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The AdS/CFT correspondence and Spectrum Generating Algebras

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We list the spectrum generating algebras for string theory and M-theory compactified on various backgrounds of the form $AdS_{d+1} \times S^n$. We identify the representations of these algebras which make up the classical supergravity spectra and argue for the presence of these spectrum generating algebras in the classical string/M-theory. We also discuss the role of the spectrum generating algebras on the conformal field theory side.

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1. Introduction

There exists convincing evidence [1,2] for a duality between string theory or M-theory on $AdS_{d+1} \times S^n$ with N units of n-form flux through S^n and a d-dimensional $SU(N)$ superconformal field theory on the boundary of AdS_{d+1}. This conjecture exists both in a weak form and in a strong form. In the weak form the space $AdS_{d+1} \times S^n$, with size proportional to N, is taken to be quite large. In this limit supergravity dominates and captures all the physics of the dual large N superconformal field theory. In the strong form of the conjecture, string theory or M-theory effects need to be taken into account to properly describe the finite N superconformal field theory. Available evidence for the AdS/CFT conjecture focuses mainly on the weak form [1,2], although some progress has been made towards understanding the full stringy spectrum [3].

In order to understand string theory (M-theory is more problematic) on $AdS_{d+1} \times S^n$, the classical string action needs to be quantized in this background. This procedure should produce the discrete spectrum of string states and their masses along with rules for calculating their interactions. In this paper, we use an alternative approach to provide information on the string spectrum. We consider the eleven, ten and six-dimensional supergravity limits of M/string theory, as well as massive ten-dimensional stringy fields expanded in Kaluza-Klein (KK) modes on S^n. Even though identifying the proper independent string degrees of freedom using this method is extremely difficult, we argue that one important qualitative feature of the Kaluza-Klein reduction survives, namely the presence of a so-called *spectrum generating algebra* [4,5,6].

A spectrum generating algebra (SGA) typically does not commute with the Hamiltonian and is non-linearly realized at the level of the action, but it describes the entire spectrum of a particular physical system [4]. SGAs have been very successfully used in nuclear, atomic and molecular physics, not only in the study of spectra but also in the computation of various transition amplitudes (for more details on this subject, consult [4]).

In Kaluza-Klein reductions SGAs usually appear because the towers of harmonics used in these reductions can be fit into unitary irreducible representations (UIR) of the conformal groups of the corresponding compactified spaces (spheres, products of spheres, or any Einstein spaces which have a natural action of the conformal group) [5], [6], [7]. Since the eigenvalues of harmonics are related to the masses of the corresponding Kaluza-Klein states, the algebra of the conformal group does not commute with the Hamiltonian. For the case of compactifications on S^n the corresponding conformal group $SO(n+1,1)$ acts as a spectrum generating algebra [5,6].

The conformal generators of the SGA are not isometries of the compactification manifold. Rather, the operation of rescaling of the manifold corresponds to the "scanning" of the spectrum of the associated operator (Dirac, Laplace, etc.) on the manifold in question. In particular the spherical harmonics on S^n provide a natural UIR of the conformal group of S^n [1], which generates the spectrum of KK modes, see section 2. We will explicitly demonstrate this construction for supergravity fields. To do this, we extend the results known in the supergravity literature [6] by demonstrating how the corresponding spectra fit into UIRs of the relevant conformal group for some of the maximally symmetric examples of the AdS/CFT correspondence: IIB string theory on $AdS_5 \times S^5$ [9], M-theory on $AdS_4 \times S^7$ [10] and $AdS_7 \times S^4$ [11], and IIA or IIB string theory on $AdS_3 \times S^3 \times X^4$ [12](see section 3 and Tables 1.-4.). Our results can be also extended to the case of IIA or IIB string theory on $AdS_2 \times S^2 \times X^6$ [13].[2]

In view of the AdS/CFT correspondence we consider the map between the action of an SGA on the supergravity spectrum and the corresponding action of what we call an *operator generating algebra* (OGA) on the chiral primaries on the CFT side. Demonstrating the presence of SGAs in the supergravity spectrum allows us to argue for and better understand the extension of the SGAs to the full string/M-theory. More explicitly, we discuss the KK towers of the level one massive string states of the flat ten-dimensional (IIA or IIB) string on S^5, and show how they provide UIRs of the corresponding SGA. We expect that the action of the OGA generalizes to include operators in the CFT dual to stringy fields. In particular we discuss how the relevant OGA could act on the so-called Konishi supermultiplet [14,15] which is expected to correspond to massive string states of IIB string theory on S^5, see section 4. Finally, in section 5, we discuss the relevance of SGA in the case of the recent proposal on the finite N case and quantum deformed isometries [16].

