Dehn surgeries on knots in product manifolds

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Abstract
We show that if a surgery on a knot in a product sutured manifold yields the same product sutured manifold, then this knot is a 0– or 1–crossing knot. The proof uses techniques from sutured manifold theory.

Dedicated to the memory of Professor Andrew Lange

1 Introduction
An interesting problem on Dehn surgery is: when does a surgery on a knot yield a manifold homeomorphic to the original ambient manifold? The most famous result in this direction is the Knot Complement Theorem proved by Gordon and Luecke [8]: when the ambient manifold is $S^3$, only the unknot admits surgeries which yield $S^3$.

In this paper, we are going to study this problem for knots in surfaces times an interval. Our main result is as follows.

Theorem 1.1. Suppose $F$ is a compact surface, $K \subset F \times I$ is a knot. Suppose $\alpha$ is a nontrivial slope on $K$, and $N(\alpha)$ is the manifold obtained from $F \times I$ via the $\alpha$–surgery on $K$. If the pair $(N(\alpha), (\partial F) \times I)$ is homeomorphic to the pair $(F \times I, (\partial F) \times I)$, then one can isotope $K$ such that its image on $F$ under the natural projection $p: F \times I \to F$ has either no crossing or exactly one crossing.

The slope $\alpha$ can be determined as follows. Let $\lambda_b$ be the “blackboard” frame of $K$ associated with the previous projection. Namely, $\lambda_b$ is the frame specified by the surface $F$. When the projection has no crossing, $\alpha = \frac{1}{n}$ for some integer $n$ with respect to $\lambda_b$; when the minimal projection has exactly one crossing, $\alpha = \lambda_b$.

It is easy to see the surgeries in the statement of Theorem 1.1 do not change the homeomorphism type of the pair $(F \times I, (\partial F) \times I)$. In fact, when $K$ is a 0–crossing knot, it is clear that the $\frac{1}{n}$–surgery preserves the homeomorphism
type of the pair. When $K$ is a 1–crossing knot, we can add a one-handle to $F \times \frac{1}{2}$ near the crossing to get a Heegaard surface $F'$ for $F \times I$. $K$ can be embedded into $F'$ as in Figure 1. $F'$ splits $F \times I$ into two parts $U_0, U_1$, where $U_0$ is $F \times [0, \frac{1}{2}]$ with a one-handle added to $F \times \frac{1}{2}$, and $U_1$ is $F \times [\frac{1}{2}, 1]$ with a one-handle added to $-F \times \frac{1}{2}$. The embedding of $K$ can be chosen such that $K$ goes through each of the two one-handles exactly once. Now the blackboard frame $\lambda_b$ is the frame specified by $F'$, and the $\lambda_b$–surgery on $F'$ cancels each one-handle with a two-handle. Hence the new pair is still homeomorphic to $(F \times I, (\partial F) \times I)$.

**Definition 1.2.** Notations are as in the previous theorem. Fix a product structure on $(\partial F) \times I$. Up to an isotopy relative to $(\partial F) \times I$, this product structure uniquely extends to a product structure $\mathcal{P}$ on $F \times I$ and a product structure $\mathcal{P}_\alpha$ on $N(\alpha)$. (This fact can be proved using Alexander’s trick.) Identify $F$ with $F \times 1$. Let $i, i_\alpha: F \times 0 \to F \times 1$ be the natural identity maps with respect to $\mathcal{P}$ and $\mathcal{P}_\alpha$, respectively. We call

$$\varphi_\alpha = i \circ i_\alpha^{-1}: F \to F$$

the map induced by the $\alpha$–surgery. This map $\varphi_\alpha$ fixes $\partial F$ pointwise, and is unique up to an isotopy relative to $\partial F$. Hence $\varphi_\alpha$ can be viewed as an element in the mapping class group $\mathcal{MCG}(F, \partial F)$.

The definition of the map $\varphi_\alpha$ is justified by the following lemma.

**Lemma 1.3.** Let $Y(\alpha)$ be the manifold obtained from $F \times S^1$ by $\alpha$–surgery on $K$. Then $Y(\alpha)$ can be obtained from $F \times I$ by identifying $(x, 0)$ with $(\varphi_\alpha(x), 1)$ for any $x \in G$.

**Proof.** The manifold $F \times S^1$ is obtained from $F \times I$ by identifying $y$ with $i(y)$ for each $y \in F \times 0$. Let $y = (x, 0)$ with respect to the product structure $\mathcal{P}_\alpha$ on $N(\alpha)$, then $i_\alpha(y) = (x, 1)$ with respect to $\mathcal{P}_\alpha$. We then have

$$i(y) = \varphi_\alpha(x, 1) = (\varphi_\alpha(x), 1),$$

since we identify $F$ with $F \times 1$ in the above definition. Hence $(x, 0)$ is identified with $(\varphi_\alpha(x), 1)$ in $Y(\alpha)$ for each $x \in F$.  

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Proposition 1.4. Notations are as in Theorem 1.1. When the projection of $K$ has no crossing and $\alpha = \frac{1}{n}$,

$$\varphi_\alpha = \tau^n,$$

where $\tau$ is the righthand Dehn twist along $K \subset F$. When the minimal projection of $K$ has exactly one crossing, let $a, b, c$ be the simple closed curves obtained by resolving the crossing in two different ways as in Figure 2 and let $\tau_a, \tau_b, \tau_c$ be the righthand Dehn twists along $a, b, c$. Then

$$\varphi_\alpha = \tau_a^{-2} \tau_b \tau_c^{-1}$$

when the crossing is positive, and

$$\varphi_\alpha = \tau_a^{-2} \tau_b^{-2} \tau_c$$

when the crossing is negative.

This paper can be compared with Ni [9]. In fact, Theorem 1.4 in [9] can be restated in a form similar to Theorem 1.1.

Theorem 1.5. Suppose $F$ is a compact surface, $K \subset F \times I$ is a knot and $\alpha$ is a slope on $K$. Let $N(\alpha)$ be the manifold obtained by the $\alpha$–surgery on $K$. If $F \times \{0\}$ is not Thurston norm minimizing in $H_2(N(\alpha), (\partial F) \times I)$, then there is an ambient isotopy of $F \times I$ which takes $K$ to a curve in $F \times \{\frac{1}{2}\}$. Moreover, $\alpha$ is the frame on $K$ specified by $F \times \{\frac{1}{2}\}$.

The proof of Theorem 1.5 uses Gabai’s sutured manifold theory [2, 3, 4] and an argument due to Ghiggini [7]. Using a different method, Scharlemann and Thompson [12] get the same conclusion of Theorem 1.5 under the assumption that $F \times \{0\}$ is compressible in $N(\alpha)$.

This paper is organized as follows. In Section 2, we give some preliminaries on sutured manifold theory and foliations, as well as a characterization of one-crossing knot projections. In Section 3, we study some warm-up cases. In Section 4, we use the argument in the proof of Theorem 1.5 to reduce our

![Figure 2: A 1–crossing knot](image)
problem to the case where $F$ is a pair of pants. In Section 5, we study this case by analyzing the map induced by surgery and using a variant of the argument in Ni [9].

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2 Preliminaries

In this section, we are going to review the sutured manifold theory introduced by Gabai in [2]. We also state a uniqueness result for the Euler classes of taut foliations of fibred manifolds. In addition, we define “double primitive” knots in $F \times I$ and show that they are exactly the knots with a projection consisting of only one crossing.

2.1 Sutured manifold decompositions

Definition 2.1. A sutured manifold $(M, \gamma)$ is a compact oriented 3–manifold $M$ together with a set $\gamma \subset \partial M$ of pairwise disjoint annuli $A(\gamma)$ and tori $T(\gamma)$. The core of each component of $A(\gamma)$ is a suture, and the set of sutures is denoted by $s(\gamma)$.

