The Stability of Heavy Objects with Multiple Contacts

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Abstract. *In both robot grasping and robot locomotion, we wish to hold objects stably in the presence of gravity. We present a derivation of second-order stability conditions for a supported heavy object, employing the tool of Stratified Morse Theory. We then apply these general results to the case of objects in the plane.*

1 Introduction

We consider a heavy rigid object, $\mathcal{B}$, which is in frictionless contact with one or more rigid and immovable obstacles, $\mathcal{A}_1, \ldots, \mathcal{A}_k$, in the presence of gravity, or any other force arising from a conservative potential. Using Stratified Morse Theory, we examine the conditions under which such an object lies in a stable equilibrium.

Our analysis has a number of practical applications. First, consider the obstacles to be robot "fingers" in contact with $\mathcal{B}$. Our problem then amounts to determining whether the fingers stably support $\mathcal{B}$ in the presence of gravity. Our analysis may also be useful for studying the stability of a multi-limbed robot moving over convoluted terrain. The supporting obstacles are then interpreted as "footholds." If the robot moves quasi-statically, then at any moment it might be considered a single rigid body and the question of whether it slips and falls is the same as the problem considered here. Although some friction will be present in these applications, our analysis of the frictionless case will be maximally conservative. Frictional forces can only improve the stability of a grasp or stance.

Robotics researchers have considered the stability of an object under gravitational loading in the context of sensorless part manipulation [11, 2], quasi-static manipulation [7], and object recognition [4]. However, all of these analyses considered a single rigid object lying on a plane. In our analysis, the points of contact need not lie in a plane, and the contacts need not be flat, but can assume any curvature. We only require that the boundary of $\mathcal{B}$ is smooth in the vicinity of the contacts. Additionally, we use the elegant methods of Stratified Morse Theory (SMT) [3] to develop our results. Kriegman [5] has also used SMT to analyze the stable poses of piecewise smooth bodies lying on a plane. In some sense, our work can be considered an extension of some of Kriegman's results to the case of multiple nonplanar and nonflat contacts. In the context of "whole arm manipulation," Trinkle et al. [10] have also extensively studied the stability of a multiply contacted body under the influence of gravity. Our SMT approach formalizes and extends many of their results, generalizing to non-uniform force fields and to non-polyhedral objects. We can use these results to develop engineering "rules of thumb" which are useful in the applications described above. Finally, we will point out analogies between our results and the second order mobility theory developed in [8, 9].

2 Configuration space review

We use a configuration-space (c-space) based approach, as outlined in [8, 9], to analyze the stability of $\mathcal{B}$ about its contact configuration. The rigid object, $\mathcal{B}$, and the obstacles, $\mathcal{A}_1, \ldots, \mathcal{A}_k$, are located in an n-dimensional physical space, $\mathcal{W} = \mathbb{R}^n$ ($n = 2$ or 3). For our stability analysis, we can focus on the c-space of $\mathcal{B}$, rather than the composite c-space of the $k+1$ rigid bodies. $\mathcal{B}$'s c-space is the manifold $\mathcal{C} = \mathbb{R}^n \times SO(n)$, where $SO(n)$ is the group of rotations of $\mathbb{R}^n$. $SO(3)$ is parametrized by $\theta \in \mathbb{R}^3$, via the exponential map, where $\widehat{\theta} = \theta/\|\theta\|$ is the axis of rotation, and $\|\theta\|$ is the angle of rotation [6]. We regard $SO(2)$ as a subgroup of $SO(3)$ with rotation axis $\widehat{\theta}$ normal to the plane. The translational degrees of freedom are parametrized by $\mathbb{R}^n$. Thus $\mathcal{C}$ is parametrized by $\mathbb{R}^m$, where $m = \frac{1}{2}n(n+1)$.

Given a physical obstacle $\mathcal{A}_i$, its corresponding *c-space obstacle* (c-obstacle), denoted $\mathcal{CA}_i$, is the set of all configurations in $\mathcal{C}$ at which $\mathcal{B}$ intersects the obstacle. The boundary of $\mathcal{CA}_i$, denoted $\mathcal{S}_i$, consists of those configurations where the surfaces of $\mathcal{B}$ and $\mathcal{A}_i$ touch each other, while their interiors are disjoint. If $\mathcal{B}$ contacts $\mathcal{A}_i$ at configuration q_0, then q_0 lies in the c-obstacle boundary, $\mathcal{S}_i$. When $\mathcal{B}$ is in contact with k fingers, q_0 lies on the intersection of $\mathcal{S}_i$ for $i = 1, ..., k$. For convenience we will denote the union of all the c-obstacles, $\mathcal{CA}_1 \cup \ldots \cup \mathcal{CA}_k$, by $\mathcal{CA}$. The free configuration space, called the *freespace* $\mathcal{F}$, is the space that remains after removing the c-obstacle interiors from the c-space.

To model the rigidity of the contacting bodies, we introduce the following signed c-space distance function. It measures the distance of a configuration point q from $\mathcal{S}_i$ as follows:

$$d_i(q) \triangleq \begin{cases} \mathtt{dst}(q, \mathcal{S}_i) & \text{if } q \text{ is outside of } \mathcal{CA}_i \\ 0 & \text{if } q \text{ is on } \mathcal{S}_i \\ -\mathtt{dst}(q, \mathcal{S}_i) & \text{if } q \text{ is in the interior of } \mathcal{CA}_i \end{cases}$$

where $\mathtt{dst}$ is the Euclidean distance function. In [8] it was shown that the gradient of this distance function at $q_0 \in \mathcal{S}_i$ is n_i, the unit outward pointing normal to $\mathcal{S}_i$ at q_0. In the frictionless case, n_i can be associated with the reaction force between $\mathcal{B}$ and $\mathcal{S}_i$. Similarly, $D^2 d_i(q_0)$ is the *c-space curvature* form, or the curvature of the c-obstacle surface at q_0. It is a function of the surface curvatures of $\mathcal{B}$ and $\mathcal{S}_i$ at the contact. A closed form formula for $D^2 d_i(q_0)$ can be found in [8].

