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Nonadiabatic elimination of auxiliary modes in continuous quantum measurements

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In quantum measurement or control processes, there are often auxiliary modes coupling to the quantum system that we are interested in—they together form a bath or an environment for the system. The bath can have finite memory (non-Markovian), and simply ignoring its dynamics (i.e., adiabatically eliminating it) will prevent us from predicting the true quantum behavior of the system. We generalize the technique introduced by Strunz *et al.* [Phys. Rev. Lett. **82**, 1801 (1999)], and develop a formalism that allows us to eliminate the bath nonadiabatically in continuous quantum measurements, and obtain a non-Markovian stochastic master equation for the system that we focus on. This formalism also illuminates how to design the bath—acting as a quantum filter—to effectively probe interesting system observables (e.g., the quantum-nondemolition observable).

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Introduction. Recent developments in techniques of high-precision metrology have allowed quantum-level measurement and control of matters at all scales, ranging from microscopic atoms [1] to macroscopic mechanical oscillators [2]. In these experiments, the atoms or mechanical oscillators, as objects of interest (or the system), are usually coupled to auxiliary degrees of freedom (or the bath) (e.g., the cavity mode in cavity QED systems), which are in turn coupled to external readout devices. It is often desirable to obtain a self-contained equation for the system only, by eliminating bath degrees of freedom, especially when we want to study quantum dynamics of the system or implement feedback control. In the literature, the usual approach is to ignore the dynamics of bath modes by assuming that they follow the system instantaneously, and can be adiabatically eliminated. However, this becomes inadequate when bath modes evolve at scales longer than the system and thus have finite memory (i.e., when the dynamics becomes non-Markovnian).

One way to account for a non-Markovian bath is the Feynman-Vernon influence functional method [3]. Diósi and Strunz *et al.* [4–6] developed an equivalent (but much simpler) method by unraveling the bath evolution into possible quantum trajectories. These trajectories are shown to drive a non-Markovian stochastic Schrödinger equation (SSE), which averages into the exact non-Markovian master equation. This has been applied extensively to study open quantum dynamics of non-Markovian systems. In their model, the system-bath coupling is general, but the quantum measurement process is not included as *a priori*. Even though the SSE in the Markovian limit can be interpreted as the conditional evolution of the system's pure state under a continuous measurement, yet in general, the non-Markovian SSE does not allow such an interpretation, as discussed by Diosi [7] and Wiseman *et al.* [8].

Here, we explicitly include the quantum measurement process in the general system-bath coupling model by coupling the bath to an additional continuous probe field, a quantum Wiener process [9], that is projectively measured by a detector [10], as shown in Fig. 1. If the measurement results of the detector are ignored (i.e., averaged over all possibilities) the probe field can be viewed as an additional Markovian bath, which is similar to the pseudomode model [11,12] for studying general non-Markovian dynamics. Given different measurement outcomes, the system and the bath are projected into different conditional quantum states. By generalizing the Diosi-Strunz approach, we eliminate the bath from the joint system-bath evolution, obtaining a non-Markovian stochastic master equation (SME) for the system only. In contrast to the Markovian case, here the system undergos a mixed-state conditional evolution, agreeing with the general statement by Wiseman *et al.* [8].

In the following, we first present the general formalism, and then apply it to several models of experimental interests that can be solved exactly. To treat the general system-bath coupling, we use the perturbation method with the coupling strength as the expansion parameter, and present the leading-order result, which is valid in the weak-coupling limit. We further show that this result can serve as a guideline for designing the appropriate bath, as a quantum filter, to effectively measure desired observables of the system. In particular, we will consider the quantum-nondemolition (QND) measurements.

Model. The total Hamiltonian for our model reads

$$\hat{H} = \hat{H}_s + \hat{H}_b + \hat{H}_{\rm int} + \sum_k \hbar\sqrt{\gamma_k}\,[\hat{a}_k\,\hat{b}^\dagger_{\rm in}(t) + \hat{a}^\dagger_k\,\hat{b}_{\rm in}(t)],$$
$$\hat{H}_b \equiv \sum_k \hbar\,\omega_k \hat{a}^\dagger_k \hat{a}_k, \quad \hat{H}_{\rm int} \equiv \sum_k \hbar\, g_k(\hat{L}\hat{a}^\dagger_k + \hat{L}^\dagger \hat{a}_k). \qquad (1)$$

Here $\hat{H}_s$, $\hat{H}_b$, and $\hat{H}_{\rm int}$ are the system, bath, and interaction Hamiltonian, respectively; $\hat{a}_k$ and ω_k are the annihilation operator and eigenfrequency of the kth bath mode with $[\hat{a}_k, \hat{a}^\dagger_{k'}] = \delta_{kk'}$; the system-bath coupling is not restricted to the rotating-wave approximation, as $\hat{L}$ can be Hermitian; g_k is the coupling constant between the system operator $\hat{L}$ and the kth bath mode $\hat{a}_k$; $\hat{b}_{\rm in}(t)$ are annihilation operators for the input probe field at different times and $[\hat{b}_{\rm in}(t), \hat{b}^\dagger_{\rm in}(t')] = \delta(t-t')$; and γ_k is the coupling strength between the bath and the probe field. We exclude those modes that are not coupled to the probe field—they effectively introduce thermal decoherence and can be included by adding Lindblad terms in the final master equation. In addition, we only consider one probe field, and can be easily generalized to multiple probe fields.

