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COMPLEX BIFURCATION FROM REAL PATHS*

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This paper is dedicated to Edward L. Reiss on the occasion of his 60*th birthday.*

Abstract. A new bifurcation phenomenon, called complex bifurcation, is studied. The basic idea is simply that real solution paths of real analytic problems frequently have complex paths bifurcating from them. It is shown that this phenomenon occurs at fold points, at pitchfork bifurcation points, and at isola centers. It is also shown that perturbed bifurcations can yield two disjoint real solution branches that are connected by complex paths bifurcating from the perturbed solution paths. This may be useful in finding new real solutions.

A discussion of how existing codes for computing real solution paths may be trivially modified to compute complex paths is included, and examples of numerically computed complex solution paths for a nonlinear two point boundary value problem, and a problem from fluid mechanics are given.

Key words. bifurcation theory, nonlinear differential equations, path following

AMS(MOS) subject classifications. 35B32, 34C30, 58F14, 58C28

1. Introduction. We study a new bifurcation phenomenon that we call complex bifurcation. The basic idea is simply that real solution paths of real analytic problems frequently have complex paths bifurcating from them. Indeed the phenomenon is not rare since it occurs at fold points, at pitchfork bifurcation points, and at isola centers. If the problem of interest is finite-dimensional with nonlinear algebraic equations, then our observation is simply that real roots may become complex as some parameter varies. Thus indeed discrete approximations of nonlinear functional equations frequently exhibit this phenomenon. However, these discrete complex paths need not be spurious since, as we show, the underlying problem in a Banach space setting also exhibits the same behavior. The occurrence of some complex bifurcations in finite-dimensional homotopy problems was considered in [2].

The importance of complex paths bifurcating from real paths is not yet fully understood. However, since disjoint real solution paths can be connected by complex paths, the latter may be useful in finding new solutions. As an example of this phenomenon, we show that perturbed bifurcations can yield two disjoint real solution branches that are connected by complex paths bifurcating from the perturbed solution paths. We also show how minor modifications to existing real algorithms can be introduced to compute the complex solution branches and we show some results from the use of such modifications.

To make this paper fairly complete we briefly recall the local path continuation theory for simple folds. We then examine simple quadratic folds and exhibit complex bifurcation from them. We show how the same can be done at isola centers and pitchfork bifurcations. More general results on complex paths are then developed. The extension of real codes to compute complex paths is presented. Finally, some examples are given.

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2. Real path following; simple folds. We consider smooth nonlinear problems of the form

$$F(u, \lambda)=0, \qquad F: \mathbb{B} \times \mathbb{R} \rightarrow \mathbb{B}; \tag{2.1}$$

where $u \in \mathbb{B}$ is a real Banach space and $\lambda \in \mathbb{R}$ is a real parameter. To study paths or families of solutions of such problems we start with the following basic definition.

DEFINITION 2.2. A point $(u^0, \lambda^0) \in \mathbb{B} \times \mathbb{R}$ is a regular solution of (2.1) if

(a) $F(u^0, \lambda^0)=0$,

(b) $F_u(u^0, \lambda^0) \equiv F_u^0$ is nonsingular.

By means of the implicit function theorem we get Theorem 2.3.

THEOREM 2.3. *Through each regular solution* (u^0, λ^0) *there exists a unique smooth path of solutions*

$$\Gamma^0: \{u(\lambda), \lambda\}, \qquad |\lambda-\lambda^0|<\delta$$

for some $\delta>0$.

The standard proof of the implicit function theorem yields a constructive procedure to determine the path Γ^0. The path may be continued using this procedure until some singular point is encountered. The "generic," or most common type, of singularity is that in Definition 2.4.

DEFINITION 2.4. A point $(u^0, \lambda^0) \in \mathbb{B} \times \mathbb{R}$ is a simple singular solution of (2.1) if

$$F(u^0, \lambda^0)=0; \tag{2.4a}$$

$$F_u^0 \text{ is singular with: } \quad \dim \mathcal{N}(F_u^0)=\operatorname{codim} \mathcal{R}(F_u^0)=1; \tag{2.4b}$$

$$F_u^0 \text{ Fredholm with index zero.} \tag{2.4c}$$

Here $\mathcal{N}(F_u^0)$ and $\mathcal{R}(F_u^0)$ are the null space and range, respectively, of the linear operator F_u^0.

To enable us to continue a solution path through a simple singular point we need more information. The most common additional constraint is that of Definition 2.5.

DEFINITION 2.5. A point $(u^0, \lambda^0) \in \mathbb{B} \times \mathbb{R}$ is a simple fold point of (2.1) if it is a simple singular solution of (2.1) with $F_\lambda(u^0, \lambda^0) \equiv F_\lambda^0: \mathbb{R} \rightarrow \mathbb{B}$ bounded and

$$F_\lambda^0 \notin \mathcal{R}(F_u^0). \tag{2.5a}$$

To show how easily we can continue a solution path through a simple fold we present Theorem 2.6.

THEOREM 2.6. *If* (u^0, λ^0) *is a simple fold point of* (2.1) *then there exists a real smooth path of solutions.*

$$\Gamma^I: \{u(s), \lambda(s)\}, \qquad |s-s_0|<\delta \tag{2.6a}$$

where s is a real parameter and

$$u(s_0)=u^0, \quad \lambda(s_0)=\lambda^0. \tag{2.6b}$$

Further $s \not\equiv \lambda$ *(i.e.,* Γ^I *is not parametrized by* λ*) and*

$$\dot{\lambda}(s_0)=\left.\frac{d\lambda(s)}{ds}\right|_{s=s_0}=0, \tag{2.6c}$$

$$\dot{u}(s_0)=\left.\frac{du(s)}{ds}\right|_{s=s_0} \in \mathcal{N}(F_u^0). \tag{2.6d}$$

Proof. From (2.4b) it follows that there exist $\phi \in \mathbb{B}$ and $\psi^* \in \mathbb{B}^*$, the dual space of $\mathbb{B}$, such that:

$$\mathcal{N}(F_u^0) = \text{span}\{\phi\}, \tag{2.7a}$$

$$\mathcal{R}(F_u^0) = \{v \in \mathbb{B}: \psi^* v = 0\}. \tag{2.7b}$$

Furthermore (2.4c) implies that

$$\psi^* \phi \neq 0, \tag{2.7c}$$

and (2.5a) implies that

$$d \equiv \psi^* F_\lambda^0 \neq 0. \tag{2.7d}$$

Now consider the following augmented system and problem:

$$H(u, \lambda, s) \equiv \begin{pmatrix} F(u, \lambda) \\ \psi^*[u - u^0] - (s - s_0) \end{pmatrix} = 0; \qquad H: \mathbb{B} \times \mathbb{R}^2 \to \mathbb{B} \times \mathbb{R}. \tag{2.8}$$

The path Γ^I will be established by applying the Implicit Function Theorem to $H = 0$. It is clear that $H(u^0, \lambda^0, s_0) = 0$ and

$$A \equiv \frac{\partial H(u^0, \lambda^0, s_0)}{\partial(u, \lambda)} = \begin{pmatrix} F_u^0 & F_\lambda^0 \\ \psi^* & 0 \end{pmatrix}. \tag{2.9a}$$

We claim that the Fréchet derivative $A: \mathbb{B} \times \mathbb{R} \to \mathbb{B} \times \mathbb{R}$ has a bounded inverse. Let $v \in \mathbb{B}$ and $x \in \mathbb{R}$ be such that $A\binom{v}{x} = 0$. The $\mathbb{B}$ component of this relation is just

$$F_u^0 v + F_\lambda^0 x = 0.$$

Operating with ψ^* yields, since $F_u^0 v \in \mathcal{R}(F_u^0)$ and $\psi^* F_\lambda^0 \neq 0$, that $x = 0$. It follows that $v = \beta\phi$ for some $\beta \in \mathbb{R}$. Now the last component becomes $\psi^* \beta\phi = 0$ and hence $\beta = 0$. Thus the only null vector for A is the zero vector. Since $\mathbb{B} \times \mathbb{R}$ need not be finite-dimensional we must also show that for any $w \in \mathbb{B}$ and $\gamma \in \mathbb{R}$ the system

$$A\begin{pmatrix} v \\ x \end{pmatrix} = \begin{pmatrix} w \\ \gamma \end{pmatrix} \tag{2.9b}$$

has a solution. Again from the $\mathbb{B}$ component we easily get that

$$x = \psi^* w / \psi^* F_\lambda^0,$$

and hence v must satisfy

$$F_u^0 v = w - F_\lambda^0 \psi^* w / \psi^* F_\lambda^0.$$

Since the right-hand side is in $\mathcal{R}(F_u^0)$, solutions exist and all are of the form

$$v = v_0 + \beta\phi, \qquad \beta \in \mathbb{R}.$$

Here v_0 is any fixed particular solution. Using this in the last component finally yields

$$\beta = (\gamma - \psi^* v_0) / \psi^* \phi.$$

Thus existence and uniqueness are established and so A has an inverse.

We need only show that A is closed and then it follows from the Closed Graph Theorem (see [16, Thm. 4.4, p. 65]) that A^{-1} is bounded. In fact, since F_u^0 is Fredholm and F_λ^0 is bounded, it is clear from the form of A that it is a closed operator. Thus the Implicit Function Theorem is applicable to (2.8) and yields a smooth path as in (2.6a), (2.6b) satisfying $H(u(s), \lambda(s), s) = 0$ and hence:

$$F(u(s), \lambda(s)) = 0. \tag{2.10}$$

Differentiating and setting $s=s_0$ in the above yields

$$F_u^0\dot{u}(s_0)+F_\lambda^0\dot{\lambda}(s_0)=0, \tag{2.11}$$

from which (2.6c), (2.6d) follow. Of course if the parameter $s\equiv\lambda$ in any neighborhood of s_0 then $\dot{\lambda}(s_0)=1$. This contradicts (2.6c) and the proof is complete.