2. Review of representation theory of $SO(n+1,1)$

Let us consider a generic supergravity theory compactified on $AdS_m \times S^n$. All supergravity fields can be expanded into harmonic functions on S^n (this is just the physical statement of the Peter-Weyl theorem [8]). It can be shown that these harmonic functions provide a UIR of the conformal group on S^n, which is $SO(n+1,1)$. This can be seen as follows [8,6]: Let S^n denote a unit sphere $\sum x^i x^i = 1$ and let $g \in SO(n+1,1)$

$$g = \begin{pmatrix} a^i_j & b^i \\ c_j & d \end{pmatrix} \tag{2.1}$$

The action of $g \in SO(n+1,1)$ on S^n is given by

$$(gx)^i = (a^i_j x^j + b^i)(c_k x^k + d)^{-1} \tag{2.2}$$

[1] This follows from an extension of the Peter-Weyl theorem, [8], vol.2, chapter 9.

[2] We can as well consider heterotic/M-theory/F-theory compactifications on Calabi-Yau manifolds to six and four dimensions as our starting point.

and the action of $g^{-1} \in SO(n+1,1)$ on complex functions $f : S^n \to \mathbb{C}$ by

$$((g^{-1})f)(x^i) = (c_k x^k + d)^\sigma f((gx)^i) \tag{2.3}$$

where $\sigma \in \mathbb{C}$ is called the weight. Furthermore, let $L^2(S^n)$ be the Hilbert space of square integrable complex functions over S^n with the natural inner product

$$(f_1, f_2) \equiv \int_{S^n} \bar{f}_1(x) f_2(x) \sqrt{G} d^n x \tag{2.4}$$

where $\sqrt{G} d^n x$ is the $SO(n+1,1)$ invariant measure on S^n. Then it is easy to show that this inner product is preserved under the action of $g^{-1} \in SO(n+1,1)$ on f_1, f_2 defined above, provided that the weight $\sigma = -n/2 + i\rho$, where ρ is an arbitrary real number. Thus the space of harmonic functions over S^n provides a unitary irreducible representation of $SO(n+1,1)$ [8].

Since KK modes of supergravity fields on S^n are expected to fit into UIRs of the $SO(n+1,1)$ SGA, we obviously need to use the representation theory of non-compact $SO(n+1,1)$ groups to understand the physical spectrum of KK modes. Their exists a construction for the UIRs of the group $SO(n+1,1)$ [17] completely analogous to the one for UIRs of its maximal compact subgroup $SO(n+1)$. In the case of $SO(n+1)$ a unitary irreducible representation is determined by a set of numbers m_{ij}, $(1 \le i < j \le n+1)$, all of which are integer or half-integer simultaneously (there are important differences between $n+1 = 2p$ and $n+1 = 2p+1$). A vector in the representation space is denoted by $|m_{ij}>$, where m_{ij} provide a complete set of highest weight labels (named Gel'fand-Zetlin (GZ) labels) which uniquely determine an irreducible representation. The labels m_{ij} obey the following conditions

$$\begin{aligned} -m_{2k+1,1} \le m_{2k,1} \le m_{2k+1,1} \le \ldots \le m_{2k+1,k-1} \le m_{2k,k} \le m_{2k+1,k} \\ |m_{2k,1}| \le m_{2k-1,1} \le m_{2k,2} \le \ldots \le m_{2k,k-1} \le m_{2k-1,k-1} \le m_{2k,k} \ , \end{aligned} \tag{2.5}$$

where $k = 1, \ldots, p-1$ if $n+1 \in 2\mathbb{Z}$ and $k = 1, \ldots, p$ if $n+1 \in 2\mathbb{Z}+1$.

Based on this result it can be shown [17] that the UIRs of $SO(n+1,1)$ (with important differences between $n+1 = 2p$ and $n+1 = 2p+1$) are described by a set of $SO(n+1)$ GZ labels m_{ij}, satisfying certain inequalities, along with a weight $\sigma = -n/2 + i\rho$. One important property of the UIRs of $SO(n+1,1)$ is that irreducible representation of $SO(n+1)$ occur within with multiplicity one or not at all [17].

For example, in the case of $SO(2p,1)$ their exist UIRs with $SO(2p)$ content described by the following requirement on the GZ labels

$$|m_{2p,1}| \le m_{2p+1,1} \le \ldots \le m_{2p+1,p-1} \le m_{2p,p} \tag{2.6}$$

where $m_{2p+1,j} = 0, 1/2, 1, \ldots$ for $1 \leq j \leq p-1$ and the weight $\sigma = -p + i\rho$, with $\rho > 0$. These UIRs are labelled $D(m_{2p+1,1} \ldots m_{2p+1,p-1}, i\rho)$. The complete list of UIRs of $SO(n+1,1)$ in this notation is given in [17]. We follow this notation and the results of [17] in the main body of the paper. Note that in the GZ-notation these representations typically consist of a finite number of infinite towers.