Every component of $R(\gamma) = \partial M - \text{int}(\gamma)$ is oriented. Define $R_+(\gamma)$ (or $R_-(\gamma)$) to be the union of those components of $R(\gamma)$ whose normal vectors point out of (or into) $M$. The orientations on $R(\gamma)$ must be coherent with respect to $s(\gamma)$, hence every component of $A(\gamma)$ lies between a component of $R_+(\gamma)$ and a component of $R_-(\gamma)$.

Definition 2.2. Let $S$ be a compact oriented surface with connected components $S_1, \ldots, S_n$. We define

$$x(S) = \sum_i \max\{0, -\chi(S_i)\}.$$

Let $M$ be a compact oriented 3–manifold, $A$ be a compact codimension–0 sub-manifold of $\partial M$. Let $h \in H_2(M, A)$. The Thurston norm $x(h)$ of $h$ is defined to be the minimal value of $x(S)$, where $S$ runs over all the properly embedded surfaces in $M$ with $\partial S \subset A$ and $[S] = h$.

Definition 2.3. Let $(M, \gamma)$ be a sutured manifold, and $S$ a properly embedded surface in $M$, such that no component of $\partial S$ bounds a disk in $R(\gamma)$ and no component of $S$ is a disk with boundary in $R(\gamma)$. Suppose that for every component $\lambda$ of $S \cap \gamma$, one of 1)–3) holds:

1) $\lambda$ is a properly embedded non-separating arc in $\gamma$.

2) $\lambda$ is a simple closed curve in an annular component $A$ of $\gamma$ in the same homology class as $A \cap s(\gamma)$. 
3) \(\lambda\) is a homotopically nontrivial curve in a toral component \(T\) of \(\gamma\), and if \(\delta\) is another component of \(T \cap S\), then \(\lambda\) and \(\delta\) represent the same homology class in \(H_1(T)\).

Then \(S\) is called a **decomposing surface**, and \(S\) defines a **sutured manifold decomposition**

\[
(M, \gamma) \xrightarrow{S} (M', \gamma'),
\]

where \(M' = M - \text{int}(\text{Nd}(S))\) and

\[
\gamma' = (\gamma \cap M') \cup \text{Nd}(S'_+ \cap R_-(\gamma)) \cup \text{Nd}(S'_- \cap R_+(\gamma)),
\]

\[
R_+(\gamma') = ((R_+(\gamma) \cap M') \cup S'_+) - \text{int}(\gamma'),
\]

\[
R_-\gamma') = ((R_-\gamma) \cap M') \cup S'_-) - \text{int}(\gamma'),
\]

where \(S'_+\) (\(S'_-\)) is that component of \(\partial\text{Nd}(S) \cap M'\) whose normal vector points out (into) \(M'\).

**Definition 2.4.** A sutured manifold \((M, \gamma)\) is **taut**, if \(M\) is irreducible and \(R(\gamma)\) is Thurston norm minimizing in \(H_2(M, \gamma)\).

Suppose \(S\) is a decomposing surface in \((M, \gamma)\), \(S\) decomposes \((M, \gamma)\) to \((M', \gamma')\). \(S\) is **taut** if \((M', \gamma')\) is taut.

**Definition 2.5.** Suppose

\[
(M, \gamma) \xrightarrow{S} (M', \gamma')
\]

is a taut decomposition, by [2] we can extend this decomposition to a sutured manifold hierarchy of \((M, \gamma)\), from which we can construct a taut foliation \(\mathcal{F}\) of \(M\), such that \(R(\gamma)\) consists of compact leaves of \(\mathcal{F}\). We then call \(\mathcal{F}\) a **foliation induced by \(S\)**. Moreover, when \(R_+(\gamma)\) is homeomorphic to \(R_-\gamma)\), from \(M\) we can obtain a manifold \(Y\) with boundary consisting of tori by gluing \(R_+(\gamma)\) to \(R_-\gamma)\) via a homeomorphism. \(\mathcal{F}\) then becomes a taut foliation \(\mathcal{F}_1\) of \(Y\). We also say that \(\mathcal{F}_1\) is a **foliation induced by \(S\)**.

**Definition 2.6.** A decomposing surface is called a **product disk**, if it is a disk which intersects \(s(\gamma)\) in exactly two points. A decomposing surface is called a **product annulus**, if it is an annulus with one boundary component in \(R_+(\gamma)\), and the other boundary component in \(R_-\gamma)\).

We recall the main result in Gabai [3], which has been intensively used in Ni [9]. Note that the result is not stated in its original form, but it is contained in the argument in [3]. See also [9, Theorem 2.8] for a sketch of the proof.

**Definition 2.7.** An **I-cobordism** between closed connected surfaces \(T_0\) and \(T_1\) is a compact 3–manifold \(V\) such that \(\partial V = T_0 \cup T_1\) and for \(i = 0, 1\) the induced maps \(j_i: H_k(T_i) \to H_k(V)\) are injective.

**Definition 2.8.** Suppose \(M\) is a 3-manifold, \(T\) is a toral component of \(\partial M\). If all tori in \(M\) which are I-cobordant to \(T\) in \(M\) must be parallel to \(T\), then we say \(M\) is **\(T\)–atoroidal**.
Theorem 2.9 (Gabai). Let \((M, \gamma)\) be a taut sutured 3–manifold. \(T\) is a toral component of \(\gamma\), \(S\) is a decomposing surface such that \(S \cap T = \emptyset\), and the decomposition \((M, \gamma) \xrightarrow{S} (M_1, \gamma_1)\) is taut. Suppose \(M\) is \(T\)–atoroidal, then for any slope \(\alpha\) on \(T\) except at most one slope, the decomposition after Dehn filling \((M(\alpha), \gamma \setminus T) \xrightarrow{S} (M_1(\alpha), \gamma_1 \setminus T)\) is taut.

A special case of the above theorem is the case \(\gamma = \partial M\), which is the original form in [3].

2.2 Euler classes of foliations

We will need the Euler classes of foliations.

Definition 2.10. Suppose \(Y\) is a compact 3–manifold with \(\partial Y\) consisting of tori. \(\mathcal{P}\) is an oriented plane field transverse to \(\partial Y\). Let \(T(\partial Y)\) be the tangent plane field of \(\partial Y\). The line field \(\mathcal{P} \cap T(\partial Y)\) has a natural orientation induced by the orientations of \(\mathcal{P}\) and \(T(\partial Y)\), thus it has a nowhere vanishing section \(v \subset \mathcal{P}|_{\partial Y}\). Then one can define the relative Euler class

\[ e(\mathcal{P}) \in H^2(Y, \partial Y) \]

of \(\mathcal{P}\) to be the obstruction to extending \(v\) to a nowhere vanishing section of \(\mathcal{P}\). When \(\mathcal{F}\) is a foliation of \(Y\) that is transverse to \(\partial Y\), let \(T\mathcal{F}\) be the tangent plane field of \(\mathcal{F}\) and let \(e(\mathcal{F}) = e(T\mathcal{F})\).

Definition 2.11. Suppose \(C\) is a properly embedded curve in a compact surface \(F\). We say \(C\) is efficient in \(F\) if

\[ |C \cap \delta| = |[C] \cdot [\delta]|, \quad \text{for each boundary component } \delta \text{ of } F. \]

Suppose \(S\) is a properly embedding surface in compact 3–manifold \(Y\) with boundary consisting of tori. We say \(S\) is efficient in \(Y\) if \(S \cap T\) consists of coherently oriented parallel essential curves for each boundary component \(T\) of \(Y\).

