Supplementary Material for
“Gapless excitations in strongly fluctuating superconducting wires”
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MICROSCOPIC PHASE ACTION

In order to describe correlations of the order parameter in a superconducting wire we examine its microscopic action obtained from the BCS Hamiltonian by a Hubbard-Stratonovich transformation followed by an expansion around the saddle point [1, 2]. In the low temperature limit, this yields [1, 2]:

\[ S = \nu_0 A \Delta_0^2 \int_0^L dx \int_0^{1/T} dt \left\{ \frac{\nu_0^2}{2} \ln (\rho^2) - 1 \right\} + 2\xi_0^2 \rho^2 \left[ \phi'^2 + \frac{1}{v_{\phi}} \phi''^2 \right] + \xi_0^2 \left[ \rho'^2 + \frac{1}{v_{\rho}} \rho''^2 \right] \],

where \( L \) and \( A \) are the wire’s length and cross section, respectively, \( \xi_0^2 = \pi D / 8 \Delta_0 \), \( v_{\rho} = \sqrt{3\pi/2} D \Delta_0 \), \( \phi = \sqrt{\pi D \Delta_0} (2AV_c \nu_0 + 1) \propto v_{\phi} \sqrt{N_\perp} \) the phase velocity, \( V_c \) the Fourier transform of the short range Coulomb interaction, \( N_\perp = p^2 F A / \pi^2 \) is the number of one dimensional channels in the wire, \( \nu_0 \) the density of states, \( D \) the electronic diffusion constant in the normal state, and the SC order parameter is parameterized as \( \Delta = \Delta_0 \rho e^{i\phi} \), with \( \Delta_0 \), the mean field solution. Rescaling the imaginary time by \( y = v_{\rho}\tau \), the low energy excitations of the system are phase fluctuations whose action follow:

\[ S[\phi] = K/2 \int dxdy \left\{ (\partial_x \phi)^2 + (\partial_y \phi)^2 / N_\perp \right\}, \tag{1} \]

where the phase stiffness is

\[ K = \frac{4\nu_0 A \Delta_0^2 \xi_0^2}{v_{\rho}} \approx \frac{R_Q}{2R_\xi}. \tag{2} \]

The system described by this model undergoes a Kosterlitz-Thouless phase transition between a quasi-ordered phase (superconductor) and a disordered phase where phase slip pairs unbind [3]. Correlations of the order parameter in the disordered phase decay exponentially:

\[ \langle \Delta(x, \tau)\Delta^\dagger(0,0) \rangle = \Delta_0^2 e^{-x/\xi_{KT}} e^{-\tau/\tau_{KT}}, \tag{3} \]

over a typical length \( \xi_{KT} \), and time \( \tau_{KT} \). This corresponds to

\[ \langle \Delta\Delta^\dagger \rangle_{q,\Omega} = \frac{\Delta_0^2 \xi_{KT} \tau_{KT}}{(1 + q^2 \xi_{KT}^2)(1 + \Omega^2 \tau_{KT}^2)} \]. \tag{4}

LEADING ORDER CORRECTION TO THE TUNNELING DENSITY OF STATES OF A FLUCTUATING SUPERCONDUCTOR

The tDOS is given by

\[ \nu_e = -\frac{1}{\pi} \text{Im} G^R(r, r, \epsilon) = -\frac{1}{\pi} \text{Im} \int \frac{d^3p}{(2\pi)^3} G^R(p, \epsilon), \tag{5} \]

where \( G^R(r, r, \epsilon) \) is the retarded Green’s function which can be expressed to second order in the pairing amplitude:
\[ G(p, \omega_n) = G_0(p, \omega_n) + T \sum_{q, \Omega} G_0(p, \omega_n) \Lambda(q, \omega_n, \omega_n + \Omega) G_0(p + q, \omega_n + \Omega) \Lambda(q, \omega_n + \Omega, \omega_n) G_0(p, \omega_n) \langle \Delta \Delta^\dagger \rangle_{q, \Omega}. \] (6)

Here:

\[ G_0(k + q, \omega)^{-1} = i(\omega) + \frac{i}{2\tau} \text{sign}(\omega) - \xi \]

\[ \Lambda(\omega, \omega + \Omega, q) = \frac{1}{2\tau} \frac{\Theta(\omega(\omega + \Omega))}{|2\omega + \Omega| + Dq^2 + 1/\tau_\phi}, \]

and correlations of the order parameter are given by Eq. (4). The density of states is then given by

\[ \delta \nu(\epsilon) = \nu(\epsilon) - \nu_0 \nu_0 = -\frac{1}{\pi} \text{Im} \int d\omega \frac{\Theta(\omega_0(\omega_0 + \Omega))}{(2\omega_0 + \Omega)^2} \frac{\xi K_T}{2\pi^2 + q^2 \xi K_T^2} \]

\[ \approx \frac{2\pi i \text{sign}(\omega_n) T \sum_{q, \Omega} \Theta(\omega_n(\omega_n + \Omega)) \tau K_T}{|2\omega_n + \Omega| + Dq^2 + 1/\tau_\phi} \int dq \frac{\xi K_T}{2\pi^2 + q^2 \xi K_T^2} \frac{\xi K_T}{2\pi^2 + q^2 \xi K_T^2} \langle \Delta \Delta^\dagger \rangle_{q, \Omega}, \] (7)