3 Review of Stratified Morse Theory

This section reviews the essential components of Stratified Morse Theory (SMT) that are required in the sequel (additional details can be found in [3]).

Stratified Sets. A *regularly stratified* set $\mathcal{X}$ is a set $\mathcal{X} \subset \mathbb{R}^m$ which can be decomposed into a finite union of disjoint smooth manifolds[1] called *strata*, satisfying the

[1] Recall that a manifold $\mathcal{M} \subset \mathbb{R}^m$ of dimension d is a hypersurface that locally looks like $\mathbb{R}^d$, for a fixed d, $0 \le d \le m$.

IEEE International Conference
on Robotics and Automation
0-7803-1965-6/95 $4.00 ©1995 IEEE

Whitney condition. The dimension of the strata varies between zero and m. (Zero-dimensional strata are isolated point manifolds; m-dimensional strata are open subsets of $\mathbb{R}^m$.) The Whitney condition requires that the tangents of two neighboring strata "meet nicely"; in our case, this condition is almost always satisfied. $\mathcal{B}$'s freespace, $\mathcal{F}$, is typically a stratified set. The m-dimensional strata are connected components of the interior of $\mathcal{F}$, which are open subsets of $\mathbb{R}^m$. The lower dimensional strata consist of the c-obstacle boundaries and their intersections.

Morse Theory. Let f be a smooth real-valued function defined on a smooth manifold $\mathcal{M} \subset \mathbb{R}^m$. A point $x \in \mathcal{M}$ is a *critical point* of f if its derivative at x, $\nabla f(x)$, vanishes there. A *critical value* of f is the image $c = f(x) \in \mathbb{R}$, of a critical point x. The function f is a *Morse function* if its second derivative matrix $D^2 f(x)$ is non-singular at all its critical points. The *Morse index* μ of a critical point is the number of negative eigenvalues of $D^2 f(x)$ at that point. Let $M|_{\leq v}$ denote the *sublevel set* of f, i.e. $M|_{\leq v} = \{x \in M\colon\ f(x) \leq v\}$. Morse Theory is concerned with the change in topology of the sublevel sets as v varies.

Stratified Morse theory is concerned with Morse functions on regular stratified sets [3]. Let f be defined now on a stratified set $\mathcal{X} \subset \mathbb{R}^m$ (in our problem, f will be the potential energy of $\mathcal{B}$, and $\mathcal{X}$ will be $\mathcal{B}$'s freespace). The critical points of f are the union of the critical points obtained by restricting f to the individual strata. The function f is a Morse function on $\mathcal{S}$ if it is Morse in the classical sense on the stratum containing the critical point x *and* if $\nabla f(x)$ is *not* normal to any of the other strata meeting at x. The Morse index μ of each critical point is now the number of negative eigenvalues of $D^2 f(x)$ evaluated only on the stratum containing the critical point. Note that, by definition, every zero-dimensional stratum is a critical point with Morse index $\mu = 0$.

SMT guarantees that *as $v = f(x)$ varies within the open interval between two adjacent critical values of f, the sublevel sets $\mathcal{X}|_{\leq v} = \{x \in \mathcal{X} : f(x) \leq v\}$ are topologically equivalent* (homeomorphic) *to each other.* If v increases beyond a critical value $v = c$ of $f(x)$ the topological type of the sublevel set $\mathcal{X}|_{\leq v}$ changes. The new topological type can be obtained by "gluing" a new topological set onto the old sublevel set. The details of this operation are determined by the Morse index μ at the critical point and by a construction called the *lower halflink* that we now describe.

Let $\mathcal{S}$ be the stratum of $\mathcal{X}$ containing a critical point x_c. Let $D(x_c)$ be a small disc of radius δ in the ambient space, that is transversal (e.g. perpendicular) to the stratum $\mathcal{S}$, such that it intersects $\mathcal{S}$ only at the point x_c. Note that the dimension of $D(x_c)$ is necessarily the dimension of the ambient space, m, minus the dimension of the stratum $\mathcal{S}$. The *normal slice*, $N(x_c)$, through the stratum $\mathcal{S}$ at the point x_c is defined as the intersection of $D(x_c)$ with the stratified set $\mathcal{X}$,

$$N(x_c) \triangleq D(x_c) \cap \mathcal{X}.$$

The *lower halflink* is the set

$$l^- \triangleq N(x_c) \cap f^{-1}(c - \epsilon),$$

where $\epsilon > 0$ and $\epsilon \ll \delta \ll 1$. It can be shown that the topological nature of l^- does not change for all $\epsilon > 0$ sufficiently small. Finally, the cone with base set l^- and vertex x_c is denoted by $\mathsf{cone}(l^-)$. By definition $\mathsf{cone}(l^-) = \{x_c\}$ when l^- is empty.

Theorem 3.12 in [3] guarantees that the topological change in the sublevel sets at a critical point x_c consists of taking a new topological set H of the form

$$H = D^\mu \times \mathsf{cone}(l^-) \tag{1}$$

and gluing it to $\mathcal{X}|_{\leq c-\epsilon}$ along the "gluing seam",

$$G = D^\mu \times l^- \cup \mathsf{bdy}(D^\mu) \times \mathsf{cone}(l^-) \tag{2}$$

where once again μ is the Morse index at x_c, and bdy denotes boundary.