Proposition 2.12. Suppose \(Y\) is a compact 3–manifold that fibres over \(S^1\). Let \(G\) be a fibre of the fibration \(\mathcal{E}\). Suppose \(\mathcal{F}\) is a taut foliation of \(Y\) which is transverse to \(\partial Y\) such that \(G\) is a leaf of \(\mathcal{F}\). Then

\[ e(\mathcal{F}) = e(\mathcal{E}) \in H^2(Y, \partial Y)/\text{Tors}. \]
Proof. This result follows easily from the fact that the Floer homology of a fibred manifold is “monic”. Using this approach, one can even prove that the two Euler classes are equal in $H^2(Y, \partial Y)$. Here we will present a more geometric proof.

In order to prove the desired result, we only need to show that

$$\langle e(F), h \rangle = \langle e(E), h \rangle$$

for any $h \in H_2(Y, \partial Y)$. When $h = [G]$, we have

$$\langle e(F), [G] \rangle = \langle e(E), [G] \rangle = \chi(G).$$

In general, suppose $U \subset Y$ is a proper surface representing $h$ such that $U \ni G$. We can choose the representative $U$ such that $U$ is efficient in $Y$. Then $U \cap G$ can also be made efficient in $G$. Cutting $Y$ open along $G$, we get $G \times I$. Let $U \subset G \times I$ be the proper surface obtained by cutting $U$ open along $C = U \cap G$. Let $C_0, C_1 \subset G$ be proper oriented curves such that

$$-C_0 \times 0 = (\partial U) \cap (G \times 0), \quad C_1 \times 1 = (\partial U) \cap (G \times 1).$$

Since $C_0$ and $C_1$ are homologous efficient curves in $G$ relative to $\partial G$, as in the proof of [3, Lemma 0.6], we can find compact subsurfaces $V_1, V_2, \ldots, V_n$ and efficient curves

$$C_0 = \gamma_0, \gamma_1, \ldots, \gamma_{n-1}, \gamma_n = C_1$$

in $G$, such that

$$\partial V_i \setminus (\partial G) = \gamma_i \cup (-\gamma_{i-1}).$$

Let $W_i = \overline{G \setminus V_i}$. Perturbing the surface

$$\bigcup_{i=1}^n \left( (-\gamma_{i-1} \times \left[ \frac{i-1}{n}, \frac{i}{n} \right]) \cup (V_i \times \frac{i}{n}) \right)$$

slightly, we get a proper surface $V \subset G \times I$, such that

$$(\partial V) \cap (G \times 0) = -C_0 \times 0, \quad (\partial V) \cap (G \times 1) = C_1 \times 1.$$ 

Similarly, perturbing the surface

$$\bigcup_{i=1}^n \left( (\gamma_{i-1} \times \left[ \frac{i-1}{n}, \frac{i}{n} \right]) \cup (W_i \times \frac{i}{n}) \right)$$

slightly, we have a proper surface $W \subset G \times I$, such that

$$(\partial W) \cap (G \times 0) = C_0 \times 0, \quad (\partial W) \cap (G \times 1) = -C_1 \times 1.$$ 

Let $\overline{V} \subset Y$ be the proper surface obtained from $V$ by identifying $C_0 \times 0$ and $C_1 \times 1$ with $C \subset G \subset Y$. Similarly, define $\overline{W} \subset Y$. Note that

$$[\overline{V}] - [\overline{U}] = [V \cup (-U)] \in H_2(Y, \partial Y).$$
Perturbing $V \cup (-U)$ slightly, we get a properly immersed surface in $Y$ which is disjoint from the fibre $G$. So $[V \cup (-U)] = m[G]$ for some integer $m$. Using (2), in order to check (1) for $h = [U]$, we only need to check it for $h = [V]$.

Since $\mathcal{F}$ is taut, by Thurston [13, Corollary 1] we have

$$
\chi(V) \leq \langle e(\mathcal{F}), [V] \rangle,
$$

$$
\chi(W) \leq \langle e(\mathcal{F}), [W] \rangle.
$$

Adding the two inequalities together, we get

$$
\chi(V) + \chi(W) \leq \langle e(\mathcal{F}), [V] + [W] \rangle. \tag{5}
$$

By the constructions (3), (4), the result of doing oriented cut-and-pastes to $V$ and $W$ is $n$ copies of $G$. So the left hand side of (5) is $n\chi(G)$, while the right hand side is $\langle e(\mathcal{F}), nG \rangle = n\chi(G)$. So the equality holds. In particular, we should have

$$
\chi(V) = \langle e(\mathcal{F}), [V] \rangle.
$$

The same argument shows that

$$
\chi(V) = \langle e(\mathcal{E}), [V] \rangle,
$$

so (1) holds for $h = [V]$. Since we have checked (1) for all elements $h \in H_2(Y, \partial Y)$, $e(\mathcal{F})$ is equal to $e(\mathcal{E})$ up to a torsion element in $H^2(Y, \partial Y)$.

2.3 A characterization of one-crossing knot projections

In this subsection, we will give a characterization of one-crossing knot projections in terms of double primitive knots. This fact is not used in the current paper, but it is useful to bare it in mind.

**Definition 2.13.** Let $F' \subset F \times I$ be a connected surface of genus $g(F) + 1$, and $\partial F' = (\partial F) \times \frac{1}{2}$. Suppose $F'$ is a Heegaard surface. Namely, $F'$ splits $F \times I$ into two parts $U_0$ and $U_1$, such that $U_0$ is homeomorphic to $(F \times [0, \frac{1}{2}]) \cup H_1$, and $U_2$ is homeomorphic to $(F \times [\frac{1}{2}, 1]) \cup H_2$, where $H_1$ is a one-handle with feet on $F \times \frac{1}{2}$ and $H_2$ is a one-handle with feet on $-F \times \frac{1}{2}$. A knot $K \subset F \times I$ is a double primitive knot if it is isotopic to a curve on $F'$ which goes through each of $H_1, H_2$ exactly once.

**Lemma 2.14.** A knot $K \subset F \times I$ is double primitive if and only if it has a projection which has only one crossing.

**Proof.** If a knot has a one-crossing projection, then it is double primitive as shown in the introduction. Now assume $K$ is double primitive, then $K$ is embedded into a Heegaard surface $F'$ as in the above definition.

We claim that $F'$ is stabilized. Namely, there is a compressing disk $D_0 \subset U_0$ and a compressing disk $D_1 \subset U_1$ such that $|\partial D_0 \cap \partial D_1| = 1$. When $F$ is closed, this follows from the theorem of Scharlemann and Thompson [11] that the Heegaard splittings of $F \times I$ are standard. When $F$ is not closed, let $R$ be
the torus with one hole, we can glue a copy of $R$ to each component of $\partial F$, then $F$ becomes a closed surface $G$ and $F'$ becomes a Heegaard surface $G'$ in $G \times I$. Using Scharlemann and Thompson’s theorem, $G'$ is stabilized, hence there are compressing disks $D_0$ and $D_1$ in the two compression bodies separated by $G'$, such that $|\left(\partial D_0 \cap \partial D_1\right)| = 1$. Using standard arguments we can isotope $D_0$ and $D_1$ to be disjoint from the copies of $R \times I$, so $D_0 \subset U_0$, $D_1 \subset U_1$, thus our conclusion follows.

Since $g(F') = g(F) + 1$, after compressing $F'$ along $D_0$ we get a surface homeomorphic to $F$ (and hence parallel to $F \times 0$ in $F \times I$). So $F'$ is obtained from $F \times \frac{1}{2}$ by adding a one-handle, and $D_1$ is a disk whose boundary goes through the one-handle exactly once. Now the local picture of $F'$ looks exactly like in Figure 1. The knot $K$ goes through the one-handle once and intersects $\partial D_1$ once, so there is a crossing near $D_1$ and no crossing elsewhere.  

