{n}},\ \mathbf{u}(t) \in \mathbf{R}^{\bar{m}}$ and $\mathbf{y}(t) \in \mathbf{R}^{\bar{r}}$. Assume that Γ is a given subspace of $\mathbf{R}^{\bar{n}}$.

Definition 2: An initial state $\mathbf{x}_0 \in \Gamma$ is said to be *structural initial value observable with respect to* Γ (SIV observable w.r.t. Γ) if there does not exist another initial state $\mathbf{x}'_0$ in the space Γ such that the system $\mathbf{S}$ results in the same output by starting from each of the initial states $\mathbf{x}_0$ and $\mathbf{x}'_0$. Furthermore, the system $\mathbf{S}$ is called *SIV observable w.r.t.* Γ if every initial state $\mathbf{x}_0$ in Γ is SIV observable w.r.t. Γ.

It is evident that the system $\mathbf{S}$ is SIV observable w.r.t. Γ if and only if the intersection of Γ and the unobservable subspace of $\mathbf{S}$ is only the origin. Assume that $\mathbf{S}$ is controllable, unobservable with an unobservable subspace of dimension $\mu \in \mathbf{N}$, and SIV observable w.r.t. Γ. A question of interest is how to design an arbitrarily fast observer for this system if it is known *a priori* that the initial state of the system belongs to Γ. To address this question, realize the system in the Kalman observable form as

$$\begin{aligned} \begin{bmatrix} \dot{\mathbf{z}}_1(t) \\ \dot{\mathbf{z}}_2(t) \end{bmatrix} &= \begin{bmatrix} \mathbf{A}_{11} & 0 \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{z}_1(t) \\ \mathbf{z}_2(t) \end{bmatrix} + \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \end{bmatrix} \mathbf{u}(t), \\ \mathbf{y}(t) &= \mathbf{C}_1 \mathbf{z}_1(t), \end{aligned} \tag{19}$$

where $\mathbf{A}_{22} \in \mathbf{R}^{\mu\times\mu}$, the pair $(\mathbf{A}_{11}, \mathbf{C}_1)$ is observable, and the relations $\mathbf{z}_1(t) = \mathbf{T}_1\mathbf{x}(t)$ and $\mathbf{z}_2(t) = \mathbf{T}_2\mathbf{x}(t)$ hold for some appropriate similarity transformation matrix $\mathbf{T} := [\mathbf{T}_1^T\ \mathbf{T}_2^T]^T$. For simplicity, assume that $\mathbf{A}_{22}$ is a Hurwitz matrix, i.e., all unobservable modes of the systems are stable (this condition holds for the decentralized problem under investigation in this paper).

Consider an initial state $\mathbf{x}[0] \in \Gamma$. Notice that $\mathbf{z}_2(t)$ cannot be observed in the output $\mathbf{y}(t)$ and, on the other hand, since the system is SIV observable w.r.t. Γ, the signal $\mathbf{z}_2(t)$ should be recoverable from the observable state $\mathbf{z}_1(t)$. Indeed, it can be verified that there exists a linear map $\zeta(\cdot)$ (matrix) such that $\mathbf{z}_2(0) = \zeta\mathbf{z}_1(0)$ for every $\mathbf{x}_0 \in \Gamma$, where $\mathbf{z}_1(0) = \mathbf{T}_1\mathbf{x}_0$ and $\mathbf{z}_2(0) = \mathbf{T}_2\mathbf{x}_0$.

It can be verified that an arbitrarily fast observer in the standard form, i.e. a Luenberger observer, may not exist if $\bar{r} < \mu$, due to the presence of an unobservable subspace of dimension

μ. Nonetheless, it may be speculated that one can simply design an observer to recover $\mathbf{z}_1(t)$ from the observed output $\mathbf{y}(t)$ and then design a compensator to retrieve $\mathbf{z}_2(t)$ from $\mathbf{z}_1(t)$. This idea normally fails for designing an arbitrarily fast observer. The reason is that a fast observer for $\mathbf{z}_1(t)$ often leads to a large overshoot in the estimation of $\mathbf{z}_1(t)$ and since $\mathbf{z}_2(t)$ must be retrieved from $\mathbf{z}_1(t)$ using an open-loop compensator (based on the linear map ζ), there is no way to diminish the effect of overshoot quickly in the estimation of $\mathbf{z}_2(t)$. To this end, a more complex observer will be introduced in the sequel. Select a positive real τ and define

$$\Pi := e^{\mathbf{A}_{22}\tau}\zeta e^{-\mathbf{A}_{11}\tau} + \int_0^\tau e^{\mathbf{A}_{22}(\tau-t)}\mathbf{A}_{21}e^{-\mathbf{A}_{11}(\tau-t)}dt. \tag{20}$$

Consider the transfer function $(sI-\mathbf{A}_{22})^{-1}(\mathbf{A}_{21}+\Pi(sI-\mathbf{A}_{11}))$. Since this is a proper function, it can be realized in the space-state form. Let a realization of this transfer function be given by the state-space matrices $(\mathbf{M}_1,\mathbf{M}_2,\mathbf{M}_3,\mathbf{M}_4)$. Due to the stability of the matrix $\mathbf{A}_{22}$, the matrix M_1 must be Hurwitz.

Theorem 4: Assume that the system $\mathbf{S}$ starts from an unknown initial state $\mathbf{x}_0 \in \Gamma$. Given a matrix L_1 such that $\mathbf{A}_{11} + L_1\mathbf{C}_1$ is Hurwitz, consider the compensator

$$\dot{\hat{\mathbf{z}}}_1(t) = \mathbf{A}_{11}\hat{\mathbf{z}}_1(t) + \mathbf{B}_1\mathbf{u}(t) + L_1(\mathbf{C}_1\hat{\mathbf{z}}_1(t) - \mathbf{y}(t)), \quad \hat{\mathbf{z}}_1(0) = 0, \tag{21a}$$

$$\dot{\mathbf{z}}_{11}(t) = \mathbf{A}_{11}\mathbf{z}_{11}(t) + \mathbf{B}_1\mathbf{u}(t), \quad \mathbf{z}_{11}(0) = 0, \tag{21b}$$

$$\dot{\mathbf{z}}_{21}(t) = \mathbf{A}_{21}\mathbf{z}_{11}(t) + \mathbf{A}_{22}\mathbf{z}_{21}(t) + \mathbf{B}_2\mathbf{u}(t), \quad \mathbf{z}_{21}(0) = 0, \tag{21c}$$

$$\dot{\mathbf{z}}_p(t) = \mathbf{M}_1\mathbf{z}_p(t) + \mathbf{M}_2\left(\hat{\mathbf{z}}_1(t) - \mathbf{z}_{11}(t)\right)u_s(t-\tau), \quad \mathbf{z}_p(0) = 0, \tag{21d}$$

$$\mathbf{y}_p(t) = \mathbf{M}_3\mathbf{z}_p(t) + \mathbf{M}_4\left(\hat{\mathbf{z}}_1(t) - \mathbf{z}_{11}(t)\right)u_s(t-\tau), \tag{21e}$$

$$\hat{\mathbf{z}}_2(t) = \mathbf{z}_{21}(t) + \mathbf{y}_p(t), \tag{21f}$$

where $u_s(\cdot)$ is the step function. This is an SIV observer for the system $\mathbf{S}$ (where $\hat{\mathbf{z}}_i$ estimates $\mathbf{z}_i$ for $i = 1, 2$), which satisfies the following properties:

i) It is internally stable.

ii) The state estimation error converges to zero.

iii) The state estimation error is independent of the input of the system.

In addition, the observation process can be made arbitrarily fast by letting τ go to zero and pushing the eigenvalues of $\mathbf{A}_{11} + L_1\mathbf{C}_1$ towards $-\infty$.