We stress that the proof in Theorem 2.6 of the existence of the path Γ^I through the simple fold is constructive. This is but one instance of the general technique, introduced in [10], of using an augmented or inflated system to continue through singular points. Generalizations of the augmented system (2.8) are quite practical in actually computing solution paths. The obvious generalization in which the element (vector) ψ^* is replaced occasionally is a most useful extension. Of course $\psi^*\in\mathbb{B}^*$ need not be in $\mathcal{R}^\perp(F_u^0)$, when F_u^0 is singular, but it must have a component (or nontrivial projection) in this subspace. Our proof above is hardly changed for this important case.

3. Paths through quadratic folds: complex bifurcation. The geometry of a path through a simple fold can be explored by simply examining the higher derivatives of the path at the fold (as is done in the elementary theory of curves in $\mathbb{R}^2$, say). This analysis of the path (2.6a) proceeds from (2.10) by taking the second derivative and evaluating the result at $s=s_0$ to get:

$$F_u^0\ddot{u}_0+F_{uu}^0\dot{u}_0\dot{u}_0+2F_{u\lambda}^0\dot{u}_0\dot{\lambda}_0+F_{\lambda\lambda}^0\dot{\lambda}_0\dot{\lambda}_0+F_\lambda^0\ddot{\lambda}_0=0. \tag{3.1}$$

Here we use the obvious notation $\ddot{u}_0\equiv\ddot{u}(s_0)$, etc., and the fact that, following from smoothness, $F_{u\lambda}^0\dot{u}_0\dot{\lambda}_0=F_{\lambda u}^0\dot{\lambda}_0\dot{u}_0$. At a simple fold, (2.6c) holds and when we operate with ψ^* using (2.7b) in (3.1) we get

$$\psi^*\cdot F_{uu}^0\dot{u}_0\dot{u}_0+\psi^*\cdot F_\lambda^0\ddot{\lambda}_0=0.$$

Recalling (2.7d) and writing $\dot{u}_0=\alpha\phi$ from (2.6d) it follows that

$$\ddot{\lambda}_0=\ddot{\lambda}(s_0)=-\alpha^2\frac{\psi^*\cdot F_{uu}^0\phi\phi}{\psi^*\cdot F_\lambda^0},\qquad \alpha\in\mathbb{R}. \tag{3.2}$$

If $\ddot{\lambda}_0\neq 0$ then the solution path (in $\mathbb{B}\times\mathbb{R}$) at a simple fold lies locally to one side of λ_0 as follows from the Taylor expansion of $\lambda(s)$ about s_0. This is the basis for Definition 3.3.

DEFINITION 3.3. A point $(u^0,\lambda^0)\in\mathbb{R}\times\mathbb{B}$ is a simple quadratic fold for (2.1) if it is a simple fold and

$$a\equiv\psi^*\cdot F_{uu}^0\phi\phi\neq 0. \tag{3.3}$$

A schematic diagram of the local behavior of a solution path near a simple quadratic fold is shown in Fig. 1.

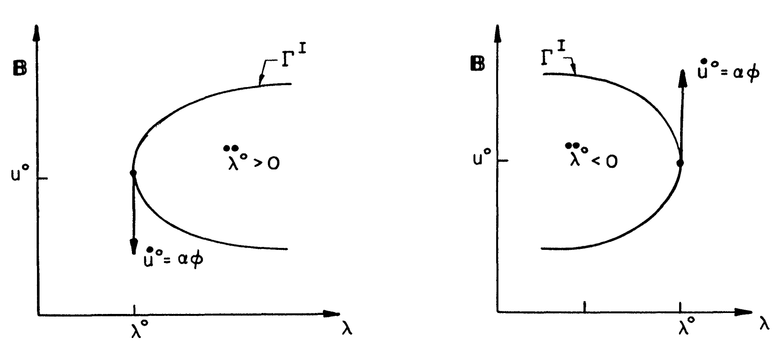

FIG. 1. *Real solution paths through simple quadratic folds.*

We shall now show that there is indeed a path that exists, locally, on the "opposite" side of the simple quadratic fold. But this path, which bifurcates from Γ^I at the fold, lies in the *complex Banach space* $\mathbb{B}\oplus i\mathbb{B}$. We must introduce this space, the complexified problem, and some related terminology to proceed. First the complex Banach space is defined as

$$\mathbb{B}\oplus i\mathbb{B}\equiv\{z=u+iv;\ u\in\mathbb{B},\ v\in\mathbb{B}\}. \tag{3.4}$$

Then the complexified problem obtained from (2.1) is

$$F(z,\lambda)\equiv F(u+iv,\lambda)\equiv f(u,v;\lambda)+ig(u,v;\lambda)=0. \tag{3.5}$$

Here we retain λ as real and $f(\cdot)$ and $g(\cdot)$ are the real and imaginary parts of F. Note that the complexified problem must reduce to the real problem (2.1) when $v=0$, so it follows that:

$$\left.\begin{aligned} &f(u,0;\lambda)\equiv F(u,\lambda) &&(3.6\text{a})\\ &g(u,0;\lambda)=0 &&(3.6\text{b})\end{aligned}\right\}\forall u\in\mathbb{B},\ \lambda\in\mathbb{R}.$$

Furthermore if, as we assume, $F(z,\lambda)$ has a Fréchet derivative with respect to z, call it $F_z(z,\lambda)$, then just as in the classical theory of analytic functions, the Cauchy-Riemann equations must hold:

$$\begin{aligned} f_u(u,v;\lambda)&=g_v(u,v;\lambda),\\ f_v(u,v;\lambda)&=-g_u(u,v;\lambda).\end{aligned} \tag{3.7}$$

We say that $F(z,\lambda)$ is analytic (in z) when F_z exists.

To understand the behavior of solution paths of (3.5) going through the fold point in the complex space, we proceed using perturbation theory. Later we shall state a precise result which contains all of the assumptions we use in this construction. We seek smooth solutions of (3.5) in the form

$$\Gamma(s):\{u(s),v(s),\lambda(s)\}. \tag{3.8}$$

This yields the real problem on $\mathbb{B}\times\mathbb{B}\times\mathbb{R}\to\mathbb{B}\times\mathbb{B}$:

$$f(u(s),v(s);\lambda(s))=0, \tag{3.9a}$$

$$g(u(s),v(s);\lambda(s))=0. \tag{3.9b}$$

At $s=s_0$ we require the path (3.8) to be at the fold or:

$$u(s_0)=u^0,\quad v(s_0)=0,\quad \lambda(s_0)=\lambda^0. \tag{3.10}$$

Then with (3.8) in (3.9) we get, on differentiating once and setting $s=s_0$:

$$\begin{aligned} f_u^0\dot{u}_0+f_v^0\dot{v}_0+f_\lambda^0\dot{\lambda}_0&=0,\\ g_u^0\dot{u}_0+g_v^0\dot{v}_0+g_\lambda^0\dot{\lambda}&=0.\end{aligned}$$

Using (3.6), which implies

$$g_u(u,0;\lambda)=0,\quad g_\lambda(u,0;\lambda)=0, \tag{3.11}$$

and (3.7), the above system becomes:

$$\begin{pmatrix} f_u^0 & 0\\ 0 & f_u^0\end{pmatrix}\begin{pmatrix}\dot{u}_0\\ \dot{v}_0\end{pmatrix}+\begin{pmatrix} f_\lambda^0\\ 0\end{pmatrix}\dot{\lambda}_0=0. \tag{3.12}$$

All solutions of this system, by virtue of (2.4), (2.5), and (3.6a), which imply that $f_\lambda^0 \notin \mathcal{R}(f_u^0)$, are of the form:

$$\dot\lambda_0 = 0, \quad \dot u_0 = \alpha\phi, \quad \dot v_0 = \beta\phi; \qquad \alpha, \beta \in \mathbb{R}. \tag{3.13}$$

The real parameters α and β are arbitrary at this stage and are determined by going to higher order.

We differentiate (3.9) twice, set $s = s_0$, multiply by ψ^* and use from (3.6) and (3.7) the relations

$$f_v^0 = -g_u^0 = 0, \quad f_{vv} = -f_{uu}, \quad f_{uv}^0 = -g_{uu}^0 = 0, \quad g_{uv} = f_{uu}$$

to get, recalling (3.13) and (2.7b):

$$\begin{aligned} &[\alpha^2 - \beta^2]\psi^* \cdot f_{uu}^0 \phi\phi + \psi^* \cdot f_\lambda^0 \ddot\lambda_0 = 0, \\ &\alpha\beta\psi^* \cdot f_{uu}^0 \phi\phi = 0. \end{aligned} \tag{3.14}$$

There are two real solutions of (3.14), since the fold is quadratic, specifically recalling (2.7d) and (3.3):

$$\begin{cases} \beta = 0 \\ \ddot\lambda_0 = -\alpha^2 \dfrac{a}{d}; \end{cases} \tag{3.15a}$$

$$\begin{cases} \alpha = 0 \\ \ddot\lambda_0 = \beta^2 \dfrac{a}{d}. \end{cases} \tag{3.15b}$$

Here α, $\beta \in \mathbb{R}$ are each still arbitrary when nonzero (accounting for the arbitrariness in the scale of s). Using (3.10), (3.13), and (3.15) our solution paths, Γ^I and Γ^{II} say, are now determined up to second order as:

$$\Gamma^I: \begin{cases} z^I(s) = u^0 + \alpha[s - s_0]\phi + \mathcal{O}([s - s_0]^2) \\ \lambda^I(s) = \lambda^0 + 0 - \alpha^2[s - s_0]^2 \dfrac{a}{d} + \mathcal{O}([s - s_0]^3) \end{cases} \tag{3.16a}$$

$$\Gamma^{II}: \begin{cases} z^{II}(s) = u^0 + i\beta[s - s_0]\phi + \mathcal{O}([s - s_0]^2) \\ \lambda^{II}(s) = \lambda^0 + 0 + \beta^2[s - s_0]^2 \dfrac{a}{d} + \mathcal{O}([s - s_0]^3). \end{cases} \tag{3.16b}$$

We see here that α on Γ^I and β on Γ^{II} are simply scale factors for the parametrization, $s - s_0$, on the paths.