GZ labels form a particularly convenient basis for understanding the harmonic analysis on Kaluza-Klein (KK) supergravity [18,19] on any coset space G/H [5]. In particular, the well known $AdS_m \times S^n$ backgrounds of KK supergravity can be understood as coset spaces, upon the Euclideanization of the relevant AdS_m spaces - $AdS_m \to S^m$. Then the spectrum of KK supergravity on $AdS_m \times S^n$ can be obtained from the harmonic analysis on $G/H = SO(m+1)/SO(m) \times SO(n+1)/SO(n)$. In this analysis [5], one fixes the H representations which describe the content of all supergravity fields, and then one expands these fields in terms of only those representations of G which contain the fixed H representations. The GZ, or highest weight labels, provide a natural basis for the implementation of this procedure [10].

More precisely, let the GZ labels of a fixed H representation be denoted by $(\alpha_1, \alpha_2, \ldots, \alpha_r)(\beta_1, \beta_2, \ldots, \beta_{q-1})$, where

$$\alpha_1 \geq \alpha_2 \geq \ldots \geq |\alpha_r|; \quad \beta_1 \geq \beta_2 \geq \ldots \geq \beta_{q-1} \tag{2.7}$$

and analogously, denote the GZ labels of a G representation by $(\gamma_1, \gamma_2, \ldots, \gamma_r)(\delta_1, \delta_2, \ldots, \delta_q)$, where [3]

$$\gamma_1 \geq \gamma_2 \geq \ldots \geq \gamma_r; \quad \delta_1 \geq \delta_2 \geq \ldots \geq |\delta_q| \;. \tag{2.8}$$

Then according to a theorem by Gel'fand and Zetlin [8] (vol. 3), the above H representation is contained in the decomposition of the above G representation provided

$$\begin{aligned} &\gamma_1 \geq \alpha_1 \geq \gamma_2 \geq \ldots \geq \gamma_r \geq |\alpha_r| \\ &\delta_1 \geq \beta_1 \geq \delta_2 \geq \ldots \geq \beta_{q-1} \geq |\delta_q| \;. \end{aligned} \tag{2.9}$$

This theorem combined with the representation theory of $SO(n+1,1)$ can be used to easily read off the corresponding UIRs of the relevant $SO(n+1,1)$ spectrum generating algebra, given the field content of a particular supergravity theory [4].

[3] Here, for illustrational purposes, we have assumed that $m+1 \in 2\mathbb{Z}+1$ such that the rank does not change between $SO(m+1)$ and $SO(m)$. Similarly, we take $n+1 \in 2\mathbb{Z}$. This also shows how the bound on the $\alpha_i, \ldots, \delta_i$ are different depending on the parity of $m+1$ $(n+1)$.

[4] For the reader's convenience we also give the relation between the Dynkin basis (commonly used in the supergravity literature) and the GZ basis. For the case of $SO(2p)$ denote the GZ basis by a set of integers $(l_1, l_2, \ldots l_p)$; then, the Dynkin labels are give by a set of integers $(a_1, a_2, \ldots, a_p)$ such that $a_1 = l_1 - l_2, \ldots, a_{p-1} = l_{p-1} - l_p, a_p = l_{p-1} + l_p$. Similarly, for the case of $SO(2p+1)$, one has $a_1 = l_1 - l_2, \ldots, a_{p-1} = l_{p-1} - l_p, a_p = 2l_p$.

3. Type IIB supergravity on $AdS_5 \times S^5$

Given these technical tools, we now turn to actual physical applications. We consider the case of IIB supergravity on $AdS_5 \times S^5$, since for this case the actual boundary CFT of the proposed duality is precisely defined; it is $\mathcal{N} = 4$ $SU(N)$ super Yang-Mills theory (SYM) in four-dimensions. Although we discuss in detail other supergravity $AdS_{d+1} \times S^n$ examples (see Tables 2. - 4.), we study the $AdS_5 \times S^5$ case (Table 1.) when we discuss the action of the SGA on the full string theory.

The bosonic sector of the ten-dimensional IIB supergravity consists of the following representations of the little group $SO(8)$:

$$\mathbf{1}_{\mathbb{C}} + \mathbf{28}_{\mathbb{C}} + \mathbf{35}_v + \mathbf{35}_c \tag{3.1}$$

(the dilaton and the axion, RR and NS 2-forms, the graviton, and the self-dual RR 4-form). The fermionic sector (spin 1/2 and spin 3/2 fields) is given by

$$\mathbf{8}_{\mathbb{C},s} + \mathbf{56}_{\mathbb{C},s}\ . \tag{3.2}$$

To understand the reduction of this spectrum on $AdS_5 \times S^5$, we first look at how the $SO(8)$ little group representations break up into representations of $SO(5) \times SO(3)$ on the *tangent bundle* of $AdS_5 \times S^5$. In particular, we want to discuss the appearance of physical modes (i.e., those modes that appear as poles in the AdS_5 bulk propagators) and illustrate the general procedure by considering only the bosonic fields [5].