Proof: It is evident that $\hat{\mathbf{z}}_1(t) \to \mathbf{z}_1(t)$, as t goes to infinity. On the other hand, one can write $\mathbf{z}_1(t) = \mathbf{z}_{11}(t) + \mathbf{z}_{12}(t)$, where $\mathbf{z}_{11}(t)$ is given by the differential equation (21b) and

$$\dot{\mathbf{z}}_{12}(t) = \mathbf{A}_{11}\mathbf{z}_{12}(t), \quad \mathbf{z}_{12}(0) = \mathbf{z}_1(0). \tag{22}$$

Furthermore, $\mathbf{z}_2(t)$ can be decomposed as $\mathbf{z}_{21}(t) + \mathbf{z}_{22}(t)$, where $\mathbf{z}_{21}(t)$ is given by (21c) and

$$\dot{\mathbf{z}}_{22}(t) = \mathbf{A}_{21}\mathbf{z}_{12}(t) + \mathbf{A}_{22}\mathbf{z}_{22}(t), \quad \mathbf{z}_{22}(0) = \mathbf{z}_2(0). \tag{23}$$

Let $\mathcal{L}\{\cdot\}$ represent the Laplace transformation. It holds that

$$\mathcal{L}\{\mathbf{z}_{22}(t)u_s(t-\tau)\} = (sI - \mathbf{A}_{22})^{-1}\mathbf{A}_{21}\mathcal{L}\{\mathbf{z}_{12}(t)u_s(t-\tau)\} + e^{-\tau s}(sI - \mathbf{A}_{22})^{-1}\mathbf{z}_{22}(\tau). \tag{24}$$

Besides

$$\begin{aligned} \mathbf{z}_{22}(\tau) &= e^{\mathbf{A}_{22}\tau}\mathbf{z}_{22}(0) + \int_0^\tau e^{\mathbf{A}_{22}(\tau-t)}\mathbf{A}_{21}\mathbf{z}_{12}(t)dt \\ &= \left(e^{\mathbf{A}_{22}\tau}\zeta + \int_0^\tau e^{\mathbf{A}_{22}(\tau-t)}\mathbf{A}_{21}e^{\mathbf{A}_{11}t}dt\right)\mathbf{z}_1(0) = \Pi\mathbf{z}_{12}(\tau) \end{aligned} \tag{25}$$

(note that the last line of the above equation is a consequence of (22)). Moreover

$$\mathbf{z}_{12}(\tau) = e^{\tau s}(sI - \mathbf{A}_{11})\mathcal{L}\{\mathbf{z}_{12}(t)u_s(t-\tau)\}. \tag{26}$$

Thus, it results from the equations (24), (25) and (26) that

$$\mathcal{L}\{\mathbf{z}_{22}(t)u_s(t-\tau)\} = (sI - \mathbf{A}_{22})^{-1}\left(\mathbf{A}_{21} + \Pi(sI - \mathbf{A}_{11})\right)\mathcal{L}\{\mathbf{z}_{12}(t)u_s(t-\tau)\} \tag{27}$$

or equivalently

$$\begin{aligned} \dot{\mathbf{z}}_o(t) &= \mathbf{M}_1\mathbf{z}_o(t) + \mathbf{M}_2\mathbf{z}_{12}(t)u_s(t-\tau), \\ \mathbf{z}_{22}(t)u_s(t-\tau) &= \mathbf{M}_3\mathbf{z}_o(t) + \mathbf{M}_4\mathbf{z}_{12}(t)u_s(t-\tau), \end{aligned} \tag{28}$$

for some state $\mathbf{z}_o$. In order to prove that the compensator (21) is an SIV observer, it is enough to notice that $\left(\hat{\mathbf{z}}_1(t) - \mathbf{z}_{11}(t)\right)u_s(t-\tau) \to \mathbf{z}_{12}(t)u_s(t-\tau)$, as $t \to \infty$. Now, it is straightforward to show that Properties (i), (ii) and (iii) hold (notice that the eigenvalues of the matrix $\mathbf{M}_1$ are identical to those of the matrix $\mathbf{A}_{22}$, which are all stable by assumption). This observer can be made arbitrarily fast in light of the following facts:

- $\mathbf{z}_1(t)$ can be recovered arbitrarily fast by means of a proper high gain matrix L_1.
- Although making the recovery process of $\mathbf{z}_1(t)$ fast would result in a large overshoot, its deteriorating effect can be nullified by the term $u_s(t-\tau)$, which discards some undesirable

part of the signal $\hat{\mathbf{z}}_{11}(t)$ (see the equations (21d) and (21e)). Thus, it is essential that τ be positive, and clearly $\tau = 0$ may not lead to an arbitrarily fast observer for $\mathbf{z}_2(t)$. ■

Corollary 1: Assume that the system $\mathbf{S}$ starts from an unknown initial state $\mathbf{x}_0 \in \Gamma$. Given matrix gains L_1 and Q (with appropriate dimensions) such that $\mathbf{A}_{11} + L_1\mathbf{C}_1$ and $\mathbf{A} + \mathbf{B}Q$ are both Hurwitz, consider the system $\mathbf{S}$ under an observer-based controller composed of the SIV observer (21) and the static controller $\mathbf{u}(t) = Q\mathbf{T}^{-1}\big[\hat{\mathbf{z}}_1(t)^T \;\; \hat{\mathbf{z}}_2(t)^T\big]^T$. The closed-loop system is stable and, more precisely, the state of the closed-loop system can be pushed towards zero arbitrarily fast by making the eigenvalues of $\mathbf{A}_{11} + L_1\mathbf{C}_1$ and $\mathbf{A} + \mathbf{B}Q$ go to $-\infty$ and letting the parameter τ go to zero.

Proof: Since the SIV observer proposed in Theorem 4 satisfies Properties (i)-(iii) stated there, the proof follows immediately from the fact that the well-known separation principle holds for the observer-based controller given in the corollary (note that the signal $\mathbf{T}^{-1}\big[\hat{\mathbf{z}}_1(t)^T \;\; \hat{\mathbf{z}}_2(t)^T\big]^T$ estimates the state $\mathbf{x}(t)$). ■

B. Decentralization of centralized controllers

For a given controller K_c, assume that $F(A_o, B_o, C_o, D_o) \neq 0$. Consider a decentralized controller K_d consisting of a high-level decentralized controller K_d^1 given in (7) and a low-level decentralized controller K_d^2 with its first $\nu - 1$ local controllers being static as given in (17) such that $\bar{F}(G_1, G_2, ..., G_{\nu-1}) \neq 0$. Augment the system $\mathcal{S}$ with the high-level controller K_d^1 and these $\nu - 1$ low-level local controllers to obtain a system $\tilde{\mathcal{S}}(G_1, G_2, ..., G_{\nu-1})$. Let $\bar{\Phi}$ be an n-dimensional subspace defined as

$$\bar{\Phi} := \left\{ \Big[0_{1\times n} \;\; \underbrace{\left[\begin{array}{cc} \boldsymbol{\eta} & 0_{1\times n_o} \end{array}\right] \left[\begin{array}{cc} \boldsymbol{\eta} & 0_{1\times n_o} \end{array}\right] \cdots \left[\begin{array}{cc} \boldsymbol{\eta} & 0_{1\times n_o} \end{array}\right]}_{\nu \text{ times}} \Big]^T \;\middle|\; \boldsymbol{\eta} \in \mathbf{R}^{1\times n} \right\} \tag{29}$$

As discussed after Theorem 3, the "decentralized implementation" of K_c amounts to designing an arbitrarily fast state-regulating controller $K_{d_\nu}^2$ for the controllable but unobservable system $\tilde{\mathcal{S}}(G_1, G_2, ..., G_{\nu-1})$ with the *artificial* initial state $\big[0_{1\times n} \; Z_0^T\big]^T \in \bar{\Phi}$. Since the intersection of the initial-state subspace $\bar{\Phi}$ with the unobservable subspace Φ (see Theorem 3) is only the origin, the system $\tilde{\mathcal{S}}(G_1, G_2, ..., G_{\nu-1})$ is SIV observable w.r.t. $\bar{\Phi}$. Hence, by taking $\mathbf{S}$ as $\tilde{\mathcal{S}}(G_1, G_2, ..., G_{\nu-1})$ and $\mathbf{x}_0$ as $\big[0_{1\times n} \; Z_0^T\big]^T$, Corollary 1 can be exploited to design an arbitrarily fast state-regulating observed-based controller $K_{d_\nu}^2$ for $\tilde{\mathcal{S}}(G_1, G_2, ..., G_{\nu-1})$, which is composed of an SIV observer w.r.t. $\bar{\Phi}$ and a static controller. Note that the parameter ξ introduced in Definition 1 is equal to (τ, L_1, Q), where τ, L_1, Q are the parameters of the observed-based controller $K_{d_\nu}^2$.