Note that Γ^I is our previously discussed real path through the fold. But Γ^{II} bifurcates from Γ^I at the fold and goes off in the complex space to first order in $[s - s_0]$. Each path lies on opposite sides of the fold point, with respect to the λ-direction. We sketch this behavior in Fig. 2.

It is also of interest to observe that the tangent to Γ^{II} in the imaginary space, $i\mathbb{B}$, has the direction $i\phi$ of the "rotated" real tangent to Γ^I.

To summarize the results we have shown above, we state the following Theorem.

THEOREM 3.17. Complex Bifurcation Theorem. *Let* $(u^0, \lambda^0) \in \mathbb{B} \times \mathbb{R}$ *be a simple quadratic fold solution of* (2.1). *Let* $F(z, \lambda)$ *be analytic. Then the complexified problem* (3.5) *has a* (*complex*) *bifurcation at* (u^0, λ^0). *One path through* (u^0, λ^0) *is the real path* Γ^I *of Theorem* 2.6 *and* (3.16a). *The other complex path* Γ^{II} *is as in* (3.16b).

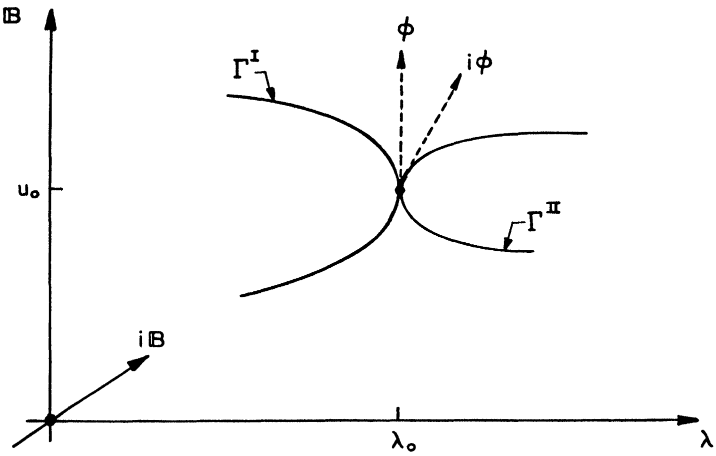

FIG. 2. *Complex bifurcation at a simple quadratic fold. The tangent to* Γ^I *is in the* ϕ*-direction, to* Γ^{II} *it is in the* $i\phi$*-direction.*

Proof. We write (3.5) or (3.9) as

$$G(U, \lambda) \equiv \begin{pmatrix} f(u, v; \lambda) \\ g(u, v; \lambda) \end{pmatrix} = 0, \quad U \equiv \begin{pmatrix} u \\ v \end{pmatrix} \in \mathbb{B} \times \mathbb{B}, \lambda \in \mathbb{R}. \tag{3.18a}$$

From (3.6) and (3.7) we get that

$$G_u^0 \equiv G_u\left(\begin{pmatrix} u^0 \\ 0 \end{pmatrix}, \lambda^0\right) = \begin{pmatrix} F_u^0 & 0 \\ 0 & F_u^0 \end{pmatrix}. \tag{3.18b}$$

Now we can easily apply the standard Lyapunov-Schmidt bifurcation theory to $G(U, \lambda)$ on the real Banach space $\mathbb{B} \times \mathbb{B}$ to get our results. The null space of G_u^0 is two-dimensional and the system (3.14) are the so-called bifurcation equations. The details can be found in [7]. Even simpler, one can apply bifurcation theory in the complex space $\mathbb{B} \oplus i\mathbb{B}$ to obtain the same results. In this formulation the null space is one-dimensional (complex). This is also carried out in [7].

As a trivial example of complex bifurcation from a quadratic fold, we consider the scalar problem:

$$F(u, \lambda) \equiv u^2 - \lambda = 0, \qquad F: \mathbb{R} \times \mathbb{R} \to \mathbb{R}. \tag{3.19}$$

The singular point $(u^0, \lambda^0) = (0, 0)$, the vertex of a parabola, is clearly a real quadratic fold point. We complexify by allowing u to be complex. Then we obtain the pair of solution paths, valid for all real s:

$$\begin{aligned} \Gamma^I&: u = s, \qquad \lambda = s^2; \\ \Gamma^{II}&: u = is, \qquad \lambda = -s^2. \end{aligned} \tag{3.20}$$

These paths bifurcate from each other at (0, 0). A trivial computation of G_u^0, G_λ^0, G_{uu}^0, ϕ and ψ^* reveals that the quadratic fold conditions are satisfied and hence Theorem 3.17 applies.

4. Further complex bifurcations. There are other types of simple singular points, (u^0, λ^0), of (2.1) at which the complexified problem has a bifurcation. These include isola centers [12] and pitchfork (or supercritical or subcritical) bifurcations. Neither of these singularities are folds, as at them:

$$F_\lambda^0 \in \mathcal{R}(F_u^0) \quad \text{or} \quad d \equiv \psi^* \cdot F_\lambda^0 = 0. \tag{4.1}$$

Using the representation (2.6) for a smooth path through (u^0, λ^0) we get (2.11) for the tangent $(\dot{u}_0, \dot{\lambda}_0)$. But by (4.1) we can introduce an element ϕ_0 satisfying

$$F_u^0\phi_0 + F_\lambda^0 = 0, \qquad \|\phi_0\| = 1. \tag{4.2}$$

Then by (2.4) the general solution of (2.11) can be written as:

$$\dot{u}_0 = \alpha_0\phi_0 + \alpha\phi, \qquad \dot{\lambda}_0 = \alpha_0, \qquad \alpha_0, \alpha \in \mathbb{R}. \tag{4.3}$$

To find α_0 and α we must go to the next order; that is, we must satisfy (3.1) using (4.3). We have, upon operating with ψ^*,

$$a\alpha^2 + 2b\alpha\alpha_0 + c\alpha_0^2 = 0, \tag{4.4}$$

where

$$\begin{aligned} a &\equiv \psi^* F_{uu}^0\phi\phi, \\ b &\equiv \psi^*[F_{uu}^0\phi\phi_0 + F_{u\lambda}^0\phi], \\ c &\equiv \psi^*[F_{uu}^0\phi_0\phi_0 + 2F_{u\lambda}^0\phi_0 + F_{\lambda\lambda}^0]. \end{aligned} \tag{4.5}$$

A simple transcritical bifurcation occurs at (u^0, λ^0) if

$$\Delta \equiv b^2 - ac > 0.$$

Then (4.4) has two real distinct roots and they determine the two tangents to the bifurcating paths. This is classical bifurcation theory. However if

$$\Delta \equiv b^2 - ac < 0, \tag{4.6}$$

the point (u^0, λ^0) is an isola center [12], [3]. Through such a point we now find that *the complexified problem has a pair of complex solution paths bifurcating from the real point* (u^0, λ^0).

Solving (4.4) for α/α_0 we get on giving up $\alpha \in \mathbb{R}$, two complex solutions and hence two complex paths bifurcating from the isola center at (u^0, λ^0). These paths have the expansions in $(s - s_0)$:

$$\Gamma^\pm: \begin{cases} u(s) = u^0 - \alpha_0[s - s_0]\left\{\left[\dfrac{b}{a} \pm i\dfrac{|\Delta|^{1/2}}{a}\right]\phi + \phi_0\right\} + \mathcal{O}([s - s_0]^2), \\ \lambda(s) = \lambda^0 + \alpha_0[s - s_0] + \mathcal{O}([s - s_0]^2). \end{cases} \tag{4.7}$$

Here α_0 is an arbitrary real nonzero scalar which serves as a scale factor for the parametrization by s. Note that this complex bifurcation is analogous to transcritical bifurcation in the real case as both branches exist over a λ-interval with λ^0 as an interior point.

A trivial example of complex bifurcation at an isola center is furnished by

$$F(u, \lambda) \equiv u^2 + \lambda^2 = 0, \qquad F: \mathbb{R} \times \mathbb{R} \to \mathbb{R}.$$

The only real solution is at $(u, \lambda) = (0, 0)$. But on allowing u to be complex while λ remains real, a pair of complex paths exist and bifurcate from each other at $(0, 0)$. These are for all $s \in \mathbb{R}$:

$$\Gamma^I: u = is, \quad \lambda = s;$$

$$\Gamma^{II}: u = -is, \quad \lambda = s.$$

We point out that the "isolas" in the above case come from considering $F(u, \lambda) = \tau$ for $\tau > 0$. The center occurs only for $\tau = 0$. For each $\tau > 0$ the isola (a circle of radius $\sqrt{\tau}$ in the (u, λ)-plane) has two simple quadratic folds from which a pair of *complex* paths bifurcate as in § 3.