4 The stability test

We now apply the methods of the last section to study the stability of a rigid body, $\mathcal{B}$, with m degrees of freedom, in contact with k rigid and immobile obstacles. The rigid body is subject to a conservative external force field, so that the potential energy of the body, $\tilde{U}(q)$, is a smooth function on the c-space $\mathcal{C}$, and depends only on the configuration q. Gravity is the canonical example of such a conservative force field, but more exotic examples include buoyancy forces (such as might be encountered by a robot operating underwater) or centrifugal forces. Let $U(q)$ be the restriction of $\tilde{U}(q)$ to the freespace $\mathcal{F}$, which admits a regular stratification.

Let q_0 denote $\mathcal{B}$'s configuration when it contacts k fingers. In c-space, q_0 lies on the c-space stratum which is the intersection of k c-obstacle surfaces; we label this stratum S. Is the point q_0 a local minimum of $U(q)$ in $\mathcal{F}$? Stratified Morse theory provides us with a rigorous answer when $U(q)$ is a Morse function on $\mathcal{F}$. Since Morse functions form an open dense set in the space of all smooth functions on $\mathcal{F}$, this will generically be the case. We will therefore assume it to be true. For completeness, we will later identify the non-generic cases in which $U(q)$ fails to be a Morse function.

Recall that q_0 is a *critical point* of $U(q)$ in the stratified set $\mathcal{F}$ iff it is a critical point on its own particular stratum S. (In particular, if S is a zero-dimensional stratum, then q_0 is automatically a critical point. This is generically true for a planar object in contact with three or more frictionless fingers, or a 3D object in contact with 6 or more fingers.) So q_0 is a critical point iff ∇U is orthogonal to $T_{q_0}S$, which is true iff ∇U lies in the subspace spanned by $n_1, \ldots, n_k$. Next we will consider the SMT requirements for such a critical point to be a local minimum.

Lemma 4.1 (Conditions for a Local Minimum) *The point q_0 is a local minimum of $U(q)$ on $\mathcal{F}$ if and only if the following two conditions are met:*

$$l^- = \emptyset \tag{3}$$
$$\mu = 0 \tag{4}$$

where μ is the Morse index and l^- is the lower halflink of q_0.

Proof: The fundamental definition of a local minimum q_0 of a function $U(q)$ is that the sublevel set $\mathcal{F}|_{\leq U(q_0)-\epsilon}$, for any $\epsilon > 0$, be empty in a sufficiently small neighborhood of q_0. The lower halflink l^- is a subset of the sublevel set and lies in a small disk around q_0. If l^- remains non-empty as we make this disk arbitrarily small, then $\mathcal{F}|_{\leq U(q_0)-\epsilon}$ contains points in every small neighborhood of q_0 and q_0 is not a local minimum. If $\mu > 0$, then q_0 is not even a local minimum on the stratum S that contains q_0. So equations 3-4 must be satisfied for q_0 to be a local minimum on $\mathcal{F}$.

Conversely, if $l^- = \emptyset$ and $\mu = 0$, the topological set H which is added to the sublevel set as $U(q)$ passes its critical value at q_0 is given by $H = D^\mu \times \mathsf{cone}(l^-) = D^0 \times \{q_0\}$, which is to say, H is the single point q_0. Furthermore, the gluing seam $G = D^\mu \times l^- \cup \mathsf{bdy}(D^\mu) \times \mathsf{cone}(l^-)$ at q_0 is empty, since $l^- = \emptyset$ and $\mathsf{bdy}(D^\mu) = \mathsf{bdy}(D^0) = \emptyset$. Both H and $\mathcal{F}|_{\leq U(q_0)-\epsilon}$ are closed sets, and since G is empty they are disconnected. Therefore $\mathcal{F}|_{\leq U(q_0)-\epsilon}$ contains no points in a sufficiently small neighborhood of $H = q_0$. □

We now have necessary and sufficient conditions for q_0 to be a local minimum of U on $\mathcal{F}$. Let us next interpret these conditions in a meaningful way. We first investigate condition 3. It is convenient to introduce some terminology from [1]. The positive span of the contact normals $n_1, \ldots, n_k$ is called the *normal cone* to $\mathcal{CA}$ at q_0 and denoted $N_C(q_0)$. Physically, it represents the set of feasible frictionless finger reaction forces.

$$N_C(q_0) \triangleq \{v = \sum_{i=1}^{k} \lambda_i n_i | \lambda_i \geq 0\}. \tag{5}$$

The cone polar to $N_C(q_0)$ is the *tangent cone* to $\mathcal{CA}$ at q_0, denoted by $T_C(q_0)$.

$$T_C(q_0) \triangleq \{v \in T_{q_0}\mathcal{C} | v \cdot n_i \leq 0 \quad \text{for} \quad n_1, \ldots, n_k\} \tag{6}$$

Let $q = \alpha(t)$ be a trajectory of $\mathcal{B}$ in $\mathcal{F}$ such that $\alpha(0) = q_0$ and the velocity $\dot{q} = \dot{\alpha}(0)$ lies in the cone tangent to the normal slice N—i.e., $\alpha(t)$ locally lies in N. If the potential energy $U(q)$ is instantaneously decreasing along $\alpha(t)$, then $\alpha(t)$ must pass through some point in l^-, and at least one trajectory of decreasing $U(q)$ must pass though each point in l^-. So condition 3 is equivalent to requiring that all trajectories which locally lie in N lead to non-decreasing $U(q)$. For any trajectory, if $\nabla U \cdot \dot{q} > 0$, then $U(q)$ is instantaneously increasing; if $\nabla U \cdot \dot{q} < 0$, then $U(q)$ is instantaneously decreasing. If $\nabla U \cdot \dot{q} = 0$, we need to consult higher-order terms in the Taylor expansion of $U(q)$. The set of instantaneous velocities $\{\dot{q}\}$ of trajectories which locally lie in N is the intersection of $-T_C(q_0)$ with $\text{span}(n_1, \ldots, n_k)$. We want to determine the sign of $\nabla U \cdot \dot{q}$ for all $\dot{q}$ in this set.