Remark 1: It can be deduced from the equation (10) in the proof of Theorem 1 and the observer-based nature of the low-level sub-controller K_d^2 that $x_{d_i}(t) \rightarrow x_c(t)$ as t tends to infinity. This implies that each local controller for the system $\mathcal{S}$ is equipped with an internal observer to asymptotically estimate the global state of the system.

Remark 2: A question arises as whether ε in Definition 1 can be set to zero or it must be strictly positive. Based on the proposed formulation, this question amounts to checking the perfect regulation and bounded peaking properties of the system $\tilde{S}(G_1, G_2, ..., G_{\nu-1})$. If this system possesses these properties, then ε can be taken as zero. For a detailed discussion on perfect regulation, one can refer to [34].

Remark 3: Although this work develops a method for the decentralized implementation of a centralized controller, it can be easily extended to the decentralized *overlapping* implementation of a centralized controller as well. This can be carried out using the bijective transformation given in [33] between decentralized overlapping and decentralized control systems.

C. *Optimal decentralized controller*

The objective of this part is to study how the results of the present work can be exploited to tackle the important problem of the optimal decentralized controller design. To this end, consider a simple interconnected system $\mathcal{S}$ with two subsystems characterized by the parameters

$$A = \begin{bmatrix} 8 & 1 \\ -8 & -2 \end{bmatrix}, \quad B_1 = C_1^T = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad B_2 = C_2^T = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{30}$$

and the initial state $x_0 = \begin{bmatrix} x_1(0) & x_2(0) \end{bmatrix}^T$. Let the centralized controller K_c be given as

$$u(t) = \begin{bmatrix} -15 & 9 \\ 9 & -6 \end{bmatrix} y(t). \tag{31}$$

This controller minimizes the performance index

$$J = \int_0^\infty \left(x(t)^T Q x(t) + u(t)^T R u(t) \right) dt \tag{32}$$

for the system $\mathcal{S}$, where

$$Q = \begin{bmatrix} 0.0063 & -0.0793 \\ -0.0793 & 0.9937 \end{bmatrix}, \quad R = \begin{bmatrix} 0.0826 & 0.1264 \\ 0.1264 & 0.2090 \end{bmatrix}. \tag{33}$$

Consider the problem of designing an affine time-invariant decentralized controller K_d to minimize the cost function J under the assumption that each local station (controller) knows the initial state of its own subsystem, but does not know the initial state of the other subsystem (for instance, the local controller of S_1 can be designed in terms of $x_1(0)$, but $x_2(0)$ is unknown for this local controller). Define J_c and J_d as the optimal performance indices in the centralized and decentralized cases, respectively. Due to the uniqueness of the solution of the underlying Riccati equation in this example and the assumption that the local controller of the first subsystem does not know $x_2(0)$, it is easy to verify that there exists no affine time-invariant decentralized controller that is able to result in the performance index J_c. Thus, one would speculate that $J_d > J_c$. However, the goal is to show that $J_d = J_c$, although there exists no affine decentralized controller with finite parameters such that its corresponding cost is equal to J_d. In other words, it is desired to prove that there is a sequence of decentralized controllers whose performance indices converge to J_c.

To prove the above-mentioned statement, consider a decentralized controller K_d composed of two interconnected sub-controllers K_d^1 and K_d^2, where K_d^1 is given in (7) and K_d^2 is a static decentralized controller as

$$u_{d_1}(t) = \begin{bmatrix} \xi & \xi^2 \end{bmatrix}^T y_{d_1}(t), \quad u_{d_2}(t) = \begin{bmatrix} -\xi^2 & \xi \end{bmatrix}^T y_{d_2}(t), \tag{34}$$

where ξ is a scalar tuning parameter that will be specified later. In light of Theorem 1, if the sub-controllers $K_{d_1}^1$ and $K_{d_2}^1$ are initialized as

$$x_{d_1}(0) = x_{d_2}(0) = \begin{bmatrix} x_1(0) & x_2(0) \end{bmatrix}^T, \tag{35}$$

then the controllers K_d and K_c both generate the same input and state for the system $\mathcal{S}$. Nonetheless, since it is already assumed that the first local controller cannot use $x_2(0)$ in light of the interconnected structure of the system, this initialization is impossible. Assume that $K_{d_1}^1$ and $K_{d_2}^1$ are initialized as

$$x_{d_1}(0) = \begin{bmatrix} x_1(0) & 0 \end{bmatrix}^T, \quad x_{d_2}(0) = \begin{bmatrix} 0 & x_2(0) \end{bmatrix}^T. \tag{36}$$

Let $x_d(t)$ and $u_d(t)$ denote the state and the input of the system $\mathcal{S}$ under the controller K_d initialized as above. Having defined $\tilde{\mathcal{S}}$ as the system $\mathcal{S}$ under K_d^1, denote the set of state-space matrices of this resultant system with $(\tilde{A}, \tilde{B}, \tilde{C})$ and its state with $\tilde{x}(t)$. Define also

$$\tilde{A}(\xi) = \tilde{A} + \tilde{B} \begin{bmatrix} \xi & \xi^2 & 0 & 0 \\ 0 & 0 & -\xi^2 & \xi \end{bmatrix}^T \tilde{C}, \quad \forall \xi \in \mathbf{R}. \tag{37}$$

Using the method proposed in [35] for checking the robust stability of a polynomially uncertain matrix, it can be shown that the matrix $\tilde{A}(\xi)$ is Hurwitz for every $\xi \geq 10$. On the other hand, the argument leading to the equation (13) can be adopted to conclude that $x_d(t) - x_c(t)$ and $u_d(t) - u_c(t)$ are equal to the state and input of the system $\mathcal{S}$ under the controller K_d, respectively, if the initial state of closed-loop system is taken as

$$\bar{x}_0 := \begin{bmatrix} 0 & 0 & 0 & x_2(0) & x_1(0) & 0 \end{bmatrix}^T. \tag{38}$$

Therefore, the relation

$$\int_0^\infty \|x_d(t) - x_c(t)\|^2 dt \leq \int_0^\infty \|\tilde{x}(t)\|^2 dt = \text{trace}\left(\bar{x}_0^T P(\xi) \bar{x}_0\right) \tag{39}$$

holds for every $\xi \geq 10$, where

$$\tilde{A}(\xi)^T P(\xi) + P(\xi)\tilde{A}(\xi) + I = 0. \tag{40}$$

Due to the relation (39) and the particular structure of $\bar{x}_0$, in order to prove that $\int_0^\infty \|x_d(t) - x_c(t)\|^2 dt$ goes to zero as ξ tends to infinity, it suffices to show that the $(4,4)$, $(4,5)$ and $(5,5)$ entries of $P(\xi)$ all attenuate to zero as ξ goes to infinity. To prove this statement, one can solve the Lyapunov equation (40) using the well-known Kronecker-product method to deduce that every entry of $P(\xi)$ is a rational function in ξ (see the proof of Lemma 2 in [36]). Since the $(4,4)$, $(4,5)$ and $(5,5)$ entries of $P(\xi)$ all turn out to be strictly proper rational functions, they attenuate to zero as ξ goes to infinity. This yields

$$\int_0^\infty \|x_d(t) - x_c(t)\|^2 dt \to 0 \quad \text{as} \quad \xi \to \infty, \tag{41}$$

or more precisely

$$\int_0^\infty \|\tilde{x}(t)\|^2 dt \to 0 \quad \text{as} \quad \xi \to \infty. \tag{42}$$

The above relation can be combined with (7c) to obtain

$$\int_0^\infty \|u_d(t) - u_c(t)\|^2 dt \to 0 \quad \text{as} \quad \xi \to \infty. \tag{43}$$

It can be concluded from (41) and (43) that K_d parameterized in terms of ξ results in a performance index J arbitrarily close to J_c (by letting ξ go to infinity).