On the tangent bundle of $AdS_5 \times S^5$ the ten-dimensional little group $SO(8)$ splits into $SO(5) \times SO(3)$. We start our discussion by decomposing the $SO(8)$ representations for the graviton, h_{ab}, and the self-dual four form, a_{abcd}, in terms of $SO(5) \times SO(3)$ [6]. We get

$$h_{ab}: \quad \mathbf{35_v} \to \mathbf{1_1} + \mathbf{1_5} + \mathbf{5_3} + \mathbf{14_1},$$

and

$$a_{abcd}: \quad \mathbf{35_c} \to \mathbf{5_1} + \mathbf{10_3},$$

respectively. We are interested in those representations of $SO(6) \times SO(3)$ which contain the above representations of h_{ab} and a_{abcd}, since $SO(6)$ is the isometry group of S^5. It is convenient to list these representations in terms of their highest weight labels under $SO(6)$. The resulting $SO(6)$ labels, with their $SO(3)$ dimensions, are

$$\begin{aligned} &(l,0,0)_1, \quad (l,0,0)_5, \quad (l,0,0)_3,\ (l,1,0)_3, \\ &(l,0,0)_1,\ (l,1,0)_1,\ (l,2,0)_1 \end{aligned}$$

[5] We thank J. de Boer for very useful discussions on this topic.

[6] We use latin indices for ten-dimensional fields, greek indices from the beginning of the alphabet for S^n and greek indices from the middle of the alphabet for AdS_{d+1}.

for the graviton and

$$(l,0,0)_1,\ (l,1,0)_1,\quad (l,1,0)_3,\ (l,1,\pm 1)_3$$

for the self-dual four form, respectively. In order to understand which modes appear as physical from the point of view of the bulk AdS_5 space we need to consider the action of the AdS_5 little group $SO(4)$ on the above representations of $SO(6)\times SO(3)$. These are uniquely lifted to representations of $SO(6)\times SO(4)$, from which we directly read off the physical modes propagating in the bulk AdS_5 space. We get

$$\begin{aligned}
h_{\mu\nu}: &\quad (l,0,0)_{(3,3)}\\
h_{\alpha\mu}: &\quad (l,1,0)_{(2,2)}\\
h_{\alpha\beta}: &\quad (l,2,0)_{(1,1)}\\
h^{\alpha}_{\alpha}: &\quad (l,0,0)_{(1,1)}\\
a_{\alpha\beta\mu\nu}: &\quad (l,1,1)_{(3,1)}+(l,1,-1)_{(1,3)}\\
a_{\alpha\beta\gamma\mu}: &\quad (l,1,0)_{(2,2)}\\
a_{\alpha\beta\gamma\delta}: &\quad (l,0,0)_{(1,1)}
\end{aligned}\tag{3.3}$$

E.g., the $(\mathbf{3},\mathbf{3})$ of $SO(4)$ is given in terms of $\mathbf{1}+\mathbf{3}+\mathbf{5}$ of $SO(3)$ and so on. Note also that there will be a mixing between modes with the same quantum numbers, such as h^{α}_{α} and $a_{\alpha\beta\gamma\delta}$. In the Tables 1.-4. we suppress this mixing and list the modes as above.

By comparing to [9] we see that group theory indeed accounts for all the physical modes. One can also easily extend this analysis to the fermionic part of the spectrum.

The KK towers of physical modes cannot in general be fit alone into UIRs of the conformal group of S^5 - $SO(6,1)$. In order to get full UIRs of $SO(6,1)$ we also need to consider gauge modes (modes that do not appear as poles in the AdS_5 bulk propagators). The most convenient procedure for the identification of KK towers of both physical and gauge modes, and the corresponding UIRs of $SO(6,1)$, is to look at the Euclidean $AdS_5\times S^5$ space as a coset space - $G/H \equiv SO(6)/SO(5)\times SO(6)/SO(5)$. We list the various KK modes in terms of $SO(5)\times SO(5)$ highest weight labels, and then determine which $SO(6)\times SO(6)$ representations contain these fixed $SO(5)\times SO(5)$ representations using the theorem of Gel'fand and Zetlin reviewed in section 2. Here it is important that we started with the full ten-dimensional tangent space $SO(10)$ and not just the little group $SO(8)$ as we would otherwise not see the gauge modes. From the $SO(6)$ highest weight, GZ labels, we directly read off the corresponding UIRs of $SO(6,1)$. These UIRs must occur; the theorem above [8] implies that a complete set of orthonormal harmonic functions on S^5 forms a UIR of the conformal group of S^5, that is $SO(6,1)$.