Conditional dynamics. At each moment, the output probe field $\hat{b}_{\rm out}(t)$ is projectively measured by the detector (e.g.,

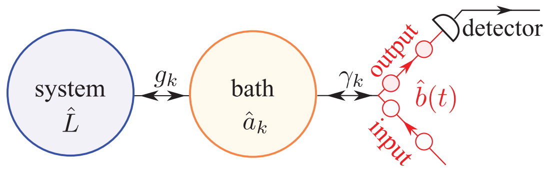

FIG. 1. (Color online) Schematics of our measurement process. The system is coupled to the bath, which in turn couples to an external probe field. The output probe field amplitude is projectively measured by a detector.

homodyne detection if the probe field is an optical field). We assume that the (amplitude) quadrature $\hat{b}_A \equiv [\hat{b}_{\rm out}(t) + \hat{b}^\dagger_{\rm out}(t)]/\sqrt{2}$ is measured with the result at moment t being $y(t)$. Given this measurement result, the system-bath system is projected into a conditional state, with the joint wave function $|\psi\rangle$ at $t+dt$ given by

$$|\psi(t+dt)\rangle = \frac{1}{P^{1/2}(y)}\langle y(t)|\hat{U}(dt)|\mathbf{0}\rangle \otimes |\psi(t)\rangle.$$

Here $\hat{U}(dt) = e^{-i\hat{H}dt/\hbar}$ is the unitary evolution operator; we assume that the input probe field (before interaction) is in the vacuum state $|\mathbf{0}\rangle$ and is separable from the joint system-bath state; $|y(t)\rangle$ is an eigenstate of $\hat{b}_A(t)$; $P(y)$ is the probability distribution for the measurement result; and $P(y) = {\rm Tr}_{pb}\{|\psi(t+dt)\rangle\langle\psi(t+dt)|\}$. By integrating over the probe field variable, we can obtain the following nonlinear Markovian SSE for the system-bath state:

$$\begin{aligned} d|\psi\rangle = &-\frac{i}{\hbar}(\hat{H}_s + \hat{H}_b + \hat{H}_{\rm int})|\psi\rangle dt - \sum_{kk'}\sqrt{\gamma_k\gamma_{k'}}\,[\hat{a}^\dagger_k\hat{a}_{k'} \\ &+ \langle\hat{a}_k - \hat{a}^\dagger_k\rangle\hat{a}_{k'} - \langle\hat{a}_k - \hat{a}^\dagger_k\rangle\langle\hat{a}_{k'} - \hat{a}^\dagger_{k'}\rangle/4]|\psi\rangle dt \\ &- \sum_k i\sqrt{\gamma_k/2}\,(2\,\hat{a}_k - \langle\hat{a}_k - \hat{a}^\dagger_k\rangle)|\psi\rangle dW, \end{aligned} \qquad (2)$$

and $y(t)dt = -i\sum_k\sqrt{\gamma_k}\langle\hat{a}_k - \hat{a}^\dagger_k\rangle dt + dW/\sqrt{2}$, which is derived from the distribution for the measurement result: $P(y) = (dt/\pi)^{1/2}\exp[-(y + i\sum_k\sqrt{\gamma_k}\langle\hat{a}_k - \hat{a}^\dagger_k\rangle)^2 dt]$ with dW the Wiener increment: $dW^2 = dt$, and $\langle\hat{a}_k\rangle \equiv \langle\psi|\hat{a}_k|\psi\rangle$. When the bath is a single-cavity mode, it gives the well-known Markovian SSE for the conditional evolution, also known as the quantum trajectory [13], of the system and cavity mode under homodyne detection [14].