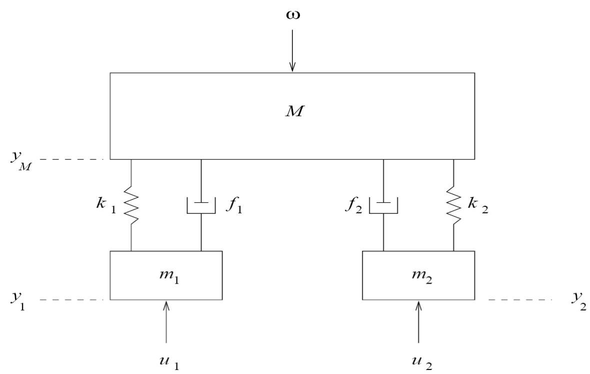


Fig. 3. The mass-spring system studied in the numerical example.

IV. Numerical example

Consider the mass-spring system $\mathcal{S}$ given in Figure 3. Regard this system as a two-channel interconnected system with the input $u_i(t)$ and the output $y_i(t)$ for its i^{th} control channel, where $i = 1, 2$. By defining the state $x(t) := [y_M(t)\ \dot{y}_M(t)\ y_1(t)\ \dot{y}_1(t)\ y_2(t)\ \dot{y}_2(t)]^T$, the state-space matrices of this system (for the nominal values given in [37]) can be obtained as

$$
\begin{aligned}
A &= \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -0.2 & -0.02 & 0.1 & 0.01 & 0.1 & 0.01 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0.1 & -1 & -0.1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0.1 & 0 & 0 & -1 & -0.1 \end{bmatrix}, \\
B_1 &= \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}^T, \quad B_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}^T, \\
C_1 &= \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.
\end{aligned} \tag{44}
$$

Assume that K_c is a centralized controller with the control law

$$
\begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ 3 & -5 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}, \tag{45}
$$

which is desired to be implemented in a decentralized fashion. Note that this controller results in a slow, oscillatory behavior for the system $\mathcal{S}$. Consider a decentralized controller K_d comprising two interconnected decentralized sub-controllers K_d^1 and K_d^2, where K_d^1 is given in (7). Recall that the present work suggests designing the low-level decentralized sub-controller K_d^2 in such

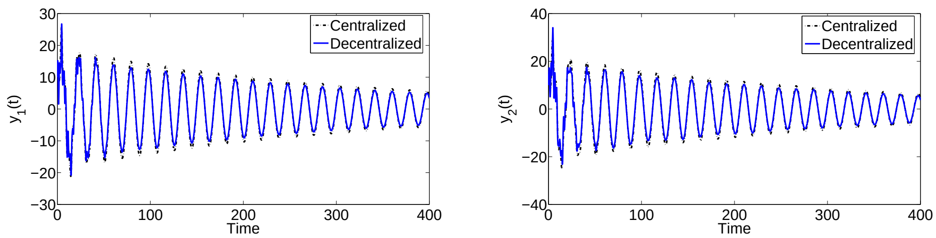


Fig. 4. The output of the system $\mathcal{S}$ under the controllers K_c and K_d.

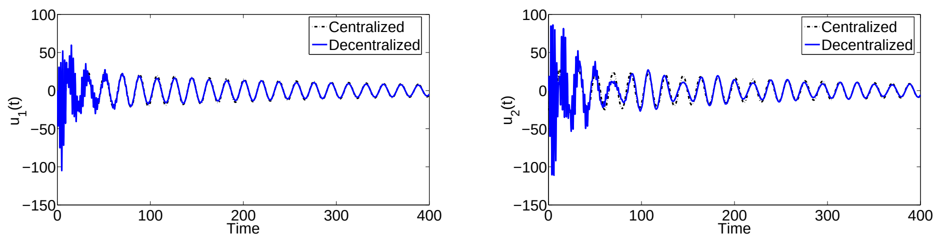


Fig. 5. The input of the system $\mathcal{S}$ under the controllers K_c and K_d.

a way that its first local controller is static while its second local controller is possibly dynamic (observer-based). However, let the possibility of designing a static low-level sub-controller K_d^2 be checked first. As stated in Theorem 2, the system $\mathcal{S}$ under K_d^1, denoted by $\tilde{\mathcal{S}}$, has 6 unobservable modes that are identical to the modes of the system $\mathcal{S}$ under the controller K_c. The "fminsearch" command of MATLAB was deployed to minimize the maximum magnitude of the observable modes of the system $\tilde{\mathcal{S}}$ under a *static* controller K_d^2, which led to a stabilizing controller K_d^2 as

$$\begin{bmatrix} u_{d_1}(t) \\ u_{d_2}(t) \end{bmatrix} = \begin{bmatrix} 10.0000 & 3.0187 & 0 & 0 \\ 0 & 0 & 10.0000 & 20.8750 \end{bmatrix}^T \begin{bmatrix} y_{d_1}(t) \\ y_{d_2}(t) \end{bmatrix}. \tag{46}$$

For the purpose of simulation, let the initial state $x(0)$ be equal to $[5\ 15\ 10\ 15\ 10\ 15]$. The output and input of the system $\mathcal{S}$ are depicted under both of the controllers K_c and K_d in Figures 4 and 5, respectively. These figures demonstrate that the controller K_d is a satisfactory decentralized implementation of K_c, as it can generate trajectories very close to the desired ones produced by K_c. This is a consequence of the fact that K_d^1 captures the dynamics of the centralized closed-loop system, and the sub-controller K_d^2 is mainly required for the internal stability of the decentralized closed-loop system. Note that in order to make the decentralized trajectories closer to the centralized ones particularly in the time interval $[0, 50]$, one needs to design an observer-based low-level controller K_d^2 using the method developed here, rather than searching for the best static low-level controller.

V. Conclusions and future work

This paper is concerned with the decentralized implementation of centralized controllers for strongly connected interconnected systems. A parameterized decentralized controller is designed for a given centralized controller associated with an interconnected system. This two-level decentralized controller is composed of two interconnected decentralized sub-controllers, where the high-level sub-controller captures the dynamics of the system under the centralized controller. The low-level decentralized sub-controller is designed in such a way that the input and state trajectories of the system under the designed (overall) decentralized controller can become arbitrarily close to those of the system under the prescribed centralized controller by tuning the free parameters of the low-level sub-controller. This is carried out using the new notion of structural initial value observability. It is shown that the developed technique can shed light on some aspect of the LQR optimal decentralized control problem.

The present work shows that every generic centralized controller is decentrally implementable, but the order of the obtained decentralized controller is high, due to the structure of its high-level sub-controller. Since every local controller of the high-level sub-controller is stable, one can use a model reduction technique to first reduce the order of the high-level sub-controller and then design the low-level sub-controller. Note that the order of the designed controller being large is a common issue even for classical centralized control problems such as H_2 or H_∞ optimal control and strong stabilization problems. The possibility of implementing a centralized controller via a decentralized controller with a prescribed bound on its order remains a subject of future research.