The cone polar to $-T_C(q_0) \cap \text{span}(n_1, \ldots, n_k)$ is $-N_C(q_0)$. We can therefore approach the question of whether equation 3 is satisfied by asking where ∇U lies with respect to the cone $N_C(q_0)$, since this information can be used to determine $\dot{U}$. For Eq. 3 to be satisfied, it is necessary for ∇U to lie in $N_C(q_0)$.

$$\nabla U = \sum_{i=1}^{k} \lambda_i n_i \text{ for } \lambda_i \geq 0 \tag{7}$$

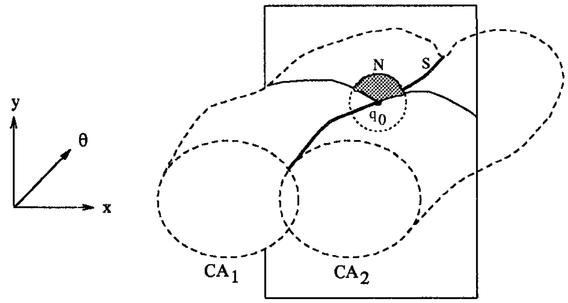

Figure 1: The intersection of the c-obstacles is denoted S. We choose a small disk which is centered on q_0 and transversal (e.g. orthogonal) to S. The normal slice N is the intersection of this disk with the freespace $\mathcal{F}$.

If ∇U lies in the *interior* of $N_C(q_0)$, then equation 3 is satisfied. So we will always be able to resolve whether or not equation 3 is satisfied except in the marginal case that ∇U lies exactly on the boundary of $N_C(q_0)$. For this case a more refined test would be necessary.

$$\nabla U \in \text{interior}(N_C(q_0)) \quad \longrightarrow \quad \ell^- = \emptyset \tag{8}$$

$$\nabla U \notin N_C(q_0) \quad \longrightarrow \quad \ell^- \neq \emptyset \tag{9}$$

Grasps which satisfy equation 7 are *equilibrium grasps* and q_0 is the equilibrium configuration. Note that equilibrium grasps implicitly satisfy the condition that $\nabla U \in \text{span}(n_1, \ldots, n_k)$, so all equilibrium points are critical points, but the converse is not true.

Up until this point our discussion has been fairly general, but we now wish to focus on a particular class of equilibrium grasps which we call *essential.*

Definition 1 *Essential grasps are equilibrium grasps for which the λ_i in Eq. 7 are (a) uniquely determined by the equilibrium condition; and (b) all strictly positive. Physically, this requirement means that all of the contacts are necessary to maintain equilibrium. This notion of essential grasps reduces to that defined in [9] in the case where* $\nabla U = 0$.

Equilibrium grasps which are not essential are termed *redundant.* Redundant grasps are also of interest, because additional contacts, which are not necessary for equilibrium, may nevertheless improve the quality of the grasp. We do not consider redundant grasps here.

For an essential grasp, since the λ_i are all positive, ∇U lies in the interior of $N_C(q_0)$, so equation 3 is known to be satisfied. We now turn to the other necessary condition for q_0 to be a local minimum in $\mathcal{F}$, equation 4, which requires that q_0 be a local minimum of the restriction of $U(q)$ to the stratum S.

Recall that $d_i(q)$ is the c-space distance function for obstacle $\mathcal{A}_i$. We introduce the **gravity relative distance** d_G, which is defined as follows:

$$d_G(q) \triangleq (\sum_{i=1}^{k} \lambda_i d_i(q)) - U(q) \tag{10}$$

where the $\{\lambda_i\}$ are the equilibrium grasp coefficients. The **gravity relative curvature form** κ_G is defined:

$$\kappa_G(q_0, \dot{q}) \triangleq \dot{q}^T D^2 d_G(q_0) \dot{q} \quad \text{for} \quad \dot{q} \in T_{q_0} S \tag{11}$$

Note κ_G is a function of the contact locations and contact curvatures. Since the λ_i are uniquely defined for an essential grasp, d_G and κ_G are also uniquely defined.

Proposition 4.2 *An essential grasp is stable, i.e. q_0 is a local minimum, if all the eigenvalues of the gravity relative curvature $D^2 d_G : T_{q_0} S \mapsto T_{q_0} S$ are negative.*

Proof: Suppose q_0 is an essential equilibrium configuration. Let $q = \alpha(t)$ be a trajectory in the stratum S which passes through q_0 at $t = 0$. The velocity of the trajectory at $t = 0$ is $\dot{q} \in T_{q_0} S$ and the instantaneous acceleration is $\ddot{q}$. The stipulation that $\alpha(t)$ lies in S implies that $d_i(\alpha(t)) = 0$ for $i = 1, \cdots, k$. Taking the second derivative of this relation we find:

$$n_i \cdot \ddot{q} + \dot{q}^T D^2 d_i \dot{q} = 0 \tag{12}$$

Let us evaluate $U(t)$ along this trajectory. The first derivative $\dot{U} = \nabla U \cdot \dot{q}$ vanishes since q_0 is a critical point. The second derivative is $\ddot{U} = \nabla U \cdot \ddot{q} + \dot{q}^T D^2 U \dot{q}$. Using equations 7 and 12, we may rewrite this as

$$\ddot{U} = \dot{q}^T (D^2 U - \sum_{i=1}^{d} \lambda_i D^2 d_i) \dot{q} = -\dot{q}^T (D^2 d_G) \dot{q}. \tag{13}$$

If the gravity relative curvature, $D^2 d_G$, has all negative eigenvalues, then clearly $\ddot{U}$ is positive for all choices of $\dot{q} \in T_{q_0} S$. Then q_0 is a local minimum of the restriction of $U(q)$ to S, and Eq. 4 is satisfied. Since Eq. 3 is always satisfied for equilibrium grasps, q_0 is a local minimum of $U(q)$ on $\mathcal{F}$ and the grasp is stable. □