The bifurcation from isola centers can only occur when $ac > 0$. When $a = 0$, the real problem may exhibit "pitchfork" bifurcation. In this case we get from (4.4) that

$$\alpha = -\frac{c}{2b}\alpha_0, \quad \text{arbitrary real if } b \neq 0; \tag{4.8a}$$

or else

$$\alpha_0 = 0, \quad \text{no condition on } \alpha. \tag{4.8b}$$

To determine α in the latter case we must again go to a higher order in our expansion. Thus we differentiate (2.10) three times, set $s = s_0$, and operate with ψ^* to get, using $a = 0$, $\alpha_0 = 0$, and (4.3):

$$\psi^*[3F^0_{uu}\phi\ddot{u}_0 + 3F^0_{u\lambda}\phi\ddot{\lambda}_0 + F^0_{uuu}\phi\phi\phi\alpha^2]\alpha = 0. \tag{4.9}$$

Using these same quantities in (3.1) yields

$$F^0_u\ddot{u}_0 + F^0_{uu}\phi\phi\alpha^2 + F^0_\lambda\ddot{\lambda}_0 = 0. \tag{4.10}$$

To solve (4.10) for $\ddot{u}_0$, which exists since $a = d = 0$, we introduce ϕ_1, as the solution of:

$$F^0_u\phi_1 + F^0_{uu}\phi\phi = 0, \qquad \|\phi_1\| = 1. \tag{4.11}$$

Then we have, on recalling (4.2), the general solution of (4.10) as:

$$\ddot{u}_0 = \beta_0\phi_0 + \beta_1\phi_1 + \beta\phi; \tag{4.12a}$$

$$\beta_0 = \ddot{\lambda}_0, \quad \beta_1 = \alpha^2, \quad \beta = \text{arb}. \tag{4.12b}$$

Using (4.12) in (4.9) yields with b as in (4.5) and

$$B \equiv 3\psi^* F^0_{uu}\phi\phi_1 + \psi^* F^0_{uuu}\phi\phi\phi, \tag{4.13}$$

the relation:

$$3b\beta_0 + B\alpha^2 = 0. \tag{4.14}$$

We thus see by again giving up $\alpha \in \mathbb{R}$ in (4.3) that α can be real or pure imaginary, depending upon the sign of $\beta_0 b/B$. In more detail the solution branches through the pitchfork bifurcation point (u^0, λ^0) at which $d = a = 0$ are obtained by using (4.3), (4.8), (4.12), and (4.14) in expansions of $(u(s), \lambda(s))$ about $s = s_0$. We can take α to be the arbitrary real parameter which scales s in case (4.8b) if we write (4.14) in the equivalent forms:

$$\beta_0 = -\frac{B}{3b}\alpha^2 \quad \text{and} \quad \beta_0 = \frac{B}{3b}\alpha^2, \qquad \beta_0 \in \mathbb{R} \quad \text{for} \quad bB \neq 0.$$

The latter case corresponds to the pure imaginary root. Then our expansions become, on replacing β in (4.12b) by $\alpha^2\beta$:

$$\Gamma^I: \begin{cases} u(s) = u^0 - \alpha_0[s - s_0]\left\{\phi_0 + \dfrac{c}{2b}\phi\right\} + \mathcal{O}([s - s_0]^2) \\ \lambda(s) = \lambda^0 + \alpha_0[s - s_0] + \mathcal{O}([s - s_0]^2) \end{cases} \tag{4.15a}$$

$$\Gamma^{II}: \begin{cases} u(s) = u^0 + \alpha[s - s_0]\phi - \dfrac{1}{2}\alpha^2[s - s_0]^2\left\{\dfrac{B}{3b}\phi_0 - \phi_1 + \beta\phi\right\} + \mathcal{O}([s - s_0]^3) \\ \lambda(s) = \lambda^0 + 0 - \dfrac{B}{6b}\alpha^2[s - s_0]^2 + \mathcal{O}([s - s_0]^3) \end{cases} \tag{4.15b}$$

$$\text{(4.15c)}\quad \Gamma^{III}:\begin{cases} u(s)=u^0+i\alpha[s-s_0]\phi+\frac{1}{2}\alpha^2[s-s_0]^2\left\{\frac{B}{3b}\phi_0-\phi_1+\beta\phi\right\}+\mathcal{O}([s-s_0]^3)\\ \lambda(s)=\lambda^0_{+0}+\frac{B}{6b}\alpha^2[s-s_0]^2+\mathcal{O}([s-s_0]^3).\end{cases}$$

Here Γ^I is the basic branch, existing on a λ-interval about λ_0. Branch Γ^{II} is the real branch bifurcating from Γ^I and it is a supercritical or subcritical bifurcation depending upon the sign of B/b. Thus near the bifurcation point Γ^{II} exists only on one side of λ^0. Branch Γ^{III} is the complex branch bifurcating from Γ^I on the opposite side of λ^0 from Γ^{II}. Note that its tangent, as in the simple quadratic fold case, is a rotation into the pure imaginary space, $i\mathbb{B}$, of the tangent to the real bifurcating branch. For each λ near λ^0 there are thus three solutions: three real, or one real and two complex. Thus it would be reasonable to call such a bifurcation point a *cubic bifurcation.* The isola center and transcritical bifurcation points could be called *quadratic bifurcations* since two solutions exist for each $\lambda\neq\lambda^0$, sufficiently close to λ^0. Indeed our results indicate that in treating complexified analytic problems we could dispense with the notion of folds, isolas and super-(or sub-) critical bifurcations and need only classify quadratic and cubic bifurcations according to $a\neq 0$ or $a=0$ and $bB\neq 0$.

A trivial example of complex bifurcation at a pitchfork bifurcation is furnished by

$$\text{(4.16)}\qquad F(u,\lambda)\equiv u(u^2-\lambda)=0.$$

There are two real paths Γ^I and Γ^{II} going through the bifurcation point $(u,\lambda)=(0,0)$ and one complex path Γ^{III}. These are given by, for all $s\in\mathbb{R}$:

$$\text{(4.17)}\qquad \begin{aligned}&\Gamma^I: u=0,\quad \lambda=s;\\ &\Gamma^{II}: u=s,\quad \lambda=s^2;\\ &\Gamma^{III}: u=is,\quad \lambda=-s^2.\end{aligned}$$

These solution branches are the exact analogs of those in (4.15).

5. Some properties of complexified solution paths. Since we have complexified a real operator into a complex analytic operator we have the Conjugate Solution Theorem.

THEOREM 5.1. The Conjugate Solution Theorem. *The complex solutions of the complexified problem* (3.5) *must occur in complex conjugate pairs.*

This result follows from the extension of the Schwarz Reflection Principle of analytic function theory [1] to the present case of analytic operators on the complex Banach space $\mathbb{B}\oplus i\mathbb{B}$. We have not been able to find this extension stated in the literature so we present the version we use as follows.

PRINCIPLE 5.2. The Reflection Principle. *Let* $F(z;\lambda)$ *of* (3.4) *and* (3.5) *be the complexification of* (2.1). *Define the complex conjugates*:

$$\bar{z}\equiv u-iv,\qquad \bar{F}(z,\lambda)\equiv f(u,v;\lambda)-ig(u,v;\lambda).$$

Then

$$\text{(5.2)}\qquad \bar{F}(\bar{z},\lambda)=F(z,\lambda)\quad \forall z\in\mathbb{B}\oplus i\mathbb{B},\quad \lambda\in\mathbb{R},$$

provided $F(z,\lambda)$ *is analytic in* z.

Proof. On the real space $\mathbb{B}$ (i.e., real "axis"), where $v\equiv 0$, we have from (3.6) that

$$F(z,\lambda)=f(u,0;\lambda)=\bar{F}(\bar{z},\lambda).$$

Now we expand the difference

$$D(z,\lambda)\equiv F(z,\lambda)-\bar{F}(\bar{z},\lambda)$$

about any point $z_0 \in \mathbb{B}$ and find that it vanishes in some neighborhood of $z_0 \in \mathbb{B} \oplus i\mathbb{B}$. Thus it vanishes off the axis. Since F_z exists on all of $\mathbb{B} \oplus i\mathbb{B}$, this argument can be continued to show that $D(z, \lambda) \equiv 0$ (see [8, p. 88]).

To prove Theorem 5.1 we let $z_0 = u_0 + iv_0$ satisfy $F(z_0, \lambda_0) = 0$. By the principle 5.2

$$F(z_0, \lambda_0) = \bar{F}(\bar{z}_0, \lambda_0) = f(u_0, -v_0; \lambda_0) - ig(u_0, -v_0; \lambda_0) = 0.$$

But then z_0 also satisfies

$$F(\bar{z}_0, \lambda_0) = f(u_0, -v_0; \lambda_0) + ig(u_0, -v_0; \lambda_0) = 0.$$

Thus our assertion regarding conjugate solutions is established.

As a consequence of the Reflection Principle it follows that most isolated real points on complex solution paths of (3.5) are bifurcation points. More precisely we have the following theorem.

THEOREM 5.3. The Complex Bifurcation Theorem. *Let the smooth complex solution path* Γ: $\{z = z(s) \equiv u(s) + iv(s),\ \lambda = \lambda(s)\}$ *of* (3.5) *have an isolated real point* $z_0 = z(s_0) = u_0 + i0$, $\lambda_0 = \lambda(s_0)$ *at which* $\dot{u}(s_0) \neq 0$. *Then* z_0 *is a bifurcation point on* Γ *and* $\bar{\Gamma}$: $\{z = \bar{z}(s),\ \lambda = \lambda(s)\}$ *is a distinct complex solution path bifurcating from* Γ *at* (z_0, λ_0).