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APPENDIX 1

A real algebraic variety is defined to be the set of real zeros of a multivariate polynomial with a positive degree. A matrix $\mathcal{M} \in \mathbf{R}^{p\times q}$ is said to be generic with respect to a given real algebraic variety (with a positive codimension) if it does not belong to that variety. Note that almost all matrices in $\mathbf{R}^{p\times q}$ are generic with respect to a fixed real algebraic variety. For simplicity, the term “generic with respect to a certain real algebraic variety” will be abbreviated as “generic” throughout this paper. The controller K_c is said to be generic if the parameter set (A_o, B_o, C_o, D_o) does not belong to a specific real algebraic variety.

To prove that Properties (i)-(vi) given in Theorem 2 hold if $F(A_o, B_o, C_o, D_o) \neq 0$ for some polynomial F, it suffices to show that each of these properties holds for a generic controller K. It will be later studied in Appendix 2 how to obtain the polynomial F.

A. Static centralized controllers

Assume for now that K_c is simply a static controller with the gain K (i.e., $u(t) = Ky(t)$). The results will be extended to a dynamic controller K_c in the next subsection. Notice that $\tilde{\mathcal{S}}$ can be represented as

$$\begin{aligned} \dot{\tilde{x}}(t) &= \tilde{A}\tilde{x}(t) + \sum_{i=1}^{\nu} \tilde{B}_i u_{d_i}(t) \\ y_{d_i}(t) &= \tilde{C}_i \tilde{x}(t), \quad \forall i \in \boldsymbol{\nu}, \end{aligned} \tag{47}$$

where

$$\begin{aligned} \tilde{A} &:= \begin{bmatrix} A & B_1K_1C & B_2K_2C & \cdots & B_\nu K_\nu C \\ 0 & A+BKC & 0 & \cdots & 0 \\ 0 & 0 & A+BKC & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & A+BKC \end{bmatrix}, \\ \tilde{B} &:= \begin{bmatrix} 0 & 0 & \cdots & 0 \\ B & 0 & \cdots & 0 \\ 0 & B & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B \end{bmatrix}, \quad \tilde{C} := \begin{bmatrix} C_1 & -C_1 & 0 & \cdots & 0 \\ C_2 & 0 & -C_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ C_\nu & 0 & 0 & \cdots & -C_\nu \end{bmatrix}, \end{aligned} \tag{48}$$

and $K_i \in \mathbf{R}^{m_i \times r}$, $\tilde{B}_i \in \mathbf{R}^{(\nu+1)n \times m}$ and $\tilde{C}_i \in \mathbf{R}^{r_i \times (\nu+1)n}$ are the i^{th} block row, the i^{th} block column and the i^{th} block row of K, $\tilde{B}$ and $\tilde{C}$, respectively.

Proof of Part (i) of Theorem 2: The goal is to prove that $\tilde{\mathcal{S}}$ is generically strongly connected. To this end, let K be a matrix such that A and $A + BKC$ have disjoint eigenvalues and that

$$\begin{bmatrix} A - \lambda_j I & B \\ K_i C & 0 \end{bmatrix} = \text{full row rank}, \quad \forall i \in \boldsymbol{\nu},\ j \in \mathbf{n}. \tag{49}$$

Note that these properties hold for a generic K, due to Assumptions 1 and 3. Assume that $p, q \in \boldsymbol{\nu},\ p \neq q$, are two indices for which $C_p(sI - A)^{-1}B_q$ is not identically zero. It is desired to show that the transfer function $\tilde{C}_p(sI-\tilde{A})^{-1}\tilde{B}_q$ is nonzero as well. One can write

$$\tilde{C}_p(sI-\tilde{A})^{-1}\tilde{B}_q = C_p(sI-A)^{-1}B_qK_qC(sI-A-BKC)^{-1}B. \tag{50}$$

Since $C_p(sI-A)^{-1}B_q$ is not identically zero, there exists a mode λ_j such that this transfer function becomes infinity at $s=\lambda_j$. In light of the above equality, $\tilde{C}_p(sI-\tilde{A})^{-1}\tilde{B}_q$ is guaranteed to be nonzero if $K_qC(sI-A-BKC)^{-1}B$ is finite and has full row rank at $s=\lambda_j$. The finiteness of $K_qC(\lambda_jI-A-BKC)^{-1}B$ follows from the assumption that A and $A+BKC$ have disjoint eigenvalues. Regarding the rank of this quantity, one can use the LUD decomposition to obtain

$$K_qC(A+BKC-\lambda_jI)^{-1}B = \text{full rank} \quad \Longleftrightarrow \quad \begin{bmatrix} A+BKC-\lambda_jI & B \\ K_qC & 0 \end{bmatrix} = \text{full rank}. \tag{51}$$

On the other hand, since the number of rows of K_qC is less than or equal to the number of columns of B, one can write

$$\begin{bmatrix} A+BKC-\lambda_jI & B \\ K_qC & 0 \end{bmatrix} = \text{full rank} \quad \Longleftrightarrow \quad \begin{bmatrix} A-\lambda_jI & B \\ K_qC & 0 \end{bmatrix} = \text{full (row) rank}. \tag{52}$$

The right side of the above statement holds in light of the foregoing assumption. Thus, $\tilde{C}_p(sI-\tilde{A})^{-1}\tilde{B}_q$ is a nonzero transfer function. So far, it is shown that if there is an edge (q,p) in the structural graph of $\mathcal{S}$, the same edge must exist in the structural graph of $\tilde{\mathcal{S}}$ too. Since the structural graph of $\mathcal{S}$ is strongly connected (by Assumption 2), this property implies that the structural graph of $\tilde{\mathcal{S}}$ is strongly connected as well. ■

Proof of Part (ii) of Theorem 2: This part follows immediately from the upper block-triangular structure of the matrix $\tilde{A}$ given in (48). ■

Proof of Part (iii) of Theorem 2: The objective is to show that the system $\tilde{\mathcal{S}}$ is controllable for a generic controller K. To this end, let K be a matrix satisfying the two generic properties stated in the proof of Part (i) of Theorem 2. It is sufficient to prove that the matrix $[\tilde{A}-\sigma I \ \tilde{B}]$ is full rank for every eigenvalue σ of $\tilde{A}$. For this purpose, two cases can be considered as follows:

- *σ is equal to λ_i, for some $i \in \boldsymbol{\nu}$:* To prove the underlying statement, it is enough to show that the null space of the matrix $[\tilde{A}-\sigma I \ \tilde{B}]$ is of dimension νm (recall that $\tilde{B} \in \mathbf{R}^{(\nu+1)n\times\nu m}$). Let $[\alpha_0^T \ \alpha_1^T \ \cdots \ \alpha_{2\nu}^T]^T$ be a vector in the null space of $[\tilde{A}-\sigma I \ \tilde{B}]$, where $\alpha_p \in \mathbf{R}^n, \ \forall p \in \{0\}\cup\boldsymbol{\nu}$

and $\alpha_q \in \mathbf{R}^m, \ \forall q \in \{\nu+1, \cdots, 2\nu\}$. One can write

$$(A - \sigma I)\alpha_0 + \sum_{j=1}^{\nu} B_j K_j C \alpha_j = 0, \tag{53a}$$

$$(A + BKC - \sigma I)\alpha_j + B\alpha_{j+\nu} = 0, \quad \forall j \in \boldsymbol{\nu}. \tag{53b}$$

By assumption, σ is not an eigenvalue of $A + BKC$. Thus, the equation (53b) yields

$$\alpha_j = -(A + BKC - \sigma I)^{-1} B \alpha_{j+\nu}, \quad \forall j \in \boldsymbol{\nu}. \tag{54}$$