To make our procedure described clearer, we choose as an example the fields which come from the reduction of the ten-dimensional graviton. We write the $SO(5)$ representations of these fields in terms of GZ labels; they are the $(0,0)_{GZ}$, $(1,0)_{GZ}$ and $(2,0)_{GZ}$

representations. From (2.9), the $(2,0)_{GZ}$ representation of $SO(5)$, a scalar on AdS_5, is contained in the $SO(6)$ representations with labels: $(l+2,2,0)_{GZ}$, $(l+2,1,0)_{GZ}$, and $(l+2,0,0)_{GZ}$ $(l \geq 0)$ which together form the $D^1(2;-5/2)$ UIR of SO(6,1). Only the symmetric tensors with $SO(6)$ labels $(l+2,2,0)_{GZ}$ are physical, matching with $h_{\alpha\beta}$ in (3.3), while the others correspond to gauge modes. The $(1,0)_{GZ}$ representation of $SO(5)$ is contained in the $SO(6)$ representations $(l+1,1,0)_{GZ}$ (physical modes matching $h_{\alpha\mu}$) and $(l+1,0,0)_{GZ}$ (gauge modes) which form the $D^1(1;-5/2)$ representation of $SO(6,1)$. Finally, the $(0,0)_{GZ}$ representation is contained in the $SO(6)$ representations $(l,0,0)_{GZ}$ which form the $D^2(-5/2)$ representation of $SO(6,1)$ (physical modes matching $h_{\mu\nu}$). The fields in this tower couple to the symmetric trace operators on the CFT side.

In the discussion above, modes which usually are ignored because they can be gauged away are *crucial* to the faithful action of the conformal group of S^5 on the Kaluza-Klein spectrum. Other gauge modes also appear in the spectrum in complete representations of the SGA. For example, an analysis of the mode expansion on AdS_5 is enough to show that the ten-dimensional graviton also yields a complete tower of vector gauge modes. We will ignore these complete towers of gauge modes, and only mention gauge modes which combine with physical modes to give UIRs of the conformal group. Generally, gauge modes are probably associated with the diagonal $U(1)$ group on the boundary, whose role in the AdS/CFT duality is still not completely understood (see for example [20]).

We now complete our analysis of the SGA representations which appear in the supergravity spectrum of $AdS_5 \times S^5$. The antisymmetric tensor, the two-form $A_{\mu\nu}$, gives rise to a tower $D^2(-5/2)$ of anti-symmetric chiral and anti-chiral tensor fields, all describing physical modes. The vector $A_{\alpha\mu}$ gives rise to the towers of vectors that make up the $D^1(1;-5/2)$ representation with only the tower with modes of the form $(l+1,1,0)_{GZ}$ in $D^1(1;-5/2)$ physical. The scalar $A_{\alpha\beta}$ gives rise to two physical KK-towers, modes of the form $(l+1,1,\pm 1)_{GZ}$, which make up the $D^0(1,1;-5/2)$ representation.

The rank-four antisymmetric self-dual tensor gives rise to chiral and anti-chiral two-forms $A_{\alpha\beta\mu\nu}$ with towers making up two $D^0(1,1;-5/2)$ representations. Each $D^0(1,1;-5/2)$ has two physical towers with $(l+1,1,\pm 1)_{GZ}$ of $SO(6)$, adding up to four towers of physical two-forms . The vector $A_{\alpha\beta\gamma\mu}$ gives a tower $D^1(1;-5/2)$ of vectors but only the $(l+1,1,0)_{GZ}$ tower of $D^1(1;-5/2)$ describes physical modes. The scalar mode, $A_{\alpha\beta\gamma\delta}$, mixes with the h^{α}_{α} scalar as can be seen from our earlier discussion, with each of the mass eigenmodes giving rise to $D^2(-5/2)$. Finally, the complex scalar, in terms of the axion and dilaton fields, gives rise to yet one more physical KK-tower of complex scalars, $D^2(-5/2)$.

The spin-1/2 field λ gives twin towers of chiral and anti-chiral spinors in the $D(1/2,1/2;-5/2+i\rho)$ representation. Each of these contains physical modes $(l+1/2,1/2,\pm 1/2)_{GZ}$, so each $D(1/2,1/2;-5/2)$ yields two towers. The chiral and anti-chiral gravitini ψ_μ also come in the representation $D(1/2,1/2;-5/2+i\rho)$, each with a total of two physical towers with modes of the form $(l+1/2, 1/2, \pm 1/2)_{GZ}$. Finally, we get KK-towers of chiral and anti-chiral spin-1/2 fields from ψ_α, each in the $D(3/2, 1/2; -5/2+i\rho)$ representation, and each of these yielding physical modes of the form $(l+3/2, 1/2, \pm 1/2)_{GZ}$. We summarize these results in Table 1. Tables 2. - 4. which contain the fields and UIRs for the cases of $AdS_4 \times S^7$, $AdS_7 \times S^4$ and $AdS_3 \times S^3$ respectively, are obtained following the same procedure.

4. Conformal field theory, operator generating algebra, massive string modes

We want to discuss what the SGA for the Kaluza-Klein states of $AdS_{d+1} \times S^n$ means on the dual CFT side. We concentrate on the AdS_5/CFT_4 correspondence, the $\mathcal{N}=4$ $SU(N)$ super Yang Mills theory in four-dimensions. Other cases are more difficult because the dual CFT is not easily described, though we believe that similar arguments to those below can be applied there as well.