3 Warm-up cases

In this section, we are going to prove some easy cases of our main theorem. When $F$ is a disk or sphere, our result follows from Gordon and Luecke’s Knot Complement Theorem [8]. When $F$ is an annulus, we have the following lemma.

**Lemma 3.1.** Theorem [1] is true when $F$ is an annulus.

**Proof.** Let $\mathcal{M}$ be the meridian of the solid torus $V = F \times I$, and let $\mathcal{L}$ be the frame of $V$ specified by $\partial V$. By Gabai [5], if $K$ is nontrivial, then $K$ is a 0– or 1–bridge braid in $F \times I$.

Capping off one boundary component of $F$ with a disk, we get a disk $D$. Let $\lambda$ be the Seifert frame of $K$ in $D \times I$ and let $\mu$ be the meridian of $K$.

If $K$ is the core of $V$, then the surgery preserves the homeomorphism type of $(F \times I, (\partial F) \times I)$ if and only if the slope is $\mu + n\lambda$ for some integer $n$.

From now on we assume the braid index of $K$ is greater than 1.

If $K$ is a 0–bridge braid, then $K$ is isotopic to $p\mathcal{L} + q\mathcal{M}$ on $\partial V$ for some $p, q \in \mathbb{Z}$. Let $\Lambda$ be the frame on $K$ specified by $\partial V$, then $\Lambda = pq\mu + \lambda$. A surgery on $K$ yields a solid torus if and only if the slope $\alpha$ of the surgery satisfies that $\Delta(\alpha, \Lambda) = 1$, namely, when the slope $\alpha$ is $\mu + n\lambda$ for some integer $n$. Now $p\alpha = pq\mu + pn\lambda$ is homologous to $\mathcal{M} + pn(p\mathcal{L} + q\mathcal{M})$ in $V \setminus K$, so the meridian of the new ambient solid torus after surgery is $(1 + pqn)\mathcal{M} + p^2n\mathcal{L}$. Since the surgery preserves the homeomorphism type of the pair $(F \times I, (\partial F) \times I)$, we must have $\Delta((1 + pqn)\mathcal{M} + p^2n\mathcal{L}, \mathcal{L}) = 1$, thus $1 + pqn = \pm 1$. Since $p > 1, n \neq 0$, we have $(p, q, n) = (2, 1, -1)$ or $(2, -1, 1)$. When $(p, q) = (2, 1)$, the slope $\alpha$ on $K$ is

$$\mu + n(pq\mu + \lambda) = (1 + pqn)\mu + n\lambda,$$

which is 1 with respect to the frame $\lambda$, and the meridian of the new ambient solid torus is $\mathcal{M} + 4\mathcal{L}$; when $(p, q) = (2, -1)$, the slope $\alpha$ on $K$ is $-1$, and the meridian of the new ambient solid torus is $\mathcal{M} - 4\mathcal{L}$. We can check $\alpha$ is the blackboard frame.
If \( K \) is a 1–bridge braid, then \( K \) is determined by 3 parameters \( \omega, b, t \) by Gabai [6]. Here \( \omega > 0 \) is the braid index, \( 1 \leq b \leq \omega - 2 \), \( t \equiv r \) (mod \( \omega \)) for some integer \( r \) with \( 1 \leq r \leq \omega - 2 \). Since the \( \alpha \)–surgery yields a solid torus, by Lemma 3.2 the slope of the surgery is \( \lambda - (t\omega + d)\mu \), where \( d \in \{b, b + 1\} \). So \( t\omega + d = \pm 1 \), which is impossible for any \( \omega, b, t \) satisfying the previous restrictions.

**Lemma 3.2.** In the above lemma, let \( \varphi_\alpha \in \text{MCG}(F, \partial F) \) be the map induced by the \( \alpha \)–surgery. If \( K \) is the core of \( F \times I \) and \( \alpha = \frac{1}{n} \), then \( \varphi_\alpha = \tau^n \), where \( \tau \) is the right hand Dehn twist in \( F \); if \( K \) is the \((2, \pm 1)\)–cable in \( F \times I \), then \( \varphi_\alpha = \tau^{\pm 4} \).

**Proof.** When \( K \) is the core of \( F \times I \), the conclusion is well-known. When \( K \) is the \((2, \pm 1)\)–cable, then from the proof of the previous lemma we know that the meridian of the new ambient solid torus is \( M \pm 4L \), hence the conclusion follows from the first case.

The following lemma is obvious.

**Lemma 3.3.** Suppose \((C \times I) \subset (F \times I)\) is a product disk or product annulus, \((C \times I) \cap K = \emptyset \). Let \( F_1 \) be the surface obtained from \( F \) by cutting \( F \) open along \( C \), let \( N_1 \) be the manifold obtained from \( N = (F \times I) \setminus \text{int}(Nd(K)) \) by cutting \( N \) open along \( C \times I \). Then the pair \((N_1(\alpha), (\partial F_1) \times I)\) is homeomorphic to \((F \times I, (\partial F) \times I)\) if and only if the pair \((N_1(\alpha), (\partial F_1) \times I)\) is homeomorphic to \((F_1 \times I, (\partial F_1) \times I)\).

**Lemma 3.4.** Theorem 1.1 is true when \( F \) is a torus.

**Proof.** Let \( C \subset F \) be a simple closed curve such that \( K \) is homologous to a multiple of \( C \). Consider the homology class \([C \times I] \in H_2(F \times I, \partial(F \times I))\), then \([C \times I] \cdot [K] = 0 \). It follows that \([C \times I]\) is also a homology class in \( H_2((F \times I) \setminus K, \partial(F \times I))\).

Let \((S, \partial S) \subset ((F \times I) \setminus K, \partial(F \times I))\) be a taut surface representing \([C \times I]\). By Theorem 2.9, \( S \) remains taut in at least one of the original \( F \times I \) and \( N(\alpha) \cong F \times I \). Hence \( S \) must be a product annulus. Cutting \( F \times I \) open along \( S \), \( K \) becomes a knot in \((\text{annulus} \times I)\). Now we can apply Lemma 3.1 and Lemma 3.2 to get our conclusion.

**Lemma 3.5.** If the conclusion of Theorem 1.1 holds for all knots whose exterior are \( \partial(Nd(K)) \)–atoroidal, then the conclusion holds for all knots in \( F \times I \).

**Proof.** By the assumption, we only need to consider the case where there is a torus in \( N = F \times I \setminus \text{int}(Nd(K)) \) which is I-cobordant but not parallel to \( \partial Nd(K) \). Let \( R \) be an “innermost” such torus.

By [9] Lemma 3.1, \( R \) bounds a solid torus \( U \) in \( F \times I \), such that \( K \subset U \). Since \( R \) is innermost in \( N \), if a torus in \((F \times I) \setminus \text{int}(U)\) is I-cobordant to \( \partial U = R \), then this torus is parallel to \( R \). Let \( V \) be the manifold obtained from \( U \) by \( \alpha \)–surgery on \( K \).

By Gabai [5], one of the following cases holds.
1) $V = D^2 \times S^1$. In this case $K$ is a 0–bridge or 1–bridge braid in $U$, and the core $K'$ of the surgery is also a 0–bridge or 1–bridge braid in $V$. Moreover, $K$ and $K'$ have the same braid index $\omega$.

2) $V = (D^2 \times S^1)\# W$, where $W$ is a closed 3–manifold and $1 < |H_1(W)| < \infty$.

3) $V$ is irreducible and $\partial V$ is incompressible.