Consider the set of allowed 1^{st}-order motions which do not oppose gravity: $M_{q_0}^{1(G)} = \{\dot{q} \in -T_C(q_0) | \nabla U \cdot \dot{q} \leq 0\}$. For an equilibrium grasp, $M_{q_0}^{1(G)}$ is exactly the subspace $T_{q_0} S$. We define the 1^{st}*-order gravity mobility index*, $m_{q_0}^{1(G)}$, to be the dimension of $M_{q_0}^{1(G)}$. For an essential equilibrium grasp, $m_{q_0}^{1(G)} = m - k$. For an equilibrium grasp, the domain of the gravity relative curvature form, $D^2 d_G$, is $M_{q_0}^{1(G)}$. We define the 2^{nd}*-order gravity mobility index*, $m_{q_0}^{2(G)}$, to be the dimension of the largest possible subspace in $M_{q_0}^{1(G)}$ on which the gravity relative curvature form κ_G is non-negative (positive semi-definite). Note that by proposition 4.2, a grasp is stable if $m_{q_0}^{2(G)} = 0$.

An analogy may be made to the 1^{st}- and 2^{nd}-order mobility indices introduced in [8] and [9]. When considering whether an equilibrium grasp permits any unstable motions, one can imagine the gravity equipotential passing through q_0 to be the impenetrable surface of a $(k+1)^{st}$ finger with distance function $-U(q)$. Free motions allowed by the actual grasp will be unstable only if they do not penetrate the imaginary "gravity finger". The gravity mobility indices of the real grasp are then precisely equivalent to the 1^{st}- and 2^{nd}-order mobility indices of the imaginary $(k+1)$-finger grasp. This analogy also illustrates that the gravity mobility indices will share useful properties of the regular mobility indices, such as coordinate invariance.

Recall that we began with the generic assumption that $U(q)$ was a Morse function on $\mathcal{F}$. This assumption can fail in two ways. First, $U(q)$ will not be a Morse function on $\mathcal{F}$ if ∇U lies exactly on the boundary of $N_C(q_0)$. In this case, some of the λ_i are zero; the fingers associated with zero λ_i are referred to as *passive.* The intersection of those c-obstacles associated with non-zero λ_i (the *active* fingers) is then a stratum which neighbors S and has a tangent space orthogonal to ∇U. Such a grasp is non-essential and cannot be immediately analyzed by SMT.

When a non-essential grasp consists of an essential grasp, plus some number of passive fingers, it is tempting to apply the gravity relative curvature form anyway, which amounts to ignoring the passive fingers and evaluating the stability of the remaining essential grasp. In general, this procedure will *underestimate* the stability of the whole grasp. However, in the special case where there is only one passive finger, we can ignore it with no loss of precision. The first-order motions allowed by the active fingers form a vector subspace of $T_{q_0}\mathcal{C}$. The addition of one passive finger only removes a half-space of these allowed motions. So a single passive finger cannot affect the stability of a grasp.

$U(q)$ can also fail to be a Morse function if any of the eigenvalues of the Hessian matrix of second derivatives of the restriction of $U(q)$ to S, evaluated at q_0, are zero. For an essential grasp, this occurs iff $D^2 d_G$ has any zero eigenvalues. In this case we would have to consider higher-order terms of the Taylor expansion of $U(q)$ on S in order to resolve the grasp stability.

5 Planar objects under gravity

As an example of the utility of our theory, in this section we apply the stability test to planar grasps in the presence of gravity. In the planar case, the $\kappa_i(\dot{q}) = \dot{q}^T D^2 d_i \dot{q}$ can be evaluated in a simple way. A first-order motion $\dot{q} \in T_{q_0}(\partial \mathcal{CA}_i)$ corresponds to an instantaneous rotation about a point in the plane somewhere along the line of the contact normal (pure translations corresponding to rotations about an axis at infinity) [8]. Let ρ_i be the distance in the plane from the i^{th} contact point to the axis of the instantaneous rotation. By convention, ρ_i is positive on the object side of the contact, and negative on the finger side. In this way we can reparametrize $\dot{q} \in T_{q_0}(\partial \mathcal{CA}_i)$ by ρ_i and the angular velocity ω. Setting $\|\omega\| = 1$, we find

$$\kappa_i(\dot{q} = (\vec{0}, \omega)) = \frac{(\rho_i \kappa_{B_i} - 1)(\rho_i \kappa_{A_i} + 1)}{\kappa_{A_i} + \kappa_{B_i}} \tag{14}$$

where κ_{A_i} and κ_{B_i} are respectively the curvatures of the finger and object at the i^{th} contact point. The requirement that the fingers do not penetrate the object yields the condition $\kappa_{A_i} + \kappa_{B_i} \geq 0$.

If $\dot{q}$ is a pure translation, then setting the translational velocity $\|v\| = 1$, we find that

$$\kappa_i(\dot{q} = (\vec{v}, 0)) = \frac{\kappa_{A_i} \kappa_{B_i}}{\kappa_{A_i} + \kappa_{B_i}} \tag{15}$$

Let $\vec{x}_{cm}$ be the location of the object's center of mass. Let ∇U_f be the force or translational component of ∇U, and let ∇U_m be the moment or rotational component. The following result is given without proof.

Proposition 5.1 *For a planar object in a uniform gravitational field, any motion $\dot{q}$ such that $\nabla U \cdot \dot{q} = 0$ corresponds to an instantaneous rotation about a point in the plane which lies on the line defined by $\{\vec{x}_{cm} + \eta \nabla U_f : \eta \in \mathbb{R}\}$ Then*

$$\dot{q}^T D^2 U \dot{q} = -\nabla U_f \cdot \vec{\rho}_{cm} \tag{16}$$

where $\vec{\rho}_{cm}$ is the vector from the axis of rotation to the center of mass, and where $\|\omega\| = 1$. (By the first part of the proposition, $\vec{\rho}_{cm}$ has either the same direction as ∇U, or the antipodal direction.)