Proof. It is clear from the Reflection Principle that $\bar{\Gamma}$ is a solution path for (3.5) and that it contains the real point z_0. We need only show that the path $\bar{\Gamma}$ is distinct from Γ. However since

$$u(s) = u(s_0) + (s - s_0)\dot{u}(s_0) + O([s - s_0]^2), \qquad \dot{u}(s_0) \neq 0$$

we are assured that

$$u(s_0 + \varepsilon_+) + iv(s_0 + \varepsilon_+) \neq u(s_0 - \varepsilon_-) + iv(s_0 - \varepsilon_-)$$

for $\varepsilon_+ > 0$, $\varepsilon_- > 0$ and $\varepsilon_+ \leqq \varepsilon$, $\varepsilon_- \leqq \varepsilon$ for some ε sufficiently small. Also $v(s) \neq 0$ in a deleted interval about s_0. Thus Γ and $\bar{\Gamma}$ have distinct complex arcs for $|s - s_0| < \varepsilon$, $s \neq s_0$; and hence they are distinct paths.

We point out that the previous complex bifurcations observed at real quadratic folds and at real pitchfork bifurcations are not of the type covered in Theorem 5.3. The folds or bifurcation points in these cases are not isolated real points since they lie on real paths. The complex bifurcations from isola centers may, however, be of the form covered by the theorem. The condition $\dot{u}(s_0) \neq 0$ is not necessary but is used to give a simple proof and because it occurs most commonly. All that is required, however, is that for all sufficiently small subarcs, Γ_ε, of Γ centered about (z_0, λ_0), the point set $\{\Gamma_\varepsilon\}$ is not symmetric with respect to reflection in the subspace $\mathbb{B} \times \mathbb{R}$ of $(\mathbb{B} \oplus i\mathbb{B}) \times \mathbb{R}$.

6. Complex connections of perturbed bifurcations. Many real bifurcation problems depend upon two (or more) real parameters, say:

$$F(u, \lambda, \tau) = 0, \qquad F: \mathbb{B} \times \mathbb{R}^2 \to \mathbb{B}.$$

For some special value of one of the parameters, say $\tau = \tau^0$, the resulting one parameter problem can contain a pair of paths bifurcating, at a simple bifurcation point $(u, \lambda) = (u^0, \lambda^0)$. As τ is varied from τ^0 this bifurcation generally breaks and yields two paths which no longer meet at (u^0, λ^0) but rather have one or two simple folds nearby. This is called perturbed or imperfect bifurcation. As we shall show below, the perturbed disjoint real paths are frequently *connected* by a complex path. This generally occurs for perturbed transcritical bifurcation where two folds are formed but not for perturbed pitchfork bifurcations. In the latter case only one fold is formed, but as we have seen in § 4 a complex path already exists and it is preserved under perturbations.

We point out that many computational problems or discretizations of continuous problems that originally depend on only one parameter take on the two parameter form where τ is a measure of the perturbations introduced by rounding errors, truncation errors, or other inaccuracies in the computations or approximations. When such effects perturb a bifurcation it can cause the loss of entire solution paths. In such cases computing the complex path can circumvent some of the difficulties and retrieve disconnected branches.

To show how perturbed bifurcations yield complex paths we employ simple algebraic models, similar to those used in §§ 3 and 4. The equivalence between the models and the full bifurcation problems can be made rigorous using normal form theory (see [5]). We shall state two such results but shall not prove them here. First we present the following theorem.

THEOREM 6.1. The Perturbed Transcritical Bifurcation Theorem. *Let* (u^0, λ^0, τ^0) *be a simple transcritical bifurcation solution of* $F(u, \lambda, \tau^0) = 0$. *That is*:

(6.1a) $F(u^0, \lambda^0, \tau^0) = 0$;

(6.1b) F_u^0 *is singular with*: $\dim \mathcal{N}(F_u^0) = \operatorname{codim} \mathcal{R}(F_u^0) = 1$;

(6.1c) F_u^0 *is Fredholm of index zero*;

(6.1d) $F_\lambda^0 \in \mathcal{R}(F_u^0)$;

(6.1e) $F_\tau^0 \notin \mathcal{R}(F_u^0)$;

(6.1f) $a \neq 0$;

(6.1g) $\Delta \equiv b^2 - ac > 0$.

Here a, b, and c are defined as in (4.5). *Then the solution paths of* $F = 0$ *near* (u^0, λ^0, τ^0) *are isomorphic to the solution paths of*

$$\xi(\xi - 2\lambda) + \tau = 0 \tag{6.2}$$

near $(\xi, \lambda, \tau) = (0, 0, 0)$.

The proof of this theorem for the complexification of $F(u, \lambda, \tau) = 0$ does not differ substantially from the proof for the real mapping, and the model system is the same. Thus considering the complexified problem, we find that there are two solutions of (6.2) for each value of λ and τ; they are

$$\begin{aligned} \xi_I(\lambda, \tau) &= \lambda + \sqrt{\lambda^2 - \tau} \\ \xi_{II}(\lambda, \tau) &= \lambda - \sqrt{\lambda^2 - \tau}. \end{aligned} \tag{6.3}$$

When $\tau < 0$ these form two real, smooth branches parametrized by λ (see Fig. 3(a)). When $\tau > 0$ there is an interval $-\sqrt{\tau} < \lambda < \sqrt{\tau}$ in which they are both complex. In this

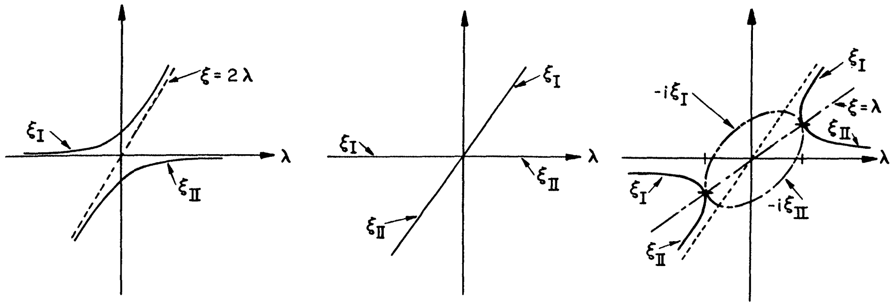

FIG. 3. *Perturbed transcritical bifurcation.* (*a*) *Split bifurcation with no folds,* $t < 0$. (*b*) *The unperturbed case,* $t = 0$. (*c*) *Split bifurcation with two folds joined by a complex isola* (*shown as* (.__.)), *here* $t > 0$.

interval a complex isola exists which connects two disjoint real branches. We sketch the behavior of the complex solution paths near a perturbed simple quadratic (transcritical) bifurcation in Fig. 3(c).

A similar result applies to the perturbations of pitchfork bifurcations. We have the following theorem.

THEOREM 6.4. The Perturbed Pitchfork Bifurcation Theorem. *Let* (u^0, λ^0, τ^0) *be a simple pitchfork bifurcation solution of* $F(u, \lambda, \tau^0) = 0$. *That is* (6.1a)-(6.1e) *hold and in addition*:

(6.4f) $$a = 0;$$

(6.4g) $$B \equiv \psi^* F^0_{uuu}\phi\phi\phi + 3\psi^* F^0_{uu}\phi\phi_1 \neq 0;$$

(6.4h) $$b \neq 0.$$

Here a and b are defined in (4.5) *and* ϕ_1 *satisfies* (4.11). *Then the solution paths of* $F = 0$ *near* (u^0, λ^0, τ^0) *are isomorphic to the solution paths of*

(6.5) $$\xi(\xi^2 - 3\lambda) + 2\tau = 0$$

near $(\xi, \lambda, \tau) = (0, 0, 0)$.

This theorem also goes over without change for the complexified problem. Thus there are three solutions of (6.5) for each value of λ and τ; they are:

(6.6) $$\begin{aligned} \xi_I(\lambda, \tau) &= A + B, \\ \xi_{II}(\lambda, \tau) &= -\frac{1}{2}(A+B) + i\frac{\sqrt{3}}{2}(A-B), \\ \xi_{III}(\lambda, \tau) &= -\frac{1}{2}(A+B) - i\frac{\sqrt{3}}{2}(A-B), \end{aligned}$$

where:

$$A = \sqrt[3]{-\tau + \sqrt{\tau^2 - \lambda^3}}, \qquad B = \sqrt[3]{-\tau - \sqrt{\tau^2 - \lambda^3}}.$$

These solutions form three smooth paths. The way in which the roots in (6.6) join to form the smooth paths is shown in Fig. 4. When $\lambda^3 < \tau^2$, one of these paths is composed

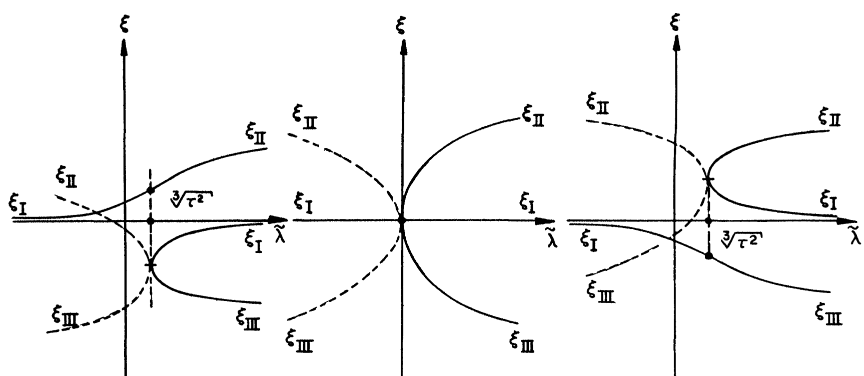

FIG. 4. *Perturbed pitchfork bifurcation.* (*a*) *and* (*c*) *The perturbed pitchfork with a complex branch bifurcating from each fold.* (*b*) *The unperturbed case,* $t = 0$, *with a complex branch bifurcating at the bifurcation point.*

of a complex conjugate pair of the roots and the other path is real. When $\lambda^3 \geqq \tau^2$, all three roots and both paths are real. The pitchfork differs from the transcritical bifurcation in that complex paths now exist for all values of the perturbation parameter. A more complete picture of the solution "surface" is given in Fig. 6, the Cusp Catastrophe.