The equations (53a) and (54) can be combined to deduce that $[\alpha_0^T \ \alpha_{\nu+1}^T \ \alpha_{\nu+2}^T \ \cdots \ \alpha_{2\nu}^T]^T$ is in the null space of the matrix

$$\left[\begin{array}{cccc} A - \sigma I & -B_1 K_1 C(A + BKC - \sigma I)^{-1} B & \cdots & -B_\nu K_\nu C(A + BKC - \sigma I)^{-1} B \end{array} \right]. \tag{55}$$

It follows from the argument made in the proof of Part (i) of Theorem 2 that the column space of the above matrix is identical to the column space of the matrix $[A - \sigma I \ B_1 \ \cdots \ B_\nu] = [A - \sigma I \ B]$ (because $K_j C(A + BKC - \sigma I)^{-1} B$ has full row rank, for every $j \in \boldsymbol{\nu}$). Since the matrix $[A - \sigma I \ B]$ is full rank, it can be concluded that the null space of the matrix given in (55) is of dimension νm. As a result, the vector $[\alpha_0^T \ \alpha_{\nu+1}^T \ \alpha_{\nu+2}^T \ \cdots \ \alpha_{2\nu}^T]^T$ belongs to a νm-dimensional space. This observation and the equation (54) lead to the conclusion that the null space of the matrix $[\tilde{A} - \sigma I \ \tilde{B}]$ is of dimension νm. This completes the proof.

- *σ is equal to $\bar{\lambda}_i$, for some $i \in \mathbf{n}$:* Assume that there exists a nonzero vector $[\beta_0 \ \beta_0 \ \cdots \ \beta_\nu]$ such that $\beta_p \in \mathbf{R}^{1 \times n}$, $\forall p \in \{0\} \cup \boldsymbol{\nu}$, and

$$\left[\begin{array}{cccc} \beta_0 & \beta_0 & \cdots & \beta_\nu \end{array} \right] \left[\begin{array}{cc} \tilde{A} - \sigma I & \tilde{B} \end{array} \right] = 0. \tag{56}$$

This implies that

$$\beta_0 (A - \sigma I) = 0 \tag{57a}$$

$$\beta_j \left[\begin{array}{cc} A + BKC - \sigma I & B \end{array} \right] + \beta_0 \left[\begin{array}{cc} B_j K_j C & 0 \end{array} \right] = 0, \quad \forall j \in \boldsymbol{\nu}. \tag{57b}$$

On the other hand, σ is not an eigenvalue of A (by the assumption made earlier). This observation, together with the equation (57a), yields that $\beta_0 = 0$. Since the pair (A, B) is controllable, it can be concluded from the equation (57b) and the equality $\beta_0 = 0$ that $\beta_1 = \beta_2 = \cdots = \beta_\nu = 0$, which violates the assumption that $[\beta_0 \ \beta_0 \ \cdots \ \beta_\nu]$ is nonzero. ■

Lemma 1: Given $i \in \{1, 2, ..., \nu - 1\}$ and $j \in \mathbf{n}$, the matrix

$$\begin{bmatrix} A - \bar{\lambda}_j I & B_1 K_1 C & B_2 K_2 C & \cdots & B_i K_i C \\ 0 & A + BKC - \bar{\lambda}_j I & 0 & \cdots & 0 \\ 0 & 0 & A + BKC - \bar{\lambda}_j I & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & A + BKC - \bar{\lambda}_j I \\ C_1 & -C_1 & 0 & \cdots & 0 \\ C_2 & 0 & -C_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ C_i & 0 & 0 & \cdots & -C_i \end{bmatrix} \tag{58}$$

is full rank for a generic controller K.

Proof: Let K be a matrix gain for which the following conditions are met:

- K is a block-diagonal matrix whose (l, l) block entry K_{ll} is of dimension $m_l \times r_l$, for every $l \in \boldsymbol{\nu}$.
- The eigenvalues of $A + \sum_{l=1}^{i} B_l K_{ll} C_l$ and $A + BKC$ are disjoint .
- The pair $(A + BKC, C_l)$ is observable, for every $l \in \boldsymbol{\nu}$.

Note that such a matrix K exists in light of the inequality $i < \nu$ together with Assumptions 1 and 2 (see [32]). Since the identity $B_l K_l C = B_l K_{ll} C_l$ holds for all $l \in \boldsymbol{\nu}$, it is easy to show that the rank of the matrix in (58) is greater than or equal to the rank of the matrix in (59) given on top of the next page (note that all of the block entries of the latter matrix are equal to zero, except for the ones on the block diagonal and in the first block column). It follows from the aforementioned assumptions on the gain K that every block diagonal entry of the matrix given in (59) has full column rank. As a result, this matrix has full column rank as well, so does the matrix given in (58). So far, it is shown that the matrix (58) is full rank for a particular choice of K. Now, it is easy to verify that this implies that the matrix (58) must be full rank for a generic K. ■

Proof of Part (iv) of Theorem 2: The first objective is to show that the mode λ_j of the system $\tilde{\mathcal{S}}$ is observable, for every $j \in \boldsymbol{\nu}$. Let K be a matrix such that the sets of eigenvalues of $A + BKC$ and A are disjoint (note that this property holds for a generic K). Consider a vector $[\alpha_0^T \ \alpha_1^T \ \cdots \ \alpha_\nu^T]^T$ in the null space of $[(\tilde{A} - \lambda_j I)^T \ \tilde{C}^T]^T$, where $\alpha_i \in \mathbf{R}^n, \ \forall i \in \{0\} \cup \boldsymbol{\nu}$. It is

$$\begin{bmatrix} \begin{bmatrix} A+\sum_{l=1}^{i} B_l K_{ll} C_l - \bar{\lambda}_j I \end{bmatrix} & \begin{bmatrix} 0 \end{bmatrix} & \cdots & \begin{bmatrix} 0 \end{bmatrix} \\ \begin{bmatrix} A+BKC-\bar{\lambda}_j I \\ 0 \end{bmatrix} & \begin{bmatrix} A+BKC-\bar{\lambda}_j I \\ -C_1 \end{bmatrix} & \cdots & \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \vdots & \vdots & \ddots & \vdots \\ \begin{bmatrix} A+BKC-\bar{\lambda}_j I \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \end{bmatrix} & \cdots & \begin{bmatrix} A+BKC-\bar{\lambda}_j I \\ -C_i \end{bmatrix} \end{bmatrix} \tag{59}$$

intended to prove that this vector is equal to zero. To this end, one can write the following set of equations:

$$(A+BKC-\lambda_j I)\alpha_i = 0, \quad \forall i \in \boldsymbol{\nu}, \tag{60a}$$

$$C_i(\alpha_0 - \alpha_i) = 0, \quad \forall i \in \boldsymbol{\nu}, \tag{60b}$$

$$(A-\lambda_j I)\alpha_0 + \sum_{i=1}^{\nu} B_i K_i C \alpha_i = 0. \tag{60c}$$

The equation (60a) yields that $\alpha_1 = \alpha_2 = \cdots = \alpha_\nu = 0$. Now, it can be concluded from the equations (60b) and (60c) that $(A-\lambda_j I)\alpha_0 = C\alpha_0 = 0$. Since the pair (A, C) is observable, this implies that α_0 must be zero. As a result, λ_j is observable.