Elimination of bath modes. To nonadiabatically eliminate bath modes, we apply the method by Strunz *et al.* [6] and choose unnormalized coherent-state representation $|\boldsymbol{\alpha}\rangle \equiv \exp[-\sum_k \alpha_k\hat{a}^\dagger_k]|\mathbf{0}\rangle$ for the bath modes. One can obtain the equation for $|\psi(\boldsymbol{\alpha}^*)\rangle \equiv \langle\boldsymbol{\alpha}|\psi\rangle$ by using $\langle\boldsymbol{\alpha}|\hat{a}_k|\psi\rangle = \partial_{\alpha^*_k}|\psi(\boldsymbol{\alpha}^*)\rangle$ and $\langle\boldsymbol{\alpha}|\hat{a}^\dagger_k|\psi\rangle = \alpha^*_k|\psi(\boldsymbol{\alpha}^*)\rangle$. The reduced density matrix for the system is given by

$$\hat{\rho}_s = {\rm Tr}_b[|\psi\rangle\langle\psi|] = \int d^2\boldsymbol{\alpha}\, e^{-|\boldsymbol{\alpha}|^2}|\psi(\boldsymbol{\alpha}^*)\rangle\langle\psi(\boldsymbol{\alpha})|. \qquad (3)$$

By using the fact that $\int d^2\boldsymbol{\alpha}\,\alpha_k e^{-|\boldsymbol{\alpha}|^2}|\psi(\boldsymbol{\alpha}^*)\rangle\langle\psi(\boldsymbol{\alpha})| = \int d^2\boldsymbol{\alpha}\, e^{-|\boldsymbol{\alpha}|^2}\partial_{\alpha^*_k}|\psi(\boldsymbol{\alpha}^*)\rangle\langle\psi(\boldsymbol{\alpha})|$ and Eq. (3), we obtain a non-Markovian SME for the system

$$\begin{aligned} d\hat{\rho}_s = &-\frac{i}{\hbar}[\hat{H}_s, \hat{\rho}_s]\,dt - \sum_k g_k([\hat{L}^\dagger, \hat{\varrho}_k] - [\hat{L}, \hat{\varrho}^\dagger_k])\,dt \\ &+ \sum_k\sqrt{2\gamma_k}(\hat{\varrho}_k + \hat{\varrho}^\dagger_k - {\rm Tr}_s\{\hat{\varrho}_k + \hat{\varrho}^\dagger_k\})\,dW, \end{aligned} \qquad (4)$$

and $y(t)dt = \sum_k\sqrt{\gamma_k}\,{\rm Tr}_s[\hat{\varrho}_k + \hat{\varrho}^\dagger_k]dt + dW/\sqrt{2}$, where we have introduced:

$$\hat{\varrho}_k \equiv i\int d^2\boldsymbol{\alpha}\, e^{-|\boldsymbol{\alpha}|^2}\partial_{\alpha^*_k}|\psi(\boldsymbol{\alpha}^*)\rangle\langle\psi(\boldsymbol{\alpha})|. \qquad (5)$$

Here the non-Markovianity only arises when eliminating the bath that has memory about the system. Equations (4) and (5) will give a self-contained SME governing the system, if $\hat{\varrho}_k$ can be written in terms of $\hat{\rho}_s$ and other system operators. To derive $\hat{\varrho}_k$, we use the approach in Ref. [6] by introducing the system operator $\hat{O}_k$ through

$$\partial_{\alpha^*_k}|\psi(\boldsymbol{\alpha}^*)\rangle \equiv -i\,\hat{O}_k(t, \boldsymbol{\alpha}^*)|\psi(\boldsymbol{\alpha}^*)\rangle. \qquad (6)$$

For most cases considered here, $\hat{O}_k$ is independent of $\boldsymbol{\alpha}^*$, and $\hat{\varrho}_k$ takes a very simple form

$$\hat{\varrho}_k = \hat{O}_k(t)\,\hat{\rho}_s. \qquad (7)$$

In general, $\hat{\varrho}_k$ is a superoperator of $\hat{\rho}_s$ determined by $\hat{O}_k$. Systematic procedures for deriving $\hat{O}_k$ (without the measurement) have been developed by Yu *et al.* [15], and applied to systems with different system Hamiltonians. Their method can be generalized to our case by using the interaction picture $|\psi(\boldsymbol{\alpha}^*)\rangle_I = \hat{U}^{-1}(t)|\psi(\boldsymbol{\alpha}^*)\rangle$ with a nonunitary evolution operator: $\hat{U}(t) = \exp[-(i/\hbar)(\hat{H}_s + \hat{H}_b - i\hbar\sum_{kk'}\sqrt{\gamma_k\gamma_{k'}}\hat{a}^\dagger_k\hat{a}_{k'})t]$. We can then determine $\hat{O}_k$ from the following consistency condition [6,15]:

$$\frac{d}{dt}\left[\partial_{\alpha^*_k}|\psi(\vec{\boldsymbol{\alpha}}^*)\rangle_I\right] = \partial_{\alpha^*_k}\left[\frac{d}{dt}|\psi(\vec{\boldsymbol{\alpha}}^*)\rangle_I\right]. \qquad (8)$$

To illustrate this formalism, we first study models in which $\hat{\varrho}_k$ can be exactly solved. We later treat the general system-bath coupling with the perturbation method.