These models of perturbed bifurcation are algebraic equations for ξ, parameterized by the two real parameters λ and τ. Models for perturbed higher order bifurcations are also algebraic, and models for multiple bifurcations are systems of algebraic equations. Most such systems of algebraic equations have the property that the *number* of roots (counting multiplicities and roots at infinity) does not change with the parameters. When this holds for the model or for the so-called algebraic bifurcation equations, it accounts for the complex connections in perturbed bifurcations. That is, in the ball about the singular solution in which the mapping to the normal form is valid, the number of complex solutions is independent of both parameters. Thus when a bifurcation breaks, creating an interval in which there are no real solutions, a complex connection must exist in the interval between the disjoint real branches. The case when the algebraic bifurcation equations have a continuum of solutions (so-called common intersection components) is not completely understood; this occurs frequently at multiple bifurcations (i.e., when the nullspace of F_u^0 has higher dimension).

7. Numerical methods for computing complex solution paths. On computers that have complex arithmetic the complex solution paths may be computed using almost the same algorithms and computer codes used for real solution paths. Below we briefly describe the pseudo-arclength continuation algorithm of Keller [11] and describe how it may be modified to compute complex solution paths. A solution path $\Gamma(s) = (u(s), \lambda(s))$, parameterized by pseudo-arclength s, satisfies

$$\begin{aligned} F(u(s), \lambda(s)) &= 0, \\ N(u(s), \lambda(s)) &= 0, \end{aligned} \tag{7.1}$$

where N is the pseudo-arclength constraint:

$$N(u, \lambda) \equiv \dot{u}^*(s_0)[u - u(s_0)] + \dot{\lambda}(s_0)[\lambda - \lambda(s_0)] - (s - s_0).$$

Thus points on $\Gamma(s)$ lie at the intersection of solutions of $F = 0$ with a hyperplane perpendicular to the tangent to $\Gamma(s)$ at s_0 and a distance $(s - s_0)$ from $\Gamma(s_0)$.

From an initial solution $\Gamma(0)$ the entire path $\Gamma(s)$ is computed. The solution at $\Gamma(s+\delta)$ is approximated using an Euler predictor,

$$\Gamma(s+\delta) \sim \Gamma(s) + \delta \cdot (\dot{u}(s), \dot{\lambda}(s)), \tag{7.2}$$

where the tangent vector $(\dot{u}(s), \dot{\lambda}(s))$ satisfies:

$$\begin{aligned} F_u\dot{u} + F_\lambda \dot{\lambda} &= 0, \\ \|\dot{u}\|^2 + |\dot{\lambda}|^2 &= 1. \end{aligned} \tag{7.3}$$

This approximation is corrected using Newton's method, which requires that the linear system

$$\begin{pmatrix} F_u & F_\lambda \\ \dot{u}^* & \dot{\lambda} \end{pmatrix} \begin{pmatrix} \Delta u \\ \Delta \lambda \end{pmatrix} = -\begin{pmatrix} F \\ N \end{pmatrix} \tag{7.4}$$

be solved for the correction $(\Delta u, \Delta\lambda)$. If F is a finite difference approximation to some continuous problem, the structure of F_u (e.g., block-tridiagonal) can be exploited by using the bordering algorithm [11]. If F_u is nonsingular the bordering algorithm computes t_1 and t_2 by solving

$$\begin{aligned} F_u t_1 &= -F \\ F_u t_2 &= -F_\lambda, \end{aligned} \tag{7.5}$$

and then the Newton correction is formed as:

$$\begin{aligned} \Delta u &= t_1 + \Delta\lambda t_2, \\ \Delta\lambda &= -(\dot{u}^* t_1 + N)/(\dot{u}^* t_2 + \dot{\lambda}). \end{aligned} \tag{7.6}$$

This requires one LU decomposition of F_u and two back substitutions per correction.

From the LU decomposition the sign of the determinant, $\det F_u$, may be easily found. Simple singular points on $\Gamma(s)$ can be detected by monitoring this sign. If it changes between $s = s_0$ and $s = s_1$, a simple singular point must exist in that interval, and a simple bisection can be used to locate the singular point to the desired accuracy. The tangents at the singular point may then be estimated using a local expansion, and the whole process is then repeated for each of the bifurcating branches.

One way of adapting this algorithm to solve the complexified problem is to rewrite it as the real system

$$H(u, v; \lambda) \equiv \begin{pmatrix} f(u, v; \lambda) \\ g(u, v; \lambda) \end{pmatrix} = 0. \tag{7.7}$$

If f_u is banded, the Fréchet derivative which must be factored is of the form

$$\begin{pmatrix} f_u & f_v \\ g_u & g_v \end{pmatrix} = \begin{pmatrix} \ddots & \ddots \\ \ddots & \ddots \end{pmatrix}.$$

Compared to the original real problem, the bandwidth has been increased to $m + w$, and the number of equations has doubled (here w is the bandwidth, and m is the number of equations in the real problem). In order to decrease the bandwidth, the real and imaginary parts of each component of u should be grouped together. This reduces the bandwidth to $2w$.

At a singular point this formulation can be cumbersome. The Fréchet derivative $H_{(u,v)}$ will always have nullspaces with even dimension, since if (ϕ_R, ϕ_I) is a null vector, $(\phi_I, -\phi_R)$ will also be a null vector. This means that when solutions are computed using (7.7), all singular points will be multiple bifurcation points. These multiple bifurcations may be classified using the conditions given in the previous sections. The conditions for the quadratic fold are fairly simple, but those for the pitchfork bifurcation appear quite involved when written in terms of the derivatives of f and g. In addition, the determinant of $H_{(u,v)}$ will not change sign at a singular point since each eigenvalue is double. This makes the detection of singular points difficult.

Another, better way of computing complex solutions is to use complex arithmetic. However, some changes are necessary to use the real algorithm. First we note that the pseudo-arclength constraint for the complexified problem is a *real* scalar constraint, say:

$$N(z, \lambda) \equiv \operatorname{Re}\{\dot{z}^*(s_0)[z - z(s_0)] + \dot{\lambda}(s_0)[\lambda - \lambda(s_0)] - (s - s_0)\} = 0. \tag{7.8}$$

This occurs since we retain the parameter, λ, as real and only complexify $u \to u + iv \equiv z$. Thus the linear systems that must be solved for Newton's method are of the form

$$\begin{aligned} F_z \Delta z + F_\lambda \Delta\lambda &= -G \\ \operatorname{Re}(\dot{z}^* \Delta z + \dot{\lambda}\Delta\lambda) &= -N. \end{aligned} \tag{7.9}$$

If F_z is nonsingular, the bordering algorithm now computes complex t_1 and t_2 by solving:

$$F_z t_1 = -F, \qquad F_z t_2 = -F_\lambda. \tag{7.10}$$

The Newton correction is now formed as:

$$\Delta z = t_1 + \Delta\lambda t_2, \qquad \Delta\lambda = -\mathrm{Re}\,(\dot{z}^* t_1 + N)/\mathrm{Re}\,(\dot{z}^* t_2 + \dot{\lambda}). \tag{7.11}$$

Only in computing $\Delta\lambda$ need we alter the real bordering algorithm to use it for complex continuation (apart from using complex arithmetic and norms).

Singular points may be detected using the complex determinant, which passes through zero in the complex plane as a singular point is passed. Since the solution paths are smooth the argument of the (complex) determinant must jump by 180° as the singular point is passed. Thus for sufficiently small δ the quantity

$$\begin{aligned}&\mathrm{Re}\,(\det(s))\,\mathrm{Re}\,(\det(s+\delta)) + \mathrm{Im}\,(\det(s))\,\mathrm{Im}\,(\det(s+\delta))\\ &\quad = \mathrm{Re}\,[\det(s)\det(s+\delta)]\end{aligned} \tag{7.12}$$

will change sign as a simple singular point is passed (i.e., provided $\Gamma(s)$ and $\Gamma(s+\delta)$ are on opposite sides of the singular point). Here $\det(s)$ is the determinant of the Jacobian at the point $(z(s), \lambda(s))$.

8. Examples. In this section we present several examples of the bifurcation of complex solution paths from real solution paths. The first set of examples are the complexification of the first four elementary catastrophes. The second is a two point boundary value problem which exhibits the most common singular solutions, and which, if a poor discretization is used to solve it, also illustrates the breaking of these bifurcations. Finally we present two examples from computations of the flow between rotating coaxial disks. Here complex connections between disjoint real paths have been found and they are not due to perturbations induced by the computations.

We have also applied these techniques to the flow of viscous incompressible fluids between concentric cylinders; that is to the Taylor-vortex problem. As is well known, Taylor vortices bifurcate from Couette flow as the Reynolds number increases and the bifurcation is of the supercritical pitchfork type. For Reynolds numbers above the bifurcation or critical value, we have computed complex flows. Thus we have found complex solutions of the Navier-Stokes equations in the (r, z)-cylindrical geometry. We do not present these results here as they did not reveal any new real solutions and the computations were done on a very crude mesh [7]. The practicality of our new techniques for such computationally intensive problems must depend upon the availability of computing power and the skill of those setting up the computations.