The second objective is to show that the repeated mode $\bar{\lambda}_j$ of the system $\tilde{\mathcal{S}}$ with multiplicity ν is only $\nu-1$ times observable, for every $j \in \mathbf{n}$. For this purpose, let K be a matrix for which the three conditions given in the proof of Lemma 1 are satisfied (such as being block diagonal) and, in addition, the eigenvalues of $A+BKC$ are all distinct. Consider the matrix $[(\tilde{A}-\bar{\lambda}_j I)^T \ \tilde{C}^T]$. Replace the last block column of this matrix with the sum of all block columns of the matrix, remove its last block row and then re-arrange its block rows to obtain

$$\begin{bmatrix} \begin{bmatrix} \Pi \end{bmatrix} & \begin{bmatrix} A+BKC-\bar{\lambda}_j I \\ \vdots \\ A+BKC-\bar{\lambda}_j I \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & \cdots & 0 \end{bmatrix} & \begin{bmatrix} A+BKC-\bar{\lambda}_j I \end{bmatrix} \end{bmatrix}, \tag{61}$$

where Π is equal to the matrix given in (58) for $i = \nu - 1$. The first observation is that the rank of the above matrix is less than or equal to the rank of the matrix $[(\tilde{A}-\bar{\lambda}_j I)^T \ \tilde{C}^T]$, due to the performed operations. Furthermore, the sub-matrix Π has full column rank in light of Lemma 1 and $\bar{\lambda}_j$ is a single eigenvalue of $A+BKC$ (by assumption). As a result, the rank of the matrix (61) is at least $\nu n-1$, and so is the rank of the matrix $[(\tilde{A}-\bar{\lambda}_j I)^T \ \tilde{C}^T]$. On the other hand, it is easy to verify that this observability matrix loses column rank (this can be seen by adding its block columns and checking the rank of the resulting block column), which makes its rank be at most $\nu n-1$. Hence, the rank of the matrix $[(\tilde{A}-\bar{\lambda}_j I)^T \ \tilde{C}^T]$ must be exactly $n\nu-1$, meaning that $\bar{\lambda}_j$ is exactly $\nu-1$ times observable for this choice of K. Let α_i denote a right eigenvector of $A+BKC$ associated with $\bar{\lambda}_i$, for every $i \in \mathbf{n}$. It is evident that the vector

$$\Big[\ \underbrace{\alpha_i^T \quad \alpha_i^T \quad \cdots \quad \alpha_i^T}_{\nu+1 \text{ times}} \ \Big]^T \tag{62}$$

is in the null space of $[(\tilde{A}-\bar{\lambda}_i I)^T \ \tilde{C}^T]$. Since the eigenvalues of $A+BKC$ are all distinct, this matrix has n independent eigenvectors; therefore, every vector in Φ can be written as a linear combination of the vectors in the form of (62) for $i \in \mathbf{n}$. This implies that Φ is in the unobservable subspace of $\tilde{\mathcal{S}}$. Due to the fact that this system has n unobservable modes and the dimension of Φ is exactly n, it can be concluded that the unobservable subspace of $\tilde{\mathcal{S}}$ is the same as Φ.

So far, it is proved that for a particular gain K, the repeated mode $\bar{\lambda}_j$ is only $\nu-1$ times observable and the unobservable subspace of the system is Φ. Now, it is straightforward to argue that these results are both valid for a generic controller K, on noting that:

- $\bar{\lambda}_j$ is at least one time unobservable for every arbitrary matrix K (this can be shown by adding up the block columns of the matrix $[(\tilde{A}-\bar{\lambda}_j I)^T \ \tilde{C}^T]$, as before).
- Since $\bar{\lambda}_j$ is only one time unobservable for a particular K, it must be at most one time unobservable for a generic K. ■

Proof of Part (v) of Theorem 2: The goal is to show that λ_j, $j \in \boldsymbol{\nu}$, is not a DFM of the system $\tilde{\mathcal{S}}$. Since λ_j is a controllable and observable mode of the system $\tilde{\mathcal{S}}$ for a generic controller K

(due to Parts (iii) and (iv) of the theorem), it is enough to prove that the inequality

$$\operatorname{rank}\begin{bmatrix} \tilde{A}-\lambda_j I & \tilde{B}_{i_1} & \dots & \tilde{B}_{i_p} \\ \tilde{C}_{i_{p+1}} & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{C}_{i_\nu} & 0 & \dots & 0 \end{bmatrix} \geq n(\nu+1) \tag{63}$$

holds for every permutation $(i_1, i_2, ..., i_\nu)$ of the set $\{1, 2, ..., \nu\}$ and $p \in \{1, ..., \nu-1\}$. The validity of the above relation will be shown only for the permutation $(i_1, i_2, ..., i_\nu) = (1, 2, ..., \nu)$, as the proof is similar for other permutations. For this purpose, consider a vector $[\alpha_0^T \ \alpha_1^T \ \cdots \alpha_{\nu+p}^T]^T$ in the null space of the matrix given in the left side of the inequality (63), where $\alpha_l \in \mathbf{R}^n, \ \forall l \in \{0\} \cup \boldsymbol{\nu}$ and $\alpha_q \in \mathbf{R}^m, \ \forall q \in \{\nu+1, \cdots, \nu+p\}$. One can write

$$(A-\lambda_j I)\alpha_0 + \sum_{i=1}^{\nu} B_i K_i C \alpha_i = 0, \tag{64a}$$

$$(A+BKC-\lambda_j I)\alpha_i + B\alpha_{i+\nu} = 0, \quad \forall i \in \{1, 2, ..., p\}, \tag{64b}$$

$$(A+BKC-\lambda_j I)\alpha_i = 0, \quad \forall i \in \{p+1, p+2, ..., \nu\}, \tag{64c}$$

$$C_i(\alpha_0 - \alpha_i) = 0, \quad \forall i \in \{p+1, p+2, ..., \nu\}. \tag{64d}$$

Let K be a generic matrix satisfying the two properties that the eigenvalues of A and $A+BKC$ constitute disjoint sets and that the relation (49) holds. The equation (64c) yields

$$\alpha_{p+1} = \alpha_{p+2} = \cdots = \alpha_\nu = 0 \tag{65}$$

Furthermore, it follows from the equation (64b) that

$$\alpha_i = -(A+BKC-\lambda_j I)^{-1} B \alpha_{i+\nu}, \quad \forall i \in \{1, 2, ..., p\}. \tag{66}$$

Therefore, one can deduce from the equations (64), (65) and (66) that the vector $[\alpha_0^T \ \alpha_1^T \ \cdots \ \alpha_p^T]^T$ is in the null space of the matrix

$$\begin{bmatrix} A-\lambda_j I & -B_1K_1C(A+BKC-\lambda_j I)^{-1}B & \cdots & -B_pK_pC(A+BKC-\lambda_j I)^{-1}B \\ C_{p+1} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ C_\nu & 0 & \cdots & 0 \end{bmatrix} \tag{67}$$

On the other hand, since $K_l C(A+BKC-\lambda_j I)^{-1}B$ has full row rank for every $l \in \boldsymbol{\nu}$ (see the proof of Part (i) of Theorem 2), the column space of the above matrix is identical to the column space of

$$\begin{bmatrix} A-\lambda_j I & B_1 & \cdots & B_p \\ C_{p+1} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ C_\nu & 0 & \cdots & 0 \end{bmatrix}. \tag{68}$$

Recall that the rank of this matrix is greater than or equal to n (by Assumption 1). This means that the rank of the matrix given in (67) is at least equal to n, which indicates that the dimension of its null space is at most pm. Thus, the vector $[\alpha_0^T \ \alpha_1^T \ \cdots \ \alpha_p^T]^T$ is contained in a pm-dimensional space. This fact, together with the relations (65) and (66), yields that the vector $[\alpha_0^T \ \alpha_1^T \ \cdots \alpha_{\nu+p}^T]^T$ is in a pm-dimensional space as well, which immediately proves the inequality (63).