We start from the fact that each supergravity KK tower corresponds to a set of chiral primaries on the CFT side with appropriate $SO(6)$ R-charges [2]. Chiral primaries appear in the trace of a symmetric product of $\mathcal{N}=4$ chiral superfields [14]. For example, the traceless part of the following operator

$$Tr(W^{(i_1} W^{i_2} \cdot \ldots \cdot W^{i_p)}) \tag{4.1}$$

corresponds to the KK states of IIB supergravity on $AdS_5 \times S^5$, where W is the $\mathcal{N}=4$ chiral superfield. We have shown that these KK states belong to UIRs of the $SO(6,1)$ SGA. Given the map between KK modes and CFT chiral primaries, we naturally expect that the complete set of UIRs of the $SO(6,1)$ SGA listed in Table 1 corresponds to

$$Tr(W^{i_1}) \oplus Tr(W^{(i_1} W^{i_2)}) \oplus \ldots \oplus Tr(W^{(i_1} W^{i_2} \cdot \ldots \cdot W^{i_p)}) \oplus \ldots \tag{4.2}$$

Note that there exists an ambiguity as to whether or not W^i transforms in $SU(N)$ or $U(N)$ [2][14]. This ambiguity is most likely related to the inclusion of gauge modes in the complete $SO(6,1)$ UIRs. Taking the lowest component of (4.2), the operators made up of the traceless part of

$$Tr(\phi^{(i_1} \phi^{i_2} \cdot \ldots \cdot \phi^{i_p)}) \tag{4.3}$$

(ϕ is the $\theta^0 \bar{\theta}^0$ component of the $\mathcal{N}=4$ chiral superfield W) fit into the $D^2(-5/2)$ of representation of $SO(6,1)$. Modulo subtleties involving gauge modes and the extra $U(1)$, the other components of (4.1) fill out the remaining UIRs listed in Table 1. This CFT counterpart of the spectrum generating algebra of KK supergravity we call operator generating algebra (OGA).

It is natural to ask whether this operator generating algebra extends to all the operators in $\mathcal{N}=4$ SYM, including the non-chiral ones which correspond to massive string modes. In order to check this, we would have to classify and organize all the non-chiral operators on the CFT side. We do not know of any such classification. What we do know is that part of the $SO(6,1)$ OGA acts on the chiral primaries by tensoring with a superfield in the **6** of $SO(6)$ and symmetrizing. How does this procedure generalize to non-chiral primaries? We sketch a natural proposal as follows. Given an operator $Tr(O(W))$, a set of UIRs of the $SO(6,1)$ OGA is generated by the following operators

$$Tr(O(W)W^{i_1}) \oplus Tr(O(W)W^{(i_1}W^{i_2)}) \oplus \ldots \oplus Tr(O(W)W^{(i_1}W^{i_2} \cdot \ldots \cdot W^{i_p)}) \oplus \ldots \quad (4.4)$$

The fact that operators such as $Tr(O(W)W^{i_1})$ are not necessarily irreducible and give direct sums of $SO(6)$ representations is useful for generating operators dual to both physical modes and gauge modes. Unfortunately, since we have not explicitly determined the generators of the proposed $SO(6,1)$ OGA, we cannot actually prove that various operators belong to UIRs of this OGA. The issue also arises as to how to deal with the possible mixing of different operators within the same UIR, a problem which already exists for the chiral primary operators.

Let us illustrate how our proposal for an OGA might work by considering the so-called Konishi multiplet on the CFT side. In terms of the SYM superfields, this multiplet is written as $Tr(W_iW^i)$ [14]. It has been suggested that the Konishi multiplet corresponds to massive string states propagating in AdS_5 [14,15]. Consider the scalar operator in the Konishi multiplet which is the $\theta^4\bar{\theta}^4$ component of $Tr(W_iW^i)$ and transforms in the $\mathbf{105}=(4,0,0)_{GZ}$ of $SO(6)$. Suppose we assume that it sits naturally at the bottom of a $(l+4,0,0)_{GZ}$ KK tower of $SO(6)$. On the AdS side this is what we would expect from scalars coming from a ten-dimensional four-tensor reduced on S^5. A good candidate for the appropriate $SO(6,1)$ UIR is then $D^1(4;-5/2)$. It is made up of the towers

$$(l+4,0,0)_{GZ},\ (l+4,1,0)_{GZ},\ (l+4,2,0)_{GZ},\ (l+4,3,0)_{GZ},\ (l+4,4,0)_{GZ}. \quad (4.5)$$

None of the extra towers in (4.5) have operators which can appear in the Konishi multiplet, but if we take into account the whole set of operators given by $Tr(W_iW^iW^{(i_1} \cdot \ldots \cdot W^{i_p)})$, then at $p=1,2,3,4$ we find operators (dual to bulk scalars) with $SO(6)$ weights

$$(4,1,0)_{GZ},\ (4,2,0)_{GZ},\ (4,3,0)_{GZ},\ (4,4,0)_{GZ} \quad (4.6)$$

which could sit at the bottoms of the extra towers. It is important to note that we do not know whether the operators above are dual to gauge modes or physical modes, since we lack a precise rule for making this distinction. Still, our primitive fit for the Konishi multiplet is an indication that there might exist an OGA, $SO(6,1)$, on the CFT side which organizes even the non-chiral operators.