Since $V \subset N(\alpha) \cong F \times I$, Cases 2) and 3) can not happen, so the only possible case is 1). Thus the core of $U$ is a knot such that a surgery on the knot yields the pair $(N(\alpha), (\partial F) \times I)$ which is homeomorphic to $(F \times I, (\partial F \times I)$. Moreover, $N \setminus \text{int}(U)$ is $\partial U$–atoroidal. By our assumption, the core of $U$ is a 0–crossing or 1–crossing knot in $F \times I$.

If the core of $U$ is isotopic to $\eta \times \{\frac{1}{2}\}$ for some simple closed curve $\eta \subset F$, let $G \subset F$ be a tubular neighborhood of $\eta$, then $K$ lies in $G \times I$ after an isotopy. Let $M = (G \times I) \setminus \text{int}(\text{Nd}(K))$. By Lemma 3.3 $(M(\alpha), (\partial G) \times I)$ is homeomorphic to $(G \times I, (\partial G) \times I)$. Applying Lemma 3.1 we find that $K$ is the $(2, \pm 1)$–cable of the core of $G \times I$, and the slope $\alpha$ is the blackboard frame $\lambda_0$.

If the core of $U$ is a 1–crossing knot, then the blackboard frame $\lambda'_0$ on $\partial U$ is the meridian of $V$, so $\lambda'_0$ cobounds a punctured disk with $\omega$ oriented copies of $\alpha$ in $U \setminus \text{int}(\text{Nd}(K))$. Moreover, the meridian $\mu'$ on $\partial U$ cobounds a punctured disk with $\omega$ oriented copies of $\mu$ in $U \setminus \text{int}(\text{Nd}(K))$. Since $|\lambda'_0| \cdot |\mu'| = 1$, considering the intersection of the two punctured disks we conclude that $\omega = 1$. Hence $\partial U$ is parallel to $\partial \text{Nd}(K)$, a contradiction.

In light of the above lemma, from now on we assume the exterior of the knot $K$ is $\partial(\text{Nd}(K))$–atoroidal.

### 4 Comparing Euler classes of foliations

Let $E$ be a maximal (up to isotopy) compact essential subsurface of $F$, such that $K$ can be isotoped in $F \times I$ to be disjoint from $E \times I$. Let $G = \overline{F \setminus E}$.

The goal of this section is to prove the following proposition.

**Proposition 4.1.** The subsurface $G$ is either an annulus or a pair of pants.

Let $T = \partial(\text{Nd}(K))$, $\gamma = ((\partial G) \times I) \cup T$. Let

$$N = (F \times I) \setminus \text{int}(\text{Nd}(K)), M = (G \times I) \setminus \text{int}(\text{Nd}(K)).$$

Then the sutured manifold $(M, \gamma)$ contains no product disks or product annuli. For a proper surface $S \subset M$, let $\partial_i(S) = S \cap (G \times i)$, $i = 0, 1$.

Let $X = (G \times S^1) \setminus \text{int}(\text{Nd}(K))$ be the manifold obtained from $M$ by gluing $G \times 1$ to $G \times 0$ via the identity map of $G$. Suppose $\xi$ is a slope on $K$. Let $N(\xi), M(\xi), X(\xi)$ be the manifolds obtained from $N, M, X$ by $\xi$–filling on $T$, respectively. Let $K(\xi) \subset M(\xi)$ be the core of the new solid torus.

By Lemma 3.3 $X(\xi)$ is a surface bundle over $S^1$ with fibre $G$ when $\xi = \infty$ or $\alpha$. We then let $\delta'(\xi)$ be the fibration of $X(\xi)$. 

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Lemma 4.2. $K \subset F \times I$ is as in Theorem 1.1. $N$ is $T$-atoroidal. Suppose $S \subset M$ is a taut surface such that $S \cap T = \emptyset$ and there exists a curve $C \subset F$ with $\partial_0 S = -C \times 0, \partial_1 S = C \times 1$. Let $S \subset X$ be the surface obtained from $S$ by gluing $\partial_0 S$ to $\partial_1 S$ via the identity map. Let $\mathcal{F}$ be a taut foliation of $X$ induced by $S$. Then

$$\langle e(\mathcal{F}), [S] \rangle = \langle e(\mathcal{\xi}^{\prime}(\xi)), [S] \rangle = \chi(S)$$

for some $\xi \in \{\infty, \alpha\}$.

**Proof.** By Theorem 2.9, $S$ remains taut in $M(\xi)$ for some $\xi \in \{\infty, \alpha\}$. Let $\mathcal{F}^{\prime}$ be a taut foliation of $X(\xi)$ induced by $S$. By Proposition 2.12,

$$e(\mathcal{F}^{\prime}) = e(\mathcal{\xi}^{\prime}(\xi)) \in H^2(X(\xi), \partial X(\xi); \mathbb{Q}).$$

Since both $\mathcal{F}$ and $\mathcal{F}^{\prime}$ are induced by $S$, we have

$$\chi(S) = \langle e(\mathcal{F}), [S] \rangle = \langle e(\mathcal{F}^{\prime}), [S] \rangle = \langle e(\mathcal{\xi}^{\prime}(\xi)), [S] \rangle.$$

\[ \square \]

**Proposition 4.3.** $K \subset F \times I$ is as in Theorem 1.1. $N$ is $T$-atoroidal. The inclusion $K \subset G \times I$ induces a map

$$i_*: H_1(K; \mathbb{Q}) \to H_1(G; \mathbb{Q}).$$

Let

$$\mathcal{V} = \{ v \in H_1(G, \partial G; \mathbb{Q}) | v \cdot i_*[K] = 0 \}.$$

Then the dimension of $\mathcal{V}$ is at most 1.

Let

$$\rho_\xi: H^2(X, \partial X; \mathbb{Q}) \to H^2(X(\xi), \partial X(\xi); \mathbb{Q})$$

be the map induced by the map of pairs

$$(X(\xi), \partial X(\xi)) \to (X(\xi), (\partial X(\xi)) \cup K(\xi)).$$

**Lemma 4.4.** Notations are as in Proposition 4.3. If the dimension of $\mathcal{V}$ is greater than 1, then there exists a properly embedded surface $H \subset X$ such that

1) $H$ is not a multiple of $[G]$,
2) $H \cap T = \emptyset$,
3) for any two elements $\varepsilon_\infty \in \rho_{\infty}^{-1}(e(\xi^{\prime}(\infty))), \varepsilon_\alpha \in \rho_{\alpha}^{-1}(e(\xi^{\prime}(\alpha)))$, we have

$$\langle \varepsilon_\infty, [H] \rangle = \langle \varepsilon_\alpha, [H] \rangle.$$
Proof. There is a natural injective map
\[ \sigma: H_1(G, \partial G) \to H_2(G \times S^1, \partial G \times S^1) \]
defined via multiplying with the \( S^1 \) factor. Moreover, all elements in \( \sigma(V) \) are represented by surfaces which are disjoint from \( K \), hence \( \sigma|\mathcal{V} \) induces an injective map
\[ \tilde{\sigma}: \mathcal{V} \to H_2(X, \partial X; \mathbb{Q}). \]
We pick two elements \( \varepsilon_{\alpha} \in \rho_{\infty}^{-1}(\varepsilon(\alpha)), \varepsilon_{\infty} \in \rho_{\alpha}^{-1}(\varepsilon(\alpha)) \). If \( \dim \mathcal{V} > 1 \), then there exists a nonzero integral element \( h \in \tilde{\sigma}(\mathcal{V}) \) such that
\[ \langle \varepsilon_{\infty}, h \rangle = \langle \varepsilon_{\alpha}, h \rangle. \]
Let \( H \subset X \) be a proper surface representing \( h \) such that \( H \cap T = \emptyset \). We claim that this \( H \) is what we need. We only need to check 3) since the first two conditions are obvious.
From the Mayer–Vietoris sequence
\[ \begin{array}{cccc}
H^1(K(\xi)) & \to & H^2(X, \partial X) & \xrightarrow{\rho_{\xi}} & H^2(X(\xi), \partial X(\xi))
\end{array} \]
and the fact that \( h \cdot [T] = 0 \) we conclude that \( \langle \varepsilon_{\xi}, h \rangle \) does not depend on the choice of \( \varepsilon_{\xi} \in \rho_{\xi}^{-1}(\varepsilon(\xi)) \). Hence 3) holds. \( \Box \)