Proof of Part (vi) of Theorem 2: It is shown in Part (iv) of the theorem that the mode $\bar{\lambda}_j$ with multiplicity ν is exactly $\nu-1$ times observable for a generic controller K. This implies that $\bar{\lambda}_j$ is a DFM. That $\bar{\lambda}_j$ is a single DFM can be proven in line with the argument made in the proof of Part (v) of Theorem 2 (and using Lemma 1). ■

B. Extension to dynamic centralized controllers

To generalize the results of the previous subsection to a dynamic controller K_c, define

$$\bar{A} = \begin{bmatrix} A & 0_{n\times n_o} \\ B_o C & A_o \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} B \\ 0_{n_o\times m} \end{bmatrix}, \quad \bar{C} = \begin{bmatrix} C & 0_{r\times n_o} \\ 0_{n_o\times n} & I_{n_o} \end{bmatrix}, \quad \bar{K} = \begin{bmatrix} D_o & C_o \end{bmatrix}. \tag{69}$$

It is easy to verify that the state-space matrices of the system $\tilde{\mathcal{S}}$ with the realization (47) are

$$\tilde{A} := \begin{bmatrix} A & B_1\bar{K}_1\bar{C} & B_2\bar{K}_2\bar{C} & \cdots & B_\nu\bar{K}_\nu\bar{C} \\ 0 & \bar{A}+\bar{B}\bar{K}\bar{C} & 0 & \cdots & 0 \\ 0 & 0 & \bar{A}+\bar{B}\bar{K}\bar{C} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \bar{A}+\bar{B}\bar{K}\bar{C} \end{bmatrix} \tag{70}$$

and

$$\tilde{B} := \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \bar{B} & 0 & \cdots & 0 \\ 0 & \bar{B} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \bar{B} \end{bmatrix}, \quad \tilde{C} := \begin{bmatrix} C_1 & -\bar{C}_1 & 0 & \cdots & 0 \\ C_2 & 0 & -\bar{C}_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ C_\nu & 0 & 0 & \cdots & -\bar{C}_\nu \end{bmatrix}, \tag{71}$$

instead of the ones given in (48), where $\bar{K}_i \in \mathbf{R}^{m_i \times (r+n_o)}$ and $\bar{C}_i \in \mathbf{R}^{r_i \times (n+n_o)}$ are the i^{th} block rows of $\bar{K}$ and $\bar{C}$, respectively, for every $i \in \boldsymbol{\nu}$. It can be observed that $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ for a general controller K_c (given in (70) and (71)) resemble the corresponding matrices for a static controller K_c (given in (48)); more precisely:

- $\bar{A}$, $\bar{B}$, $\bar{C}$ and $\bar{K}$ used in (70) and (71) correspond to A, B, C and K used in (48).
- By fixing A_o and B_o, the parameters of the controller are C_o and D_o, which are combined in the matrix $\bar{K}$. In the static case, the parameter of the controller is K, i.e. the counterpart of $\bar{K}$.

Note that the above formulation is a well-known technique used in a number of papers, such as [13], to convert a dynamic decentralized controller problem to a static one. After recognizing this analogy between the static and dynamic cases, one can pursue the arguments made in the previous subsection to prove the stated results for a general controller K_c. For instance, the only modifications needed in the proofs of Parts (i) and (iii) of Theorem 2 are to replace the conditions

$$\begin{bmatrix} A - \lambda_j I & B \\ K_i C & 0 \end{bmatrix} = \text{full row rank}, \quad \forall i \in \boldsymbol{\nu},\ j \in \mathbf{n}, \quad \text{and} \quad sp(A) \cap sp(A + BKC) = \text{empty} \tag{72}$$

with

$$\begin{bmatrix} \bar{A} - \lambda_j I & \bar{B} \\ \bar{K}_i \bar{C} & 0 \end{bmatrix} = \text{full row rank}, \quad \forall i \in \boldsymbol{\nu},\ j \in \mathbf{n}, \quad \text{and} \quad sp(A) \cap sp(\bar{A} + \bar{B}\bar{K}\bar{C}) = \text{empty}, \tag{73}$$

where $sp(\cdot)$ is the spectral operator returning the set of eigenvalues of a matrix. Now, it suffices to notice that the new conditions hold for a generic controller K_c.

APPENDIX 2

This part aims to propose a method to find a nonzero polynomial F (whose coefficients are dependent only on the parameters of $\mathcal{S}$, rather than K_c) such that Properties (i)-(vi) stated in Theorem 2 hold whenever $F(A_o, B_o, C_o, D_o) \neq 0$. To this end, note that Property (ii) holds for all controllers K_c. Besides, Properties (v) and (vi) together imply Properties (iii) and (iv). Hence, it suffices to only consider Properties (i), (v) and (vi). The method proposed in [38] can be used for this purpose (because the underlying problem is a special case of the one tackled in [38]).

Given a controller K_c, notice that Properties (v) and (vi) hold if and only if there exists a block diagonal matrix $\tilde{K}$ (whose i^{th} block diagonal entry is of dimension $m_i \times r_i$, for every $i \in \boldsymbol{\nu}$) such that $\det(sI - \tilde{A} - \tilde{B}\tilde{K}\tilde{C})\det(sI - \bar{A} - \bar{B}\bar{K}\bar{C})^{-1}$ and $\det(sI - A)\det(sI - \bar{A} - \bar{B}\bar{K}\bar{C})^{\nu-1}$ with the variable "s" have no common zero [13]. It can be concluded from the proof of Part (iv) of Theorem 2 that $\det(sI - \tilde{A} - \tilde{B}\tilde{K}\tilde{C})\det(sI - \bar{A} - \bar{B}\bar{K}\bar{C})^{-1}$ is a polynomial (as opposed to a non-polynomial rational function). In line with the proof of Theorem 1 in [38], define the polynomial $P(A_o, B_o, C_o, D_o, \tilde{K})$ as the determinant of the Sylvester matrix associated with the polynomials $\det(sI-\tilde{A}-\tilde{B}\tilde{K}\tilde{C})\det(sI-\bar{A}-\bar{B}\bar{K}\bar{C})^{-1}$ and $\det(sI-A)\det(sI-\bar{A}-\bar{B}\bar{K}\bar{C})^{\nu-1}$. Due to Sylvester's theorem, $P(A_o, B_o, C_o, D_o, \tilde{K})$ is nonzero if and only if the two polynomials mentioned above are co-prime (have no common zero). By fixing the terms A_o, B_o, C_o, D_o, the polynomial $P(A_o, B_o, C_o, D_o, \tilde{K})$ can be arranged in such a way that the monomials depend only on the entries of $\tilde{K}$ and the coefficients possibly depend on A_o, B_o, C_o and D_o. Since Properties (v) and (vi) hold for a generic K_c, it can be deduced that this resultant polynomial has at least one nonzero coefficient. Denoting this coefficient with $F_1(A_o, B_o, C_o, D_o)$, it follows from the argument made in [38] for a general scenario that Properties (v) and (vi) both hold if $F_1(A_o, B_o, C_o, D_o) \neq 0$. On the other hand, the proof of Part (i) of Theorem 2 and the discussion leading to (73) yield that Property (i) is guaranteed to hold if $\det(sI-A)$ and $\det(sI-\bar{A}-\bar{B}\bar{K}\bar{C})$ are co-prime, and in addition

$$\begin{bmatrix} \bar{A} - \lambda_j I & \bar{B} \\ \bar{K}_i \bar{C} & 0 \end{bmatrix} = \text{full row rank}, \quad \forall i \in \boldsymbol{\nu},\ j \in \mathbf{n}. \tag{74}$$

As before, it is easy to obtain a nonzero polynomial $F_2(A_o, B_o, C_o, D_o)$ such that these conditions all hold if $F_2(A_o, B_o, C_o, D_o) \neq 0$. Now, notice that the polynomial $F(A_o, B_o, C_o, D_o)$ can be taken as the least common multiple of $F_1(A_o, B_o, C_o, D_o)$ and $F_2(A_o, B_o, C_o, D_o)$.