Let us now address these issues from the AdS side. What happens with stringy, massive modes on the AdS_5 side? These modes do not have protected anomalous dimensions on the CFT side. This is clear, since if we expand the stringy fields in S^5 spherical harmonics their ten-dimensional masses will contribute α' terms to their KK reduced AdS masses. The non-linear nature of the equations relevant to stringy modes will contribute further corrections and will also mix modes with the same $SO(6)$ quantum numbers. Nevertheless, using the theorem explained above [8,6], the orthonormal basis of harmonic functions on S^5 provides UIRs of the SGA $SO(6,1)$. Thus, we expect that even the massive stringy modes can be fit in UIRs of this SGA. There is, however, a subtlety here: to *prove* that $SO(6,1)$ is the SGA of the full IIB string theory on $AdS_5 \times S^5$ we need to identify *all* the KK modes generated by the massive stringy modes, and then fit them explicitly (as we have done with the massless KK modes) in the relevant UIRs of $SO(6,1)$.

One way of getting to the string theory on $AdS_5 \times S^5$ is to start with the flat ten-dimensional string and then perturb it with an RR operator (as in [21]) such that the theory flows to the $AdS_5 \times S^5$ background [3]. One can contemplate a connection between the large radius limit of $AdS_5 \times S^5$ and the flat ten-dimensional space, by taking $N \to \infty$ and keeping g_{YM} finite on the CFT side. In this limit the states from the AdS_5 side should presumably map into states propagating in the flat ten-dimensional space, the corresponding vertex operators should match, etc. The KK reduction of the massive string modes on S^5 from flat ten dimensions should get rearranged into the massive spectrum of the string on $AdS_5 \times S^5$. Also, on both sides there should exist a natural action of the conformal group of S^5. If we can show that this group acts as an SGA on the quantum ten-dimensional spectrum reduced on S^5, we expect the conformal group of S^5 to appear as an SGA for the string theory quantized directly about the $AdS^5 \times S^5$ background.

Let us examine the KK reduction of the first massive level of the flat IIB string on S^5. The multiplet transforms in the $(\mathbf{44}+\mathbf{84}+\mathbf{128})^2$ of $SO(9)$ and has 256^2 states. We consider perturbations around the classical solution caused by the presence of massive string modes in this multiplet. We apply the same harmonic analysis used on the supergravity modes, and decompose the $SO(9)$ representations coming from $(\mathbf{44}+\mathbf{84}+\mathbf{128})^2$ in terms of $SO(5) \times SO(4)$ representations [7]. As before, we work in the GZ basis, which enables us to read off the corresponding UIRs of $SO(6,1)$. We consider briefly one example of this particular procedure. If we look at $\mathbf{44} \times \mathbf{44}$ we find a $\mathbf{450}$ of $SO(9)$, in addition to other representations of $SO(9)$ which we will ignore for now. The $\mathbf{450}$ is a four-tensor field in ten dimensions. It decomposes into - among others - a $(4,0)_{GZ}$ of $SO(5)$ which is contained in $(l+4,4,0)_{GZ}$, $(l+4,3,0)_{GZ}$, $(l+4,2,0)_{GZ}$, $(l+4,1,0)_{GZ}$, $(l+4,0,0)_{GZ}$ of

[7] Note that we automatically get representations of the little group, $SO(4)$, of AdS_5.

$SO(6)$ and generates the $D^1(4;-5/2)$ UIR of $SO(6,1)$. This particular UIR appeared in our discussion of the Konishi multiplet. The same procedure can be extended to all fields at this massive level, and to all massive levels.

In our analysis group theory has supplied us with details about the spectrum. However, there are subtleties which can only be addressed by examining the corrected classical equations of motion; proper identification of physical and gauge modes as well as the mixing of various KK modes. These phenomena happen already at the massless level, so they are not surprising. These subtleties do not change the fact that physical and/or gauge modes form UIRs of $SO(6,1)$! So, the conclusion seems to be that the $SO(6,1)$ SGA from supergravity extends to the full string theory. Of course, in order to prove this statement one would have to examine all massive modes explicitly, and address the question of mixing and identification of physical and gauge modes.

5. Discussion

To conclude, in this paper we have listed the spectrum generating algebras for string theory and M-theory compactified on various backgrounds of the form $AdS_{d+1} \times S^n$. We have identified the representations of these algebras which make up the classical supergravity spectra and we argued for the existence of these spectrum generating algebras in the classical string/M-theory. We also discussed the role of the spectrum generating algebras on the conformal field theory side in the framework of AdS/CFT correspondence.

One case we have not explicitly considered but which can be analyzed in the same way is the $AdS_2 \times S^2$ background. The corresponding boundary theory is some sort of conformal quantum mechanics, which is not well understood [13]. Whatever that boundary theory might be, there should exist an $SO(3,1)$ SGA algebra on the supergravity/string side and a corresponding OGA on the conformal quantum mechanics side.