Using Eq. (4) to describe the phase fluctuations in a phase-slip proliferated wire, in the low energy limit \( \tau_\phi \ll \tau K_T \) we may approximate Eq. (7) as

\[ I(\omega_n) \approx \frac{2\pi i \text{sign}(\omega_n) T \sum_{q, \Omega} \Theta(\omega_n(\omega_n + \Omega)) \tau K_T}{|2\omega_n + \Omega| + Dq^2 + 1/\tau_\phi} \int dq \frac{\xi K_T}{2\pi^2 + q^2 \xi K_T^2} \frac{\xi K_T}{2\pi^2 + q^2 \xi K_T^2} \frac{\xi K_T}{2\pi^2 + q^2 \xi K_T^2} \]

\[ \approx \frac{\pi i \text{sign}(\omega_n) \xi K_T^2}{|2\omega_n + \Omega|^2} \left\{ \frac{i}{4\pi} \left[ \frac{1}{2 \pi T} + \frac{\omega_n}{2 \pi T} \right] - \frac{\omega_n}{2 \pi T} - \frac{i}{2 \pi T \tau K_T} \right\} + \frac{1}{2} \coth \frac{1}{2 \pi T \tau K_T}, \] (8)

where \( \Psi(z) \) is the digamma function.

**LEADING ORDER CORRECTION TO THE SELF ENERGY**

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**FIG. 1:** The leading order correction to the self energy, given by Eq. (9). The solid line is the bare electronic Green’s function, \( G_0 \), the double wavy is the renormalized pairing interaction, \( \langle \Delta \Delta^\dagger \rangle \), and the dashed lines are the impurity scattering.

The leading order correction to the self energy, shown in Fig. 1 is given by \( G^{-1} = G_0^{-1} - \Sigma \) with:

\[ \Sigma = \sum_q T \sum_{\Omega} G(k + q, \omega + \Omega) \langle \Delta \Delta^\dagger \rangle_{q, \Omega} A^2(\omega, \omega + \Omega, q). \] (9)

The integral over fermionic momentum is dominated by \( \xi \approx 1/\tau \). Since, \( \omega_\tau, \Omega_\tau, Dq^2 \tau \ll 1 \), we can approximate \( \tilde{G}(k + q, \omega + \Omega) \approx \tilde{G}(k, \omega) \). This gives

\[ \Sigma \approx \tilde{G}(k, \omega) \sum_q T \sum_{\Omega} \frac{\Theta(\omega(\omega + \Omega))}{4\pi^2 (2\omega + \Omega + Dq^2 + 1/\tau_\phi)^2} \langle \Delta \Delta^\dagger \rangle_{q, \Omega}, \]

\[ \equiv \tilde{G}(k, \omega) A(\omega). \] (10)
Using this expression for the self energy we can write the Green’s function as:

\[ G(k, \omega)^{-1} = i(\omega) + i \frac{2}{\pi} \text{sign}(\omega) - \xi_k - \Sigma(\omega) \]

(11)

\[ = i\tilde{\omega} - \xi - \frac{1}{i\tilde{\omega} + \xi}A(\omega), \]

(12)

where \( \tilde{\omega} = \omega + \frac{1}{2\pi} \text{sign}(\omega) \). The density of states is given by:

\[ \nu(i\omega) = \frac{i\nu}{\pi} \int dkG(k, \omega) = \frac{i\nu}{\pi} \int d\xi \frac{i\tilde{\omega} + \xi}{\tilde{\omega}^2 + \xi^2 + A(\omega)} \]

(13)

where the odd integral over \( \xi \) vanishes. In the limit of \( \omega\tau \ll 1 \) we have:

\[ \nu(i\omega) \approx \nu \frac{i\text{sign}(\omega)}{\sqrt{\tilde{\omega}^2 + A(\omega)}} \]

(14)

In order to evaluate \( 4\tau^2A(\omega) \), we note that \( 4\tau^2A(\omega) \) is given by Eq. (7). Using Eq. (8) in the limit \( T \tau_K \ll 1 \) we find:

\[ 4\tau^2A(\omega) = \frac{\Delta_0^2}{2} \frac{1}{(2\omega + 1/\tau_\phi)^2} \left\{ \frac{i}{4\pi} \left[ \Psi\left(\frac{1}{2} + \frac{\omega}{2\pi T} \right) + \Psi\left(\frac{1}{2} - \frac{\omega}{2\pi T} \right) - \Psi\left(\frac{1}{2} - \frac{\omega}{2\pi T} \right) - \Psi\left(\frac{1}{2} + \frac{\omega}{2\pi T} \right) \right] + \frac{1}{2} \coth\left(\frac{1}{2\tau_K T} \right) \right\} \]

(15)

Here we have assumed \( \omega \sim T \ll 1/\tau_K \). Performing the analytic continuation \( i\omega \rightarrow \epsilon + i\delta \) we find

\[ 4\tau^2A(i\omega \rightarrow \epsilon + i\delta) = \frac{\Delta_0^2}{2} \frac{1}{(-2\epsilon + 1/\tau_\phi)^2} \left\{ \frac{i}{4\pi} \left[ i\pi + 2i\pi T \tau_K T - 2i\epsilon \tau_K T \right] + 1/2 \right\} \]

(16)

The density of states is given by

\[ \nu(\epsilon) = \Im\nu(i\omega \rightarrow \epsilon + i\delta) = \Im \left[ \frac{i\nu_0}{\Delta_0} 2 \frac{(2\epsilon + 1/\tau_\phi)}{\sqrt{4 + 4\epsilon^2 \tau_K T - T \tau_K T}} \right]. \]

(17)

In the low temperature limit \( T \tau_K, T \tau_\phi \ll 1 \), we can replace \( \nu(T) = -\int d\nu(\epsilon) \frac{d\epsilon}{d\nu} \approx \nu(\epsilon = 0, T) \), leading to:

\[ \frac{\nu(T)}{\nu_0} = \frac{2\sqrt{2}}{\Delta_0 \tau_\phi(T)}. \]

(18)