8A. The elementary catastrophes. As we saw in § 6, a qualitative analysis of singularities may be obtained by reducing them to appropriate normal forms. Catastrophe theory (Thom [17], Lu [14], Poston [15], etc.) provides a complete classification of some of these normal forms as the elementary catastrophes. As an example of the complexification of real singular points, we consider the complexification of the first four elementary catastrophes:

(8.1a) Fold: $u^2 + \mu_1 = 0$;

(8.1b) Cusp: $u^3 + \mu_1 u + \mu_2 = 0$;

(8.1c) Swallowtail: $u^4 + \mu_1 u^2 + \mu_2 u + \mu_3 = 0$;

(8.1d) Butterfly: $u^5+\mu_1 u^3+\mu_2 u^2+\mu_3 u+\mu_4=0.$

The real parameters μ_j are unrestricted and, in catastrophe theory, any one of them can be viewed as the bifurcation parameter. Their complexifications are shown in Figs. 5-8 (see pp. 477-478) for cases (8.1b), (8.1c), and (8.1d), respectively. The fold, (8.1a), is already shown in Fig. 2 with μ_1 replaced by λ. The blue surface shows the usual real catastrophies depicted many times in the literature. However, the red surface shows the complex roots, say $u+iv$ by plotting $u+v$. Figure 5 shows the cusp. The yellow curve in the $\mu_1\mu_2$-plane is the projection of the fold curve in space along which the complex roots emanate from the real roots. Figure 6 is a section of the butterfly with fixed $\mu_2=\mu_3=1$. Figure 7 is the section of the swallowtail with fixed $\mu_2=1$. Figure 8 shows Fig. 7 sliced along the μ_1—and μ_3—axis and shifted a bit to open up the cut surface. The structure can thus be seen a bit better.

The catastrophes give a catalog of possible perturbations of real bifurcations. These may be used as a reference for the complex branches near the more common bifurcation points. Each catastrophe is stable to perturbations, but a section of a catastrophe is not. A given section will be perturbed to a nearby section, so although the topology of the catastrophe is unchanged, the topology of its intersection with the section may change.

Symmetries which are preserved by the perturbation may limit the possible distortions of a section. For example, the simple cubic bifurcation is the section $\mu_2=0$ of the cusp catastrophe (8.1b). A general perturbation of this cubic both translates and rotates the section breaking the pitchfork bifurcation into a regular point and a quadratic fold. However, a perturbation which preserves the reflectional symmetry about $u=0$ only causes the section to translate in the μ_1 direction preserving the pitchfork bifurcation.

8B. A nonlinear two point boundary value problem. This example illustrates complex quadratic folds as well as transcritical and pitchfork bifurcations. It appears in [11] as an illustration of pseudo-arclength continuation. The problem is

$$u_{xx}+f(x,u;\lambda)=0, \tag{8.2a}$$

$$u(0)=u(1)=0, \tag{8.2b}$$

where[1]

$$f(x,u;\lambda)\equiv 2q(\lambda)+\pi^2\lambda p(u-u_0(x,\lambda)) \tag{8.3a}$$

$$q(\lambda)\equiv\lambda^2 e^{-\lambda/2} \tag{8.3b}$$

$$p(z)\equiv z-\sum_{n=2}^{8} z^n-2z^9 \tag{8.3c}$$

$$u_0(x,\lambda)\equiv q(\lambda)x(1-x). \tag{8.3d}$$

We employ a uniform grid with centered second order difference approximations to u_{xx} at x_j and $f_j\equiv f(x_j,u_j,\lambda)$ to get a second order accurate scheme for (8.2). Some results of the use of this scheme with mesh spacing $h=1/30$, along with pseudo-arclength continuation and branch switching at bifurcations are summarized in Fig. 9. To understand these results we point out that the bifurcations in (8.2) from the trivial solution $u=u_0(x,\lambda)$ occur at the critical values $\lambda=\lambda_n=n^2$, $n=1,2,3,\cdots$. The

[1] The polynomial $p(x)$ in [11] does not match the figure in [11]. The polynomial given here in (8.3c) is the correct one for that figure.

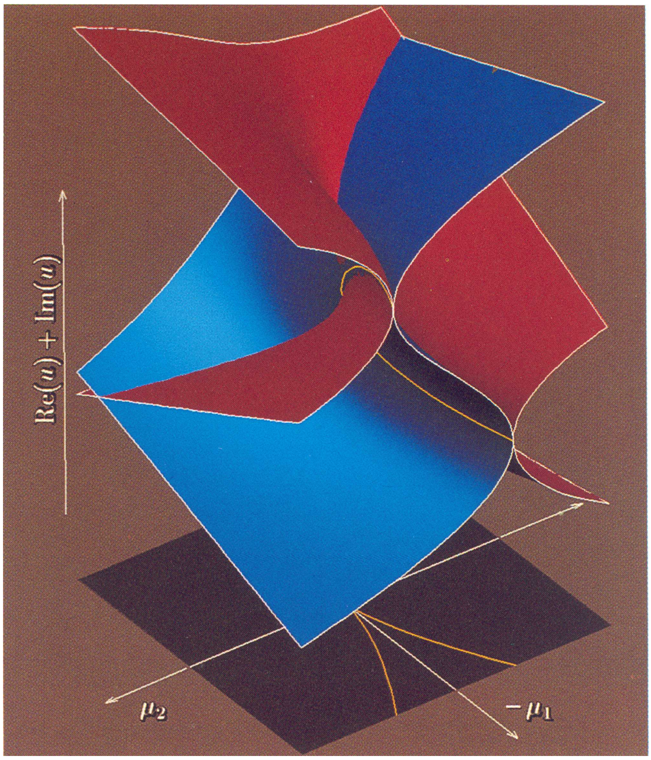

FIG. 5. *The cusp catastrophe* (8.1b).

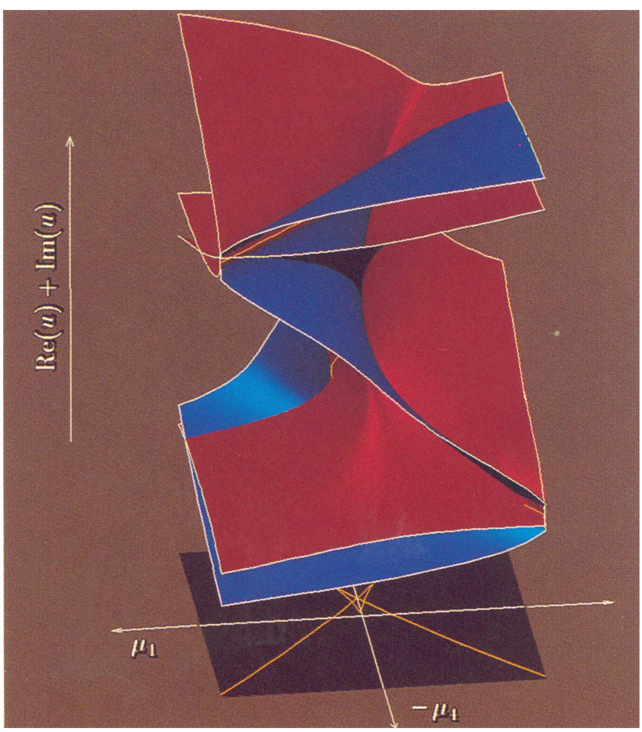

FIG. 6. *The butterfly catastrophe* (8.1d) *with* $\mu_2 = \mu_3 = 1$.

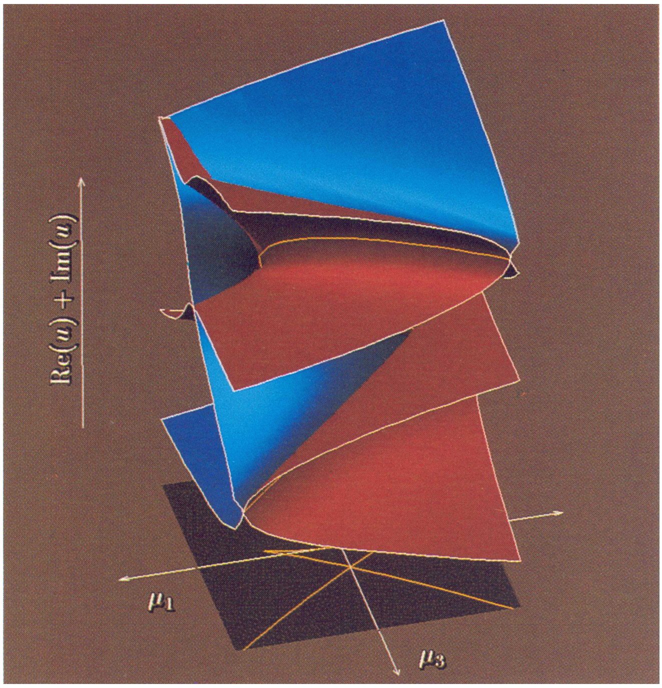

FIG. 7. *The swallowtail catastrophe* (8.1c) *with* $\mu_2 = 1$.