We conclude that for planar grasps,

$$\kappa_G = \nabla U_f \cdot \vec{\rho}_{cm} + \sum_{i=1}^{k} \lambda_i \frac{(\rho_i \kappa_{B_i} - 1)(\rho_i \kappa_{A_i} + 1)}{\kappa_{A_i} + \kappa_{B_i}} \tag{17}$$

It is convenient to scale ∇U so that $\|\nabla U_f\| = 1$ (this scaling has no effect on the stability test since it simply amounts to a choice of units for energy). The λ_i must be chosen so that equation 7 is satisfied for the normalized ∇U. With this normalization,

$$\kappa_G = \rho_{cm} + \sum_{i=1}^{k} \lambda_i \frac{(\rho_i \kappa_{B_i} - 1)(\rho_i \kappa_{A_i} + 1)}{\kappa_{A_i} + \kappa_{B_i}} \tag{18}$$

where $\rho_{cm} = \nabla U_f \cdot \vec{\rho}_{cm}$ has absolute value equal to $\|\vec{\rho}_{cm}\|$.

Let the *contact normal line*, l_i, pass through the i^{th} contact point in the direction normal to the surface of the i^{th} finger at that point. Let $\hat{v}_i$ denote the unit vector indicating the direction of l_i. Further, we denote the intersection of lines l_i and l_j by p_{ij}. Finally, we define b_{ij} as the line which passes through p_{ij} in the direction of ∇U_f. (Obviously p_{ij} and b_{ij} do not exist if l_i and l_j do not intersect. In this paper we do not consider grasps where contact normal lines are collinear and intersect at more than one point.)

The stability test for generic one-, two-, and three-fingered planar grasps is developed as follows. The positive span of the contact directions $\hat{v}_i$ is the projection of the normal cone $N_C(q_0)$ onto the translational plane in $T_{q_0}\mathcal{C}$. For the equilibrium condition 7 to be satisfied, ∇U_f must lie in this projection. Then condition 7 will hold if we choose ∇U from the set of vectors in $N_C(q_0)$ whose projection onto the translational plane is ∇U_f. Since $N_C(q_0)$ is convex, this means that for equilibrium ∇U_m must lie in a single closed interval in $\mathbb{R}$—or, equivalently, the object center of mass must lie in a particular vertical strip in the plane. This vertical strip may be just a single vertical line, it may be a finite-width strip bounded by two lines, or it may be a semi-infinite region bounded only on one side. More precisely, if ∇U_f lies in the same direction as $\hat{v}_i$, then the line l_i in the plane is a boundary of the vertical strip. If ∇U_f lies in the interior of the positive span of $\hat{v}_i$ and $\hat{v}_j$, then the line b_{ij} is a boundary of the strip.

If the center of mass lies in the interior of the allowed vertical strip, then the grasp is an essential equilibrium and condition 3 is satisfied. If the vertical strip has positive width and the center of mass lies on a boundary line of the strip, or if ∇U_f lies on the boundary of the positive span of the contact directions $\hat{v}_i$, then the grasp is an

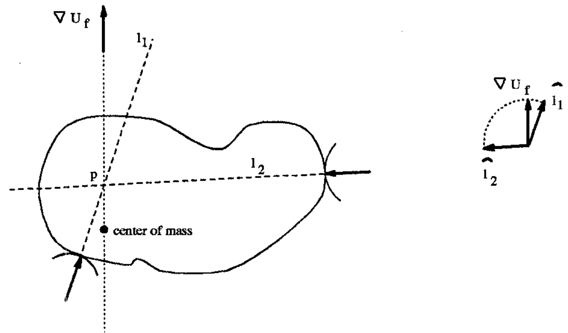

Figure 2: Contact normal lines l_1, l_2 intersect at p.

equilibrium but some of the fingers are passive; we can ignore these at the possible risk of underestimating the grasp stability, as discussed in section 4. In any case, for stability of equilibrium the grasp must also satisfy condition 4; it will do this if κ_G, as given by equation 18, is negative definite for all allowed first-order motions $\dot{q}$.

5.1 One-fingered planar grasps

Consider an object balanced on a single finger. The equilibrium condition requires that ∇U_f lie in the same direction as $\hat{v}_1$. The normal cone $N_C(q_0)$ is one-dimensional, so only a single value of ∇U_m yields equilibrium; the object center of mass must lie on the line l_1.

Using the normalization convention $\|\nabla U_f\| = 1$,

$$\kappa_G = \rho_{cm} + \frac{(\rho_1 \kappa_{B_1} - 1)(\rho_1 \kappa_{A_1} + 1)}{\kappa_{A_1} + \kappa_{B_1}} \tag{19}$$

The first-order allowed motions are all rotations about any point on the contact normal line in the plane. Let r_{cm} be the distance from the point of contact to the center of mass. Then $\rho_{cm} = r_{cm} - \rho_1$. So

$$\kappa_G = r_{cm} - \rho_1 + \frac{(\rho_1 \kappa_{B_1} - 1)(\rho_1 \kappa_{A_1} + 1)}{\kappa_{A_1} + \kappa_{B_1}} \tag{20}$$

The one-fingered grasp will be stable iff κ_G is negative-definite for all $\rho_1 \in \mathbb{R}$. Since κ_G is just a quadratic function of ρ_1, this is a matter of elementary algebra. The possible cases are summarized in table 5.1.