Assume the dimension of \( \mathcal{V} \) is greater than 1, let \( H \) be a surface as in Lemma 4.3 and suppose \( H \cap G \). Without loss of generality, we can assume no component of \( C = H \cap G \) is nullhomologous in \( H_1(G, \partial G) \), and \( H \) is efficient in \( G \times S^1 \), hence we can also assume \( H \cap G \) is efficient in \( G \).
Let \( p \in G \setminus C \) be a point. Let \( S_m(C) \) be the set of properly embedded oriented surfaces \( S \subset G \times I \), such that \( S \cap K = \emptyset, \partial_0 S = -C \times 0, \partial_1 S = C \times 1 \), and the algebraic intersection number between \( S \) and \( p \times I \) is \( m \). Similarly, let \( S_m(-C) \) be the set of properly embedded surfaces \( S \subset G \times I \), such that \( S \cap K = \emptyset, \partial_0 S = C \times 0, \partial_1 S = -C \times 1 \), and the algebraic intersection number of \( S \) with \( p \times I \) is \( m \). Since \( [C] \cdot \iota_m([K]) = 0 \), \( S_m(\pm C) \neq \emptyset \).
Suppose \( S \subset M \) is a properly embedded surface which is transverse to \( \partial G \times 0 \). For any component \( S_0 \) of \( S \), we define
\[ y(S_0) = \max\{ \frac{|S_0 \cap (\partial G \times 0)|}{2} - \chi(S_0), 0 \}, \]
and let \( y(S) \) be the sum of \( y(S_i) \) with \( S_i \) running over all components of \( S \). Let \( y(S_m(\pm C)) \) be the minimal value of \( y(S) \) for all \( S \in S_m(\pm C) \).

Lemma 4.5. When \( m \) is sufficiently large, there exist surfaces \( S_1 \in S_m(\pm C) \) and \( S_2 \in S_m(-C) \), such that they are taut.

Proof. Let \( x(\cdot) \) be the Thurston norm in \( H_2(X, \partial X) \). There exists \( N \geq 0 \), such that if \( k > N \), then \( x([H] + (k + 1)[G]) = x([H] + k[G]) + x(G) \). As in the proof of Gabai [2] Theorem 3.13, if \( Q \) is a Thurston norm minimizing surface in the
homology class \([H] + k[G]\), and \(\overline{Q} \cap G\) consists of essential curves in \(G\), then \(Q\) gives a taut decomposition of \(M\), where \(Q\) is obtained from \(\overline{Q}\) by cutting open along \(\overline{Q} \cap G\). Moreover, we can assume \(\overline{Q}\) is efficient in \(X\). Hence \(\overline{Q} \cap T = \emptyset\) and for each boundary component \(\delta\) of \(G \times i\), \(|\partial Q \cap \delta| = |[\partial Q] \cdot [\delta]|\).

Now we can apply Gabai \[3\] Lemma 0.6] to get a new taut surface \(Q'\) such that \(\partial_0 Q' = -C \times 0, \partial_1 Q' = C \times 1\). When \(m\) is sufficiently large, let \(S_1\) be the surface obtained by doing oriented cut-and-pastes of \(Q'\) with \((m - Q' \cdot (p \times I))\) copies of \(G\), then \(S_1 \in S_m(+C)\) is the surface we need. Similarly, we can find the surface \(S_2 \in S_m(-C)\).

**Correction 4.6.** In Ni \[9\], after the statement of Proposition 3.4, the author claims that there exists a circle or arc \(C \subset G\) such that \([C] \cdot i_*[K] = 0\). This claim is not true. The correct statement should be there exists an essential efficient curve \(C\) in \(G\) such that \([C] \cdot i_*[K] = 0\). The proof only needs slight changes: one can make use of the above Lemma 4.5 to find taut surfaces.

The following result is Ni \[9\] Lemma 3.6], whose proof uses the assumption that \((M, \gamma)\) contains no essential product disks or product annuli and an argument from Gabai \[4\].

**Lemma 4.7.** For any positive integers \(p, q\),

\[
y(S_p(+C)) + y(S_q(-C)) > (p + q)y(G).
\]

**Proof of Proposition 4.3.** By Lemma 4.5 when \(m\) is large there exist taut surfaces \(S_1 \in S_m(+C), S_2 \in S_m(-C)\). By Theorem 2.9, \(S_1\) remains taut in \(M(\xi_i)\) for some \(\xi_i \in \{\infty, \alpha\}, i = 1, 2\). Let \(F_i\) be a taut foliation of \(X(\xi_i)\) induced by \(S_i\).

Let \(S_1, S_2 \subset X\) be the surfaces obtained from \(S_1, S_2\) by gluing \(C \times 0\) to \(C \times 1\). We have

\[
[S_1] = [H] + m[G], \quad [S_2] = -[H] + m[G]
\]

in \(H_2(X, \partial X)\) and \(H_2(X(\xi_i), \partial X(\xi_i))\).

We have

\[
\chi(S_i) = \chi(S_i) - |\partial_0 S_i| = -y(S_i),
\]

and by Proposition 2.12

\[
\begin{align*}
\chi(S_1) &= \langle e(F_1), [S_1] \rangle \\
&= \langle e(\varepsilon(\xi_1)), [H] + m[G] \rangle,
\end{align*}
\]

\[
\begin{align*}
\chi(S_2) &= \langle e(F_2), [S_2] \rangle \\
&= \langle e(\varepsilon(\xi_2)), -[H] + m[G] \rangle.
\end{align*}
\]

By Lemma 4.4 \(\langle e(\varepsilon(\xi_1)), [H] \rangle = \langle e(\varepsilon(\xi_2)), [H] \rangle\). So

\[
\begin{align*}
\chi(S_1) + \chi(S_2) &= \langle \varepsilon(\xi_1), m[G] \rangle + \langle \varepsilon(\xi_2), m[G] \rangle \\
&= 2m\chi(G),
\end{align*}
\]

which contradicts Lemma 4.7. \(\square\)
Proof of Proposition 4.1. By Proposition 4.3, \( b_1(G) \leq 2 \), so \( G \) is an annulus, a pair of pants or a genus-one surface with one boundary component. We only need to show that the last case is not possible.

Suppose \( g(G) = 1 \) and \( |\partial G| = 1 \). Let \( C \subset G \) be a simple closed curve such that \( [C] \cdot i_*[K] = 0 \), then there exists a closed taut surface \( H \subset X \) such that \( [H] = [C \times S^1] \) and \( H \cap T = \emptyset \). Since \( M \) does not contain any product annuli, \( H \) is not a torus, hence \( H \) is not taut in \( X(\infty) \). By Theorem 2.9, \( H \) is taut in \( X(\alpha) \).

Consider the monodromy \( \varphi \) of \( X(\alpha) \), the surface \( H \subset X(\alpha) \) forces \( \varphi_*[C] = [C] \). Since \( G \) is a once-punctured torus, \( \varphi(C) \) is isotopic to \( C \). Thus there exists an annulus \( A \subset X(\alpha) \) such that \( A \cap G = C = H \cap G \), which implies that \( [H] = [A] + m[G] \) for some integer \( m \). Since \( H \) is closed, \( m = 0 \). This contradicts the facts that \( H \) is taut in \( X(\alpha) \) and that \( H \) is not a torus.