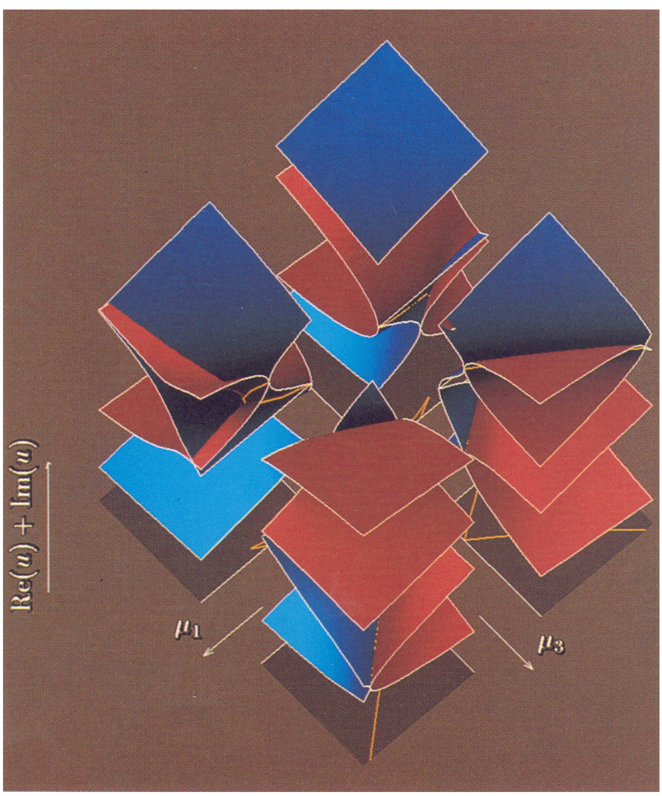

FIG. 8. *The swallowtail sliced open along the* μ_1- *and* μ_3-*axes.*

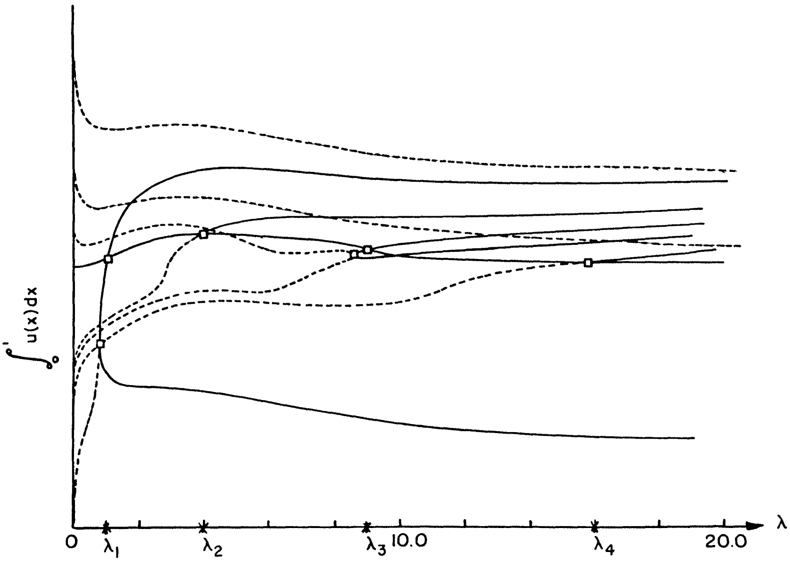

FIG. 9. *Bifurcation diagram for problem* (8.2); *centered scheme,* $O(h^2)$ *accurate. Real paths: solid curves; complex paths: dotted curves.*

odd modes, $n = 1, 3, 5, \cdots$ are all transcritical bifurcations and the even modes, $n = 2, 4, \cdots$, are all (supercritical) pitchfork bifurcations. The transcritical branches all contain simple quadratic folds quite near to, but just below, the critical values. The pitchfork branches are symmetric about the bifurcation points. Thus they appear as a single branch in our graphs since the vertical axis employs the value

$$y \equiv \int_0^1 \{\mathrm{Re}\,[u(\lambda, x)] + \mathrm{Im}\,[u(\lambda, x)]\}\, dx.$$

The complex paths emanating from the folds and from the pitchfork bifurcations are displayed as being dotted while the real paths are the solid curves. At the top of Fig. 9 two complex paths were found not yet connected to any real paths. We assume that these are associated with some higher order fold or bifurcation not included in our calculation range.

The centered scheme employed above, being second order accurate, can in principal determine the basic or trivial solution $u_0(s, \lambda)$ exactly since it is quadratic in x. This fact, plus the smoothness of the solutions and the accuracy of the thirty interval mesh, does not show any distortion or splitting of the bifurcations that might be caused by truncation errors. To show such an effect we replaced the centered scheme by an uncentered one in which f_j was replaced by $\tilde{f}_j \equiv f(x_j, u_{j-1}, \lambda)$. The results using this uncentered scheme are shown in Fig. 10. Note that a complex isola appears joining the split transcritical bifurcation near $\lambda = \lambda_1$ (this can only be seen in the enlarged sketch). The pitchfork bifurcations split yielding quadratic folds on one path and a regular path, similar to the paths in Fig. 4.

8C. The complex flow between rotating coaxial disks. The final example concerns the flow of an incompressible viscous fluid between a pair of infinite disks which are rotating about a common axis with Ω_0 and Ω_1 the rotation rates of the upper and lower disks, respectively, and d the spacing between the disks. We seek axially

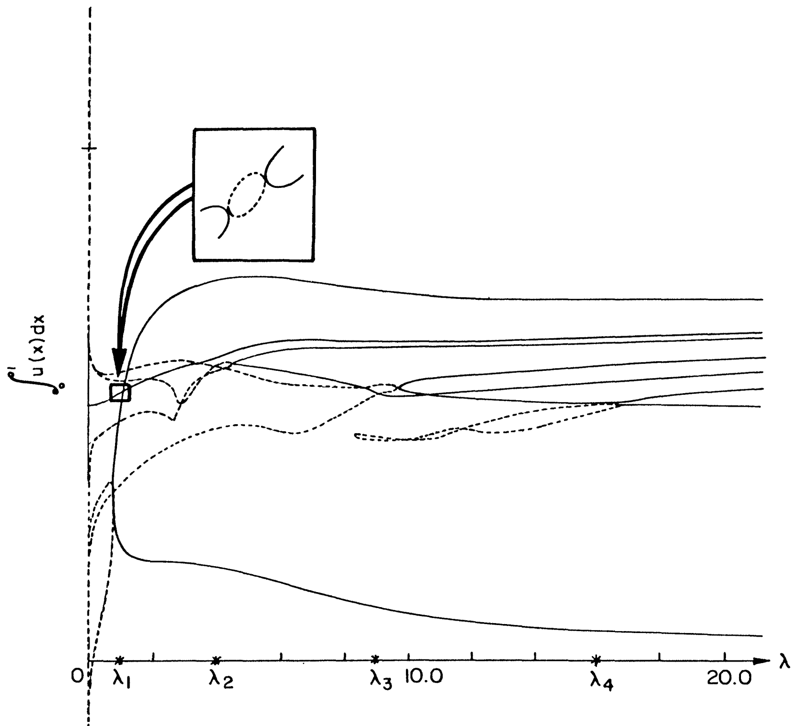

FIG. 10. *Bifurcation diagram for problem* (8.2); *noncentered scheme,* $O(h)$ *accurate. Real paths: solid curves; complex paths: dotted curves.*

symmetric solutions. In this case the Navier–Stokes equations have solutions, in cylindrical coordinates in the form:

$$u_z = \frac{1}{\Omega_0 d} f\left(\frac{z}{d}\right), \quad u_r = -\frac{1}{2\Omega_0 d} rf'\left(\frac{z}{d}\right), \quad u_\theta = \frac{1}{\Omega_0 d} rg\left(\frac{z}{d}\right), \tag{8.4}$$

provided that the functions f and g satisfy

$$\begin{aligned} f'''' &= R(ff''' + 4gg'), &\quad &\begin{cases} f(0) = f'(0) = 0 \\ f(1) = f'(1) = 0; \end{cases} \\ g'' &= R(fg' - gf'), &\quad &\begin{cases} g(0) = 1 \\ g(1) = \lambda. \end{cases} \end{aligned} \tag{8.5}$$

There are two parameters, $\lambda = \Omega_1 / \Omega_0$, the ratio of the rotational speeds of the disks, and a Reynolds number, $R = \Omega_0 d^2 / \nu$. Keller and Szeto [13] and Fier [4] have studied the dependence of the solution on both λ and R, by following paths of singular points.

Using centered differences on a grid with 100 uniform intervals, we fixed $R = 370$ and computed a path of solutions starting at $\lambda = 1$ (with a solution obtained from [4]). A schematic of the solution path in (f, g) versus λ is shown in Fig. 11. The solid curves represent real solutions and the dotted curves are complexified solutions. There are three disconnected real solution paths all connected by complex paths. The real folds labeled B, C_1, C_2, E_1, and E_2 have been previously found in [13] and [4] by continuation in *both* parameters R and λ. However, we have been able to find them and of course

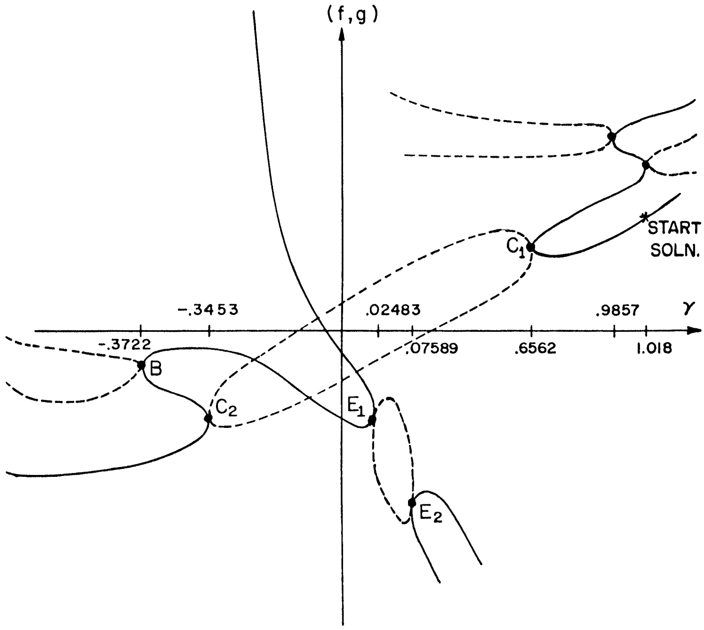

FIG. 11. *Schematic of the solution paths of the rotating disk problem at $R = 370$. Real paths: solid curves; complex paths: dotted curves.*

the three real solutions paths with R fixed, by using complex continuation. This illustrates one of the potentially powerful uses of the new techniques described in § 7.

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