Our methods should also apply to the case of string theory on $AdS \times S^n/G$ [22], where G is a discrete subgroup of the isometry group of the sphere. It would be interesting to understand the action of SGAs in this case.

One problem where we expect the concept of SGAs to have a dynamical meaning is in the computation of correlation functions within the framework of AdS/CFT duality.

Finally, an interesting question regarding SGAs concerns their interpretation in the finite N case of AdS/CFT duality (strong conjecture). Jevicki and Ramgoolam [16] have proposed that quantum deformed isometries should be relevant in this case. We note that there exists an analog of Peter-Weyl theorem for the case of $SU(2)_q$ - see [8](vol. 3). The harmonic functions for a q-deformed sphere can be also found in [8](vol. 3). It seems natural to expect that the harmonic functions over $SO(n)_q$ fit into UIRs of $SO(n+1,1)_q$, thus generalizing our previous results. In view of the proposal put forward in [16], we expect the full string theory on $AdS_{d+1} \times S^m$ to exhibit q-deformed SGAs.

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$h_{\mu\nu}$	$h_{\alpha\nu}$	$h_{\alpha\beta}$	$h^{\alpha}{}_{\alpha}$
$D^2(-5/2)$	$D^1(1;-5/2)$	$D^1(2;-5/2)$	$D^2(-5/2)$
$A_{\mu\nu}$	$A_{\alpha\mu}$	$A_{\alpha\beta}$	
$D^2(-5/2)$	$D^1(1;-5/2)$	$D^0(1,1;-5/2)$	
$a_{\alpha\beta\mu\nu}$	$a_{\alpha\beta\gamma\mu}$	$a_{\alpha\beta\gamma\delta}$	$a+i\phi$
$D^0(1,1;-5/2)$	$D^1(1;-5/2)$	$D^2(-5/2)$	$D^2(-5/2)$
ψ_μ	ψ_α	λ	
$D(1/2,1/2;-5/2+i\rho)$	$D(1/2,1/2;-5/2+i\rho)$	$D(3/2,1/2;-5/2+i\rho)$	

Table 1: The AdS_5 field content of IIB supergravity compactified on $AdS_5 \times S^5$ [9], organized in UIRs of $SO(6,1)$.

$h_{\mu\nu}$	$h_{\alpha\nu}$	$h_{\alpha\beta}$	$h^{\alpha}{}_{\alpha}, h^{\mu}{}_{\mu}$
$D^3(-7/2)$	$D^2(1;-7/2)$	$D^2(2;-7/2)$	$D^3(-7/2)$
$C_{\alpha\mu\nu}$	$C_{\alpha\beta\mu}$	$C_{\alpha\beta\gamma}$	
$D^2(1;-7/2)$	$D^1(1,1;-7/2)$	$D^0(1,1,1;-7/2)$	
ψ_μ	ψ_α	λ	
$D(1/2,1/2,1/2;-7/2+i\rho)$	$D(1/2,1/2,1/2;-7/2+i\rho)$	$D(3/2,1/2,1/2;-7/2+i\rho)$	

Table 2: The AdS_4 field content of eleven-dimensional supergravity compactified on $AdS_4 \times S^7$ [10], organized in UIRs of $SO(8,1)$ (matching [6]).

$h_{\mu\nu}$	$h_{\alpha\nu}$	$h_{\alpha\beta}$	$h^{\alpha}{}_{\alpha}$
$D^2(-2)$	$D^1(1;-2)$	$D^1(2;-2)$	$D^2(-2)$
$C_{\mu\nu\rho}$	$C_{\alpha\mu\nu}$	$C_{\alpha\beta\mu}$	$C_{\alpha\beta\gamma}$
$D^2(-2)$	$D^1(1;-2)$	$D^0(1,1;-2)$	$D^2(-2)$
ψ_μ	ψ_α		
$D(3/2,1/2;-2+i\rho)$	$D(1/2,1/2;-2+i\rho$		

Table 3: The AdS_7 field content of eleven-dimensional supergravity compactified on $AdS_7 \times S^4$ [11], organized in UIRs of $SO(5,1)$.

$h_{\mu\nu}$	$h_{\alpha\nu}$	$h_{\alpha\beta}$	
$D^1(-3/2)$	$D^0(1;-3/2)$	$D^0(2;-3/2)$	
$A_{\mu\nu}$	A_μ	A_α	ϕ
$D^0(1;-3/2)$	$D^1(-3/2)$	$D^0(1;-3/2)$	$D^1(-3/2)$
ψ_μ	ψ_α	λ	
$D(1/2;-3/2+i\rho)$	$D(3/2;-3/2+i\rho)$	$D(1/2;-3/2+i\rho)$	

Table 4: The AdS_3 field content of six-dimensional supergravity compactified on $AdS_3 \times S^3$ [12], organized in UIRs of $SO(4,1)$.

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