\( \square \)

5 Knots in pants \( \times I \)

In this section, we study the case where \( G \) is a pair of pants.

The following elementary observation is stated without proof.

Lemma 5.1. Suppose \( C_1, C_2 \subset G \) are two efficient curves consisting of essential arcs. If they are homologous in \( H_1(G, \partial G) \), then they are isotopic. \( \square \)

Let \( a, b, c \) be the three boundary components of \( G \), \( u, v, w \) be three mutually disjoint oriented arcs in \( G \) such that \( u \) connects \( b \) to \( c \), \( v \) connects \( c \) to \( a \), \( w \) connects \( a \) to \( b \). Then

\[ [u] + [v] + [w] = 0 \in H_1(G, \partial G). \]  (6)

Lemma 5.2. None of \( u, v, w \) has zero intersection number with \( i_*[K] \).

Proof. The argument is similar to the once-punctured torus case of Proposition 4.3. Assume that \( [u] \cdot i_*[K] = 0 \), then there exists a closed taut surface \( H \subset X \) such that \( [H] = [u \times S^1] \). We may assume that \( H \) is efficient in \( X \), hence \( H \) has two boundary components and \( H \cap T = \emptyset \). By Lemma 5.1 we may assume that \( H \cap G = u \).

Since \( M \) does not contain any product disks, \( H \) is not an annulus, hence \( H \) is not taut in \( X(\infty) = G \times S^1 \). By Theorem 2.9, \( H \) is taut in \( X(\alpha) \). Let \( \varphi \) be the monodromy of \( X(\alpha) \), then \( H \) forces \( \varphi(u) \) to be homologous hence isotopic to \( u \) by Lemma 5.1. Thus there exists an annulus \( A \subset X(\alpha) \) such that \( A \cap G = u = H \cap G \), which implies that \( [H] = [A] + m[G] \) for some integers. Since \( H \) has only two boundary components, \( m = 0 \). This contradicts the facts that \( H \) is taut in \( X(\alpha) \) and \( H \) is not an annulus. \( \square \)

Lemma 5.3. The intersection number of \( i_*[K] \) with each of \( u, v, w \) is \( \pm 1 \) or \( \pm 2 \).
Proof. Capping off $a$ with a disk, we get an annulus $G_a$. Now $K \subset G_a \times I$ and the $\alpha$–surgery on $K$ does not change the homeomorphism type of the pair $(G_a \times I, (\partial G_a) \times I)$. By the previous lemma, $K$ is nontrivial in $G_a \times I$. By Lemma 3.1, $K$ is the core or the $(2, \pm1)$–cable in $G_a \times I$, so $i_*[K] \cdot [u]$ is $\pm1$ or $\pm2$. The same argument applies to $v$ and $w$.

Using the previous two lemmas and (6), we may assume
\[ [u] \cdot i_*[K] = [v] \cdot i_*[K] = 1, \quad [w] \cdot i_*[K] = -2, \quad (7) \]
after reversing the orientation of $K$ and renumbering $a, b, c, u, v, w$ if necessary.

We give $a, b, c$ the boundary orientation induced from $G$, then
\[ [v] \cdot [a] = [w] \cdot [b] = -[u] \cdot [b] = [u] \cdot [c] = 1. \quad (8) \]
See Figure 3 for the homology class of $K$.

Let $\tau_a, \tau_b, \tau_c$ be the right-hand Dehn twists along (parallel copies of) $a, b, c$. The mapping class group $\text{MCG}(G, \partial G)$ of $G$ is generated by $\tau_a, \tau_b, \tau_c$. (See, for example, Farb–Margalit [1] for preliminaries on the mapping class groups of surfaces with boundary.) Since $a, b, c$ are disjoint, $\text{MCG}(G, \partial G) \cong \mathbb{Z}^3$.

Lemma 5.4. If $K$ is the $(2,1)$–cable in $G_c \times I$, then the map induced by the $\alpha$–surgery is
\[ \varphi_\alpha = \tau_a^{-2} \tau_b^{2} \tau_c^{-1}. \]
If $K$ is the $(2,-1)$–cable in $G_c \times I$, then
\[ \varphi_\alpha = \tau_a^{-2} \tau_b^{-2} \tau_c. \]

Proof. Capping off $a, b$ with two disks, $G$ becomes a disk $G_{ab}$. $K$ has a canonical frame $\lambda$, which is null-homologous in $(G_{ab} \times I) \setminus K$. Hence $\lambda$ is homologous to $l[a] + m[b]$ in $M$ for some integers $l, m$. By (7), (8) we conclude that $\lambda$ is homologous to $a - b$ in $M$. Hence $\lambda$ is also the canonical frame in $G_c \times I$, where $G_c$ is obtained from $G$ by capping off $c$ with a disk.

Figure 3: The homology class of $K$ in $G \times I$
Suppose \( \varphi_\alpha = \tau_a^p \tau_b^q \tau_c^r \). If \( K \) is the \((2,1)\)-cable in \( G_c \times I \), then by Lemma 3.1, the slope \( \alpha \) is 1 with respect to \( \lambda \).

There is a natural map

\[
q_a : \text{MCG}(G, \partial G) \to \text{MCG}(G_a, \partial G_a),
\]

where \( \text{MCG}(G_a, \partial G_a) \) is generated by \( \tau_b \). Since \( K \) is the core in \( G_a \times I \) and the slope \( \alpha \) is 1, \( q_a(\varphi_\alpha) \) must be \( \tau_b \) by Lemma 3.2. The map \( q_a \) sends both \( \tau_b \) and \( \tau_c \) to \( \tau_b \), and sends \( \tau_a \) to 1. So \( q_a(\varphi_\alpha) = \tau_b^{q+r} \), thus \( q + r = 1 \). The same argument shows that \( p + r = 1 \).

Now consider the natural map

\[
q_c : \text{MCG}(G, \partial G) \to \text{MCG}(G_c, \partial G_c) = \langle \tau_a \rangle.
\]

By Lemma 3.2, \( q_c(\varphi_\alpha) = \tau_a^4 \). Hence \( p = q = 2, r = -1 \). So we conclude that \( p + r = 1 \). The same argument works when \( K \) is the \((2, -1)\)-cable in \( G_c \times I \).

Proposition 1.4 follows from the above lemma.

The manifold \( G \times S^1 \) has a unique product structure. Let \( \omega, \omega_\alpha \subset c \times S^1 \) be \( S^1 \)-fibres with respect to the product structures on \( X(\infty) \) and \( X(\alpha) \), respectively.

**Lemma 5.5.** If \( K \) is the \((2,1)\)-cable in \( G_c \times I \), then

\[
[\omega_\alpha] = [\omega] + [c].
\]

**Proof.** The manifold \( X(\infty) \) is obtained from \( G \times I \) by identifying \((x, 0)\) with \((x, 1)\) for each \( x \in G \). By Lemma 1.3, \( X(\alpha) \) is obtained from \( G \times I \) by identifying \((x, 0)\) with \((\varphi_\alpha(x), 1)\) for each \( x \in G \). Choose parallel copies of \( a, b, c \) in \( G \), denoted \( a', b', c' \). Let \( \varphi_\alpha \) be supported in the three annuli bounded by \( a - a' \), \( b - b' \) and \( c - c' \). Pick points \( p \in c, p' \in c' \), then \( p' \times S^1 \) is an \( S^1 \)-fibre of the product structure on both \( X(\infty) \) and \( X(\alpha) \), while \( p \times S^1 \) is an \( S^1 \)-fibre of the product structure on \( X(\infty) \).
In $X(\alpha)$, we isotope $p' \times S^1$ such that it becomes a curve $\mathcal{S}$ which is the union of four segments $J, J', J_1, J_2$, where $J$ is a vertical segment in the interior of $c \times I$, $J_2 \subset G \times \epsilon$, $J_1 \subset G \times (1 - \epsilon)$, $J'$ is a vertical segment in $c' \times S^1$. See the left hand side of Figure 4.

