Exact solution of a nonlinear model of four-wave mixing and phase conjugation

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An exact solution of a nonlinear model of holographic four-wave mixing is derived. An expression for the reflectivity of a phase-conjugate mirror with depleted pumps is presented. We find that such a phase-conjugate mirror may exhibit bistability.

Theories of the interactions of light in media possessing optical nonlinearities are themselves strongly nonlinear; most theoretical approaches have depended on such linearizing assumptions as the undepleted-pumps approximation.\(^1,2\) Between the early 1960's, when second- and third-harmonic generation\(^3\) and Raman pump conversion to first Stokes \(^4\) and beam 2 with 3, gives rise to strong beam coupling. This predominance of one grating is common in many laboratory situations and is due to the directions, polarizations, and coherence relationships of the four beams relative to the nonlinear medium and to the application, in some cases, of an electric field that enhances certain gratings. We do not use the undepleted-pumps approximation. That is, the variation of all four beams is retained, and the solution is valid even when all four beams are comparable in amplitude.

The applicable coupled-wave equations are\(^10\)

\[
\begin{align*}
\frac{dA_1}{dz} &= -\gamma \frac{I_0}{I_0} (A_1 A_4^* + A_2 A_3) A_4, \\
\frac{dA_2^*}{dz} &= -\gamma \frac{I_0}{I_0} (A_1 A_4^* + A_2 A_3) A_3^*, \\
\frac{dA_3}{dz} &= \gamma \frac{I_0}{I_0} (A_1 A_4^* + A_2 A_3^*) A_2, \\
\frac{dA_4^*}{dz} &= \gamma \frac{I_0}{I_0} (A_1 A_4^* + A_2 A_3^*) A_1^*,
\end{align*}
\]

where \(\gamma\) is a complex coupling constant, which is a material parameter of the nonlinear medium.\(^11\) When \(\gamma\) is real, the phase shift between grating and fringes is \(\pi/2\). When it is purely imaginary, the grating is in phase with the fringes. \(I_0\) is the total average light intensity.

\[
I_0 = I_1 + I_2 + I_3 + I_4,
\]

where \(I_j = |A_j|^2\).

We observe the following conservation laws:

\[
\begin{align*}
A_1 A_2 + A_3 A_4 &= c, \\
I_1 + I_4 &= d_1, \\
I_2 + I_3 &= d_2.
\end{align*}
\]

Fig. 1. Scheme of the four beams involved in nonlinear interaction.
where \(c, d_1, \) and \(d_2\) are constants of integration.

With the help of these conservation laws, Eqs. (2) and (3) can be decoupled from Eqs. (4) and (5):

\[
\frac{dA_1}{dz} = \frac{-\gamma}{I_0} [A_1d_1 - A_1(I_1 + I_2) + A_2^*c], \tag{10}
\]

\[
\frac{dA_2^*}{dz} = \frac{-\gamma}{I_0} [A_1c^* - A_2^*(I_1 + I_2) + A_2^*d_2], \tag{11}
\]

\[
\frac{dA_3}{dz} = \frac{\gamma}{I_0} [A_3d_2 - A_3(I_3 + I_4) + A_4^*c], \tag{12}
\]

\[
\frac{dA_4^*}{dz} = \frac{\gamma}{I_0} [A_3c^* - A_4^*(I_3 + I_4) + A_4^*d_4]. \tag{13}
\]

By eliminating the term in \(I_1 + I_2\) between Eqs. (10) and (11), and the term in \(I_3 + I_4\) between Eqs. (12) and (13), we find the following expressions for \(A_3/A_4^*\) and \(A_4^*/A_2^*\):

\[
\frac{d}{dz} \left( \frac{A_3}{A_4^*} \right) = \frac{\gamma}{I_0} \left[ c + (d_2 - d_1) \left( \frac{A_3}{A_4^*} \right) - c^* \left( \frac{A_3}{A_4^*} \right)^2 \right], \tag{14}
\]

\[
\frac{d}{dz} \left( \frac{A_4^*}{A_2^*} \right) = \frac{-\gamma}{I_0} \left[ c + (d_1 - d_2) \left( \frac{A_4^*}{A_2^*} \right) - c^* \left( \frac{A_4^*}{A_2^*} \right)^2 \right]. \tag{15}
\]

Noting that \(I_0\) is constant because of the conservation laws, we see that these equations are immediately integrable:

\[
\frac{A_3}{A_4^*} = \frac{\left[ (\Delta - (\Delta^2 + 4|c|^2)^{1/2}) De^{-\mu z} - (\Delta + (\Delta^2 + 4|c|^2)^{1/2}) D_1 e^{\mu z} \right]}{2c^* (De^{-\mu z} - D_1 e^{\mu z})}, \tag{16}
\]

\[
\frac{A_4^*}{A_2^*} = \frac{\left[ (\Delta - (\Delta^2 + 4|c|^2)^{1/2}) E e^{-\mu z} - (\Delta + (\Delta^2 + 4|c|^2)^{1/2}) E_1 e^{\mu z} \right]}{2c (E e^{-\mu z} - E_1 e^{\mu z})}, \tag{17}
\]

where \(\Delta\) is the power flux \(d_2 - d_1\),

\[
\mu = \frac{\gamma(\Delta^2 + 4|c|^2)^{1/2}}{2I_0}, \tag{18}
\]

and \(D\) and \(E\) are constants of integration.

At this point, the problem has been transformed from the set of nonlinear differential equations [Eqs. (2)–(5)] to another set of equations [Eqs. (7)–(9), (16), and (17)], which may be solved by fitting boundary conditions.

To illustrate this solution in a particular set of boundary conditions, we consider the reflectivity of a phase-conjugate mirror. Let beams 1 and 2 be the pumping beams, beam 4 be the probe beam, and beam 3 be the phase-conjugate beam. \(I_1(0), I_4(0),\) and \(I_2(l)\) are given as boundary conditions, together with the phase-conjugation condition \(I_3(l) = 0\). After elementary manipulations of the solution equations and the boundary conditions, we find that the intensity reflectivity \(R\) is given by

\[
R = \frac{4|c|^2 T^2}{|\Delta T + (\Delta^2 + 4|c|^2)^{1/2}|^2}, \tag{19}
\]

where \(T = \tanh(\mu l)\) and \(|c|^2\) is given by the equation

\[
|c|^2 = I_1(0)I_2(l)|\Delta T + (\Delta^2 + 4|c|^2)^{1/2} + 4|c|^2 T^2 I_4(0)I_2(l) + 2|c|^2 I_4(0)(\Delta^2 + 4|c|^2)^{1/2}(T + T^*) = 0. \tag{20}
\]

This solution, of course, reduces to the known undepleted-pumps approximation theory\(^{10}\) in the limit of small \(I_4(0)\). In Fig. 2 we plot the reflectivity of a phase-conjugate mirror as a function of the coupling strength \(|\gamma|l\) for the case in which the phase shift between the grating and interference fringes is 5°.

In conclusion, then, we have solved a model of holographic four-wave mixing without using the unde-
pleted-pumps approximation; the quantitative theo-
retical study of experiments involving pump depletion
has thus been made possible without using numerical
solutions of the coupled-wave differential equations.

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11. In photorefractive media, for example, $\gamma$ depends on
    parameters such as crystal orientation, grating period,
    charge-carrier concentration, and the angle of incidence
    $\alpha$. 