5.2 Two-fingered planar grasps

We consider 2-fingered grasps such that l_1 and l_2 are not collinear. (Otherwise the grasp could not be essential for any choice of $\nabla U \neq 0$.) For equilibrium ∇U_f must lie in the positive span of $\hat{v}_1$ and $\hat{v}_2$. If l_1 and l_2 intersect at a point, p_{12}, then the object center of mass must lie on b_{12} for stability. If l_1 and l_2 are parallel, then ∇U_f must lie in the same direction as $\hat{v}_1$ and $\hat{v}_2$, and the center of mass must lie in the strip between l_1 and l_2. If l_1 and l_2 are antiparallel, then ∇U_f must lie in the same direction as one of them (say l_1) and the center of mass must lie in the half-plane bounded by l_1 which does not contain l_2.

An essential equilibrium grasp will be stable as long as the gravity relative curvature form is negative for all first-order motions in $T_{q_0}S$. There is only one such first-order motion to consider: rotation around p_{12}, if l_1 and l_2 intersect; or pure translation perpendicular to l_1 and l_2 if they are parallel/antiparallel.

Finger	*Object*		*Stability*
Convex	Convex	$\kappa_{A_1} > 0,\ \kappa_{B_1} > 0$	always unstable
Concave	Convex	$\kappa_{A_1} < 0,\ \kappa_{B_1} > 0$	stable iff $r_{cm} < (1/\kappa_{B_1})$
Convex	Concave	$\kappa_{A_1} > 0,\ \kappa_{B_1} < 0$	stable iff $r_{cm} < (1/\kappa_{B_1})$
Convex	Flat	$\kappa_{A_1} > 0,\ \kappa_{B_1} = 0$	always unstable
Flat	Convex	$\kappa_{A_1} = 0,\ \kappa_{B_1} > 0$	neutrally stable under pure translation otherwise stable iff $r_{cm} < (1/\kappa_{B_1})$

Table 1: Summary of stability results for one-fingered grasps

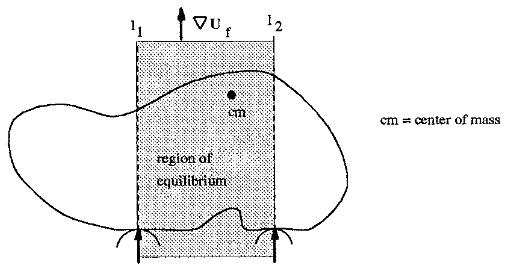

Figure 3: Contact normal lines l_1, l_2 are parallel.

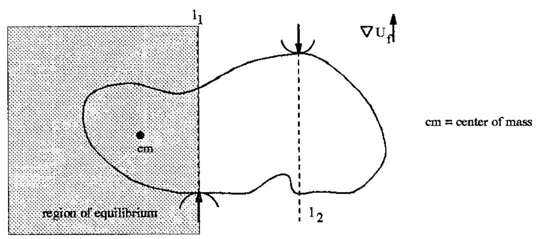

Figure 4: Contact normal lines l_1, l_2 are antiparallel.

If l_1 and l_2 intersect at p_{12}, then consider rotation around p_{12}, so that ρ_i is the distance from the ith contact point to p_{12}, and ρ_{cm} is the distance from p_{12} to the center of mass (understood to be positive if the center of mass is above p_{12}, negative if the center of mass is below p_{12}). Then the gravity relative curvature form for normalized ∇U_f is

$$\kappa_G = \rho_{cm} + \lambda_1 \frac{(\rho_1 \kappa_{B_1} - 1)(\rho_1 \kappa_{A_1} + 1)}{\kappa_{A_1} + \kappa_{B_1}} + \lambda_2 \frac{(\rho_2 \kappa_{B_2} - 1)(\rho_1 \kappa_{A_2} + 1)}{\kappa_{A_2} + \kappa_{B_2}} \tag{21}$$

where

$$\lambda_1 \hat{v}_1 + \lambda_2 \hat{v}_2 = \nabla U_f \quad \text{for} \quad \|\nabla U_f\| = 1 \tag{22}$$

and the grasp is stable if κ_G is negative.

If l_1 and l_2 are parallel or antiparallel, then the gravity relative curvature form for pure translation is

$$\kappa_G = \frac{\kappa_{A_1}\kappa_{B_1}}{\kappa_{A_1} + \kappa_{B_1}} + \frac{\kappa_{A_2}\kappa_{B_2}}{\kappa_{A_2} + \kappa_{B_2}} \tag{23}$$

and again the grasp is stable if κ_G is negative. The gravity term of the curvature form vanishes for pure translation orthogonal to ∇U_f as a consequence of our assumption of a uniform gravitational field.

If the system lies on the boundary of a region of stability then one of the fingers is passive. In this case the grasp is technically non-essential, but we can ignore a single passive finger without affecting our analysis (see section 4). The above results hold even if λ_1 or λ_2 is zero.

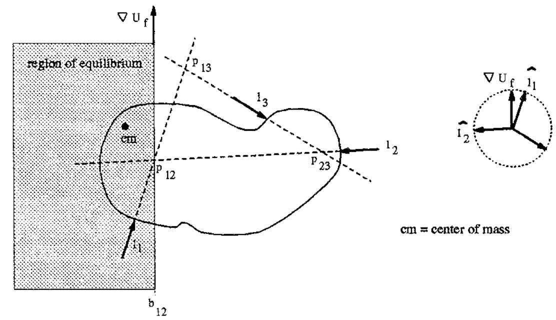

Figure 5: $\hat{v}_1, \hat{v}_2, \hat{v}_3$ positively span the plane.

5.3 Three-fingered planar grasps

We consider three-fingered planar grasps with the restrictions that none of the contact normal lines l_1, l_2, l_3 are collinear with each other, they are not all parallel, nor do they all intersect at a single point. Three-fingered grasps which violate any of these restrictions will not be essential for any choice of $\nabla U \neq 0$.

Essential three-fingered grasps imply that the stratum S is zero-dimensional, so equation 4 is automatically satisfied. Therefore, *an essential equilibrium grasp is automatically stable.* The equilibrium condition requires that ∇U_f lie in the positive span of $\hat{v}_1, \hat{v}_2, \hat{v}_3$.