As on the right hand side of Figure 4 we push the previous curve $\mathcal{S}$ down in distance $\epsilon$ to get a new curve $\mathcal{S}_-$, then $J$, becomes an arc on $G \times 1$. Using Lemma 1.3, this new arc is $\varphi_\alpha(J) = \tau_c^{-1}(J_s)$. $\mathcal{S}_-$ is a fibre of $X(\alpha)$, and it is homologous to $[p \times S^1] + [\epsilon \times 1]$. Hence our conclusion holds. 

Lemma 5.6. Let $C = v - u$. Pick a point $p \in c' \setminus \partial C$, we can then define $\mathcal{S}_m(\pm C)$ as in Section 4. Then there exists a connected surface $S \in S_1(C)$ such that $y(S) = 1$. Moreover, let $S' \subset G \times [0, 1]$ be the surface obtained from $-C \times I$ and $G \times 0$ by oriented cut-and-pastes, then $S$ is isotopic to $S'$ in $G \times [0, 1]$.

Proof. For any homology class $h \in H_2(X, (\partial G) \times S^1)$, let $x(h), x_\infty(h), x_\alpha(h)$ denote its Thurston norm in $X, X(\infty), X(\alpha)$, respectively. Let $U = -[u \times S^1], V = -[v \times S^1] \in H_2(G \times S^1, (\partial G) \times S^1)$. Since $(V - U) \cdot [K] = 0$, $V - U$ also represents an element in $H_2(X, (\partial G) \times S^1)$. Note that the Thurston norm of $h \in H_2(G \times S^1, (\partial G) \times S^1)$ is the absolute value of its algebraic intersection number with the $S^1$-fibre. Consider $V - U + m|G|$ for $m \geq 0$, using Lemma 5.5 we can compute

$$x_\infty(V - U + m|G|) = m,$$

$$x_\alpha(V - U + m|G|) = (V - U + m|G|) \cdot ([\omega] + [c]) = |m - 2|.$$

Since $x_\infty(V - U + |G|) = x_\alpha(V - U + |G|) = 1$, Theorem 2.9 implies that $x(V - U + |G|) = 1$. Let $\overline{S} \subset X$ be a taut surface in this homology class such that $\overline{S}$ is efficient in $X$. Then $\overline{S}$ is disjoint from $T$. Isotope $\overline{S}$ so that it is transverse to $G$ and its intersection with $G$ contains no trivial loops. Now $\overline{S} \cap G$ is homologous to $C$. Moreover, $\overline{S} \cap G$ can be made efficient in $G$. So $\overline{S} \cap G$ is isotopic to $C$ by Lemma 5.1. Without loss of generality, we can assume

$$\overline{S} \cap G = C \quad \text{and} \quad \overline{S} \cap ((\partial C) \times S^1) \subset G.$$

Cutting $\overline{S}$ open along $C$, we get a surface $S \in S_1(+C)$ such that $y(S) = 1$. After an isotopy of $S$, we can assume the two surfaces $S, C \times [0, 1] \subset G \times [0, 1]$ are transverse. Since $S \cap ((\partial C) \times (0, 1)) = \emptyset, S \cap (C \times (0, 1))$ consists of closed curves which bounds disks in $C \times (0, 1)$. Since $S$ is incompressible and $G \times [0, 1]$ is irreducible, we can isotope $S$ such that $S \cap (C \times (0, 1)) = \emptyset$, hence $S \cap (C \times [0, 1]) = C \times [0, 1]$. Now we glue $S$ and $C \times [0, 1]$ together along $C \times \{0, 1\}$ and perturb the resulting surface slightly, then we get a connected surface $G'$ with $x(G') = 1$ and $\partial G'$ is parallel to $(\partial G) \times 0$ in $G \times [0, 1]$. Hence $G'$ is parallel to $G \times 0$ in $G \times [0, 1]$. It follows that $S$ is isotopic to $S'$ in $G \times [0, 1]$. 

Lemma 5.7. Let $S$ be the surface obtained in Lemma 5.6. Let

$$G \times I \cong (M_1(\infty), \gamma_1)$$
be the sutured manifold decomposition associated with $S$, then $(M_1(\infty), \gamma_1)$ is a product manifold, and there is an ambient isotopy of $M_1(\infty)$ which takes $K$ to a curve in $R_+(\gamma_1)$ such that the frame of $K$ specified by $R_+(\gamma_1)$ is $\alpha$.

Proof. By Lemma 5.6 $S$ is obtained from $-C \times I$ and $G \times 0$ by oriented cut-and-pastes. So $(M_1(\infty), \gamma_1)$ is a product sutured manifold and $R_+(\gamma_1)$ is an annulus.

Let $(M_1(\alpha), \gamma_1)$ be the sutured manifold obtained from $M(\alpha)$ by decomposing along $S$. Then $M_1(\alpha)$ can also be obtained from $M_1(\infty)$ by $\alpha$–surgery on $K$.

We claim that $M_1(\alpha)$ is not taut. In fact, let $S''$ be the surface obtained from $S$ and $G \times 0$ by oriented cut-and-pastes. Let $S'' \subset X$ be the surface obtained from $S''$ by gluing $\partial_0 S''$ to $\partial_1 S''$. Then $x(\alpha, \gamma_1) = 2$ and $S''$ represents $V - U + 2[\alpha, \gamma_1]$. We already computed

$$x_\infty(V - U + 2[\alpha, \gamma_1]) = 2 > x_\alpha(V - U + 2[\alpha, \gamma_1]) = 0,$$

so $S''$ is not taut in $X(\alpha)$. Let $M''(\alpha)$ be the non-taut sutured manifold obtained by decomposing $X(\alpha)$ along $S''$.

Since $S''$ is obtained from $S$ and $G \times 0$ by oriented cut-and-pastes, and $S \cap (G \times 0) = -C \times 0$ consists of two arcs, there exist two product disks in $M''(\alpha)$ such that the result of decomposing $M''(\alpha)$ along these two disks is $(M_1(\alpha), \gamma_1)$. See the proof of Gabai [3, Theorem 3.13] for an explanation of this fact. So $(M_1(\alpha), \gamma_1)$ is not taut by Gabai [3, Lemma 0.4].

Now Theorem 1.5 implies our conclusion.

Proof of Theorem 1.7. By the results in Sections 3 and 4, we only need to consider the case $F = G$ is a pair of pants. By Lemma 5.7 $K$ lies on $R_+(\gamma_1)$, and the frame specified by $R_+(\gamma_1)$ is $\alpha$.

Since $R_+(\gamma_1)$ is an annulus, the only essential curve on it is its core. As in Figure 5 $R_+(\gamma_1)$ can be constructed in the following way. Cut $G \times \{0,1\}$ open along $(v - u) \times \{0,1\}$, we get two octagons $P_0, P_1$. There are two edges of $P_1$ which are copies of $v \times 0$ with different orientations. We call these two edges $v \times 0, -v \times 0$. Similarly, there are edges $\pm u \times 0, \pm v \times 1, \pm u \times 1$. Now we glue two product disks to $P_0, P_1$, such that one product disk connects $v \times 0$ to $-v \times 1$ and the other connects $-u \times 0$ to $u \times 1$. The annulus we get is isotopic to $R_+(\gamma_1)$. The core of this annulus is clearly a one-crossing knot in $G \times I$. The result about the frame also follows since the vertical projection $p: R_+(\gamma_1) \to G$ is an immersion.

References

Figure 5: The surface $R_+/(\gamma_1)$ containing the knot $K$