If $\hat{v}_1, \hat{v}_2, \hat{v}_3$ positively span the plane $\mathbb{R}^2$, then the region of equilibrium for the center of mass is only bounded on a single side. The vector ∇U_f must lie either in the same direction as exactly one of the contact directions $\hat{v}_i$, or in the interior of the positive span of exactly one pair of contact directions $\hat{v}_i$ and $\hat{v}_j$. If ∇U_f lies in the same direction as $\hat{v}_1$, then the center of mass must lie in the half-plane bounded by l_1 that does *not* contain p_{23}. On the other hand, if ∇U_f lies in the interior of the positive span of $\hat{v}_1$ and $\hat{v}_2$, then the grasp is stable if the center of mass lies inside the half-plane bounded by b_{12} that does *not* contain p_{13} or p_{23}.

If two of the contact normal lines are antiparallel (without loss of generality, assume that $\hat{v}_1 = -\hat{v}_3$), then a similar situation obtains, with some complications. Again the vector ∇U_f lies either in the same direction as exactly one of the $\hat{v}_i$, or inside the positive span of one of the pairs $\{\hat{v}_1, \hat{v}_2\}$ or $\{\hat{v}_2, \hat{v}_3\}$. If ∇U_f lies in the interior of the positive span of $\{\hat{v}_1, \hat{v}_2\}$, then the center of mass must lie in the half-plane bounded by b_{12} which does not contain p_{23}. If ∇U_f lies along $\hat{v}_2$, then the region of equilibrium

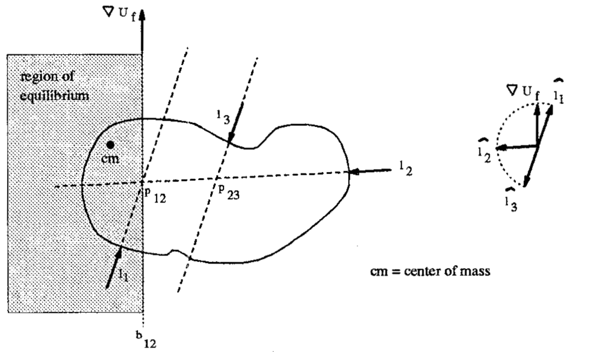

Figure 6: Two contact normal lines are antiparallel.

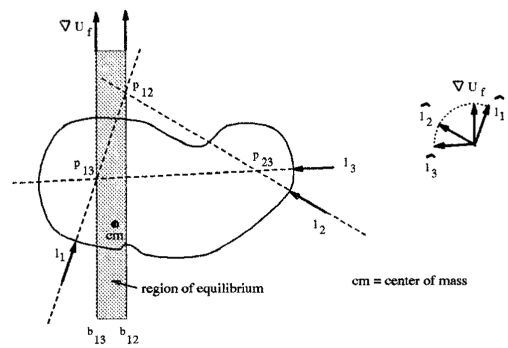

Figure 7: $\hat{v}_1, \hat{v}_2, \hat{v}_3$ do not positively span the plane and none of the contact normal lines are antiparallel.

is a half plane bounded by l_2. We can discover which half-plane by perturbing $\hat{v}_1$ or $\hat{v}_3$ by an arbitrarily small amount so that the $\hat{v}_i$ positively span the plane and p_{13} exists. The region of equilibrium is then the half-plane that does not contain p_{13}. Finally, if ∇U_f lies along $\hat{v}_1$, then finger $\mathcal{A}_2$ is passive and the grasp should be analyzed as a two-finger grasp with fingers $\mathcal{A}_1$ and $\mathcal{A}_3$.

If $\hat{v}_1, \hat{v}_2, \hat{v}_3$ do not positively span the plane, and none of the contact normal lines are antiparallel to each other, then the equilibrium region for the center of mass almost always has two boundaries (the one exception is discussed below). If ∇U_f lies in the direction of $\hat{v}_i$, then l_i will be one of the two boundaries (this could be true for one, two, or none of the l_i). If ∇U_f lies in the interior of the positive span of $\hat{v}_i$ and $\hat{v}_j$ then b_{ij} will be one of the two boundaries (this could be true for one, two, or none of the pairs of contact directions). If the center of mass lies in the strip of the plane between the two boundary lines, then the configuration is an equilibrium grasp. The only exception to this prescription occurs if ∇U_f lies in the direction of $\hat{v}_i$, and $\hat{v}_i$ does not lie in the positive span of the other two contact directions. In this case the grasp is redundant with two passive fingers and for equilibrium the center of mass must lie on the single vertical line l_i.

As stated above, for essential three-fingered grasps all equilibria are stable. However, equilibrium grasps will fail to be essential if (a) the center of mass lies exactly on one of the boundary lines of its allowed equilibrium region; or (b) ∇U_f lies on the boundary of the positive span of the three contact directions. In either of these cases we must ignore one or two passive fingers and evaluate the remaining grasp by using the gravity relative curvature form. We can ignore a single passive finger without any loss of information; if we ignore two passive fingers then we may underestimate the stability of the grasp.

6 Conclusion

We have used Stratified Morse Theory to rigorously derive second-order tests for the stability of an object held by a non-redundant number of supports, in the presence of a conservative force field such as gravity. These tests are applicable to all objects which are smooth at the points of support, and take into consideration the surface curvature of the object and supports. An important feature of the results is that the force field need not have a uniform gradient in c-space. This aids analysis of a heavy object—such as a walking robot—with a center of gravity that is quasi-statically moving while the points of support remain fixed. This feature is also necessary when dealing with more exotic conservative forces, such as the combined weight and buoyancy of an object being manipulated underwater.

To demonstrate the usefulness of the second-order tests we have applied them to some simple planar problems with uniform gravity. The tests are also fully applicable to spatial grasps. In future work we plan to address these and also consider redundant numbers of contacts.

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