Atom-cavity interaction. We consider the standard atom-cavity model, as shown in Fig. 2,with Hamiltonian

$$\hat{H} = \hbar\frac{\omega_q}{2}\hat{\sigma}_z + \hbar\,\Delta\hat{a}^\dagger\hat{a} + \hbar\, g(\hat{\sigma}_-\hat{a}^\dagger + \hat{\sigma}_+\hat{a}).$$

Here for simplicity we do not show the interaction between the cavity mode and external probe field; ω_q is the atom transition frequency and $\hat{\sigma}_z$ the Pauli matrix; Δ is the cavity detune—the difference between the cavity resonant frequency and the laser frequency. In comparison with the general Hamiltonian in Eq. (1), this corresponds to the case of $\hat{L} = \hat{\sigma}_-$ and $g_k = g\,\delta_{1k}$

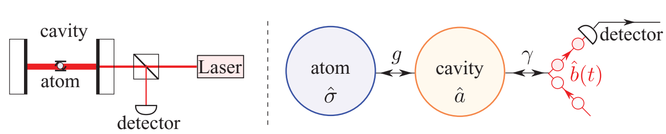

FIG. 2. (Color online) Schematics showing the atom-cavity system. A two-level atom (or a qubit) interacts with a cavity mode that is coupled to an external continuous optical field which is measured via homodyne detection.

(the bath has only one cavity mode and we will ignore subscript k afterwards). By using the consistency condition in Eq. (8), the operator $\hat{O} = f(t)\hat{\sigma}_-$ and $\hat{\varrho}$ has the following form:

$$\hat{\varrho} = f(t)\hat{\sigma}_-\hat{\rho}. \tag{9}$$

Here the time-dependent function $f(t)$ satisfies a Riccati equation: $\dot{f} - i(\omega_q - \Delta + i\gamma)f - gf^2 = g$ with the initial condition $f(0) = 0$, from the assumption that the cavity mode is initially in a vacuum state. The corresponding SME for the atom density matrix reads

$$\begin{aligned} d\hat{\rho} = &-i\left[\frac{\omega_q}{2}\hat{\sigma}_z + g\mathrm{Im}\{f\}\,\hat{\sigma}_+\hat{\sigma}_-, \hat{\rho}\right]dt \\ &- g\mathrm{Re}\{f\}\,[\hat{\sigma}_+\hat{\sigma}_-\hat{\rho} + \hat{\rho}\,\hat{\sigma}_+\hat{\sigma}_- - 2\,\hat{\sigma}_-\hat{\rho}\hat{\sigma}_+]\,dt \\ &+ \sqrt{2\gamma}[f\hat{\sigma}_-\hat{\rho} + f^*\hat{\rho}\hat{\sigma}_+ - \langle f\hat{\sigma}_- + f^*\hat{\sigma}_+\rangle\hat{\rho}]dW. \end{aligned} \tag{10}$$

This equation fully describes non-Markovian dynamics of the atom under a continuous measurement. The Markovian limit can be recovered by assuming the cavity decay rate much larger than the atom-cavity interaction rate and also the atom transition rate: $\gamma \gg g$ and $\gamma \gg \omega_q$. The cavity-mode memory then becomes negligible and

$$f(t)|_{\text{Markovian limit}} = g/\gamma, \tag{11}$$

in which case Eq. (10) reduces to the usual Markovian SME. In addition, we can also obtain the corresponding master equation if we ignore the measurement result by averaging over dW (mean of dW vanishes), we will obtain the exact master equation for an atom coupled to a damped cavity mode, a dissipative environment [16].

To confirm that Eq. (10) is the SME that correctly describes the conditional dynamics of the atom, we numerically solve and compare (i) the Markovian SSE for the joint atom-cavity wave function and (ii) the non-Makovian SME for the atom density matrix to see whether they both give the same conditional mean of $\hat{\sigma}_x$, $\hat{\sigma}_y$, and $\hat{\sigma}_z$. The numerical results are shown in Fig. 3. We have chosen $\omega_q = 1$, $\Delta = 1$, and $\gamma = 2$, and the atom-cavity initial state is $[|+\rangle_z + |-\rangle_z]/\sqrt{2} \otimes |0\rangle$. They indeed agree with each other as shown by the convergency of their accumulated numerical difference.

Linear optomechanical interaction. We now consider another exactly solvable model, the linear optomechanical interaction between a harmonic mechanical oscillator and a cavity mode. The setup is similar to Fig. 2 with the atom replaced by an oscillator, which has been studied extensively recently [2]. The Hamiltonian for the mechanical oscillator and the cavity mode reads [17–19]

$$\hat{H} = \hbar\,\omega_m(\hat{x}^2 + \hat{p}^2) + \hbar\,\Delta\hat{a}^\dagger\hat{a} + \hbar\,g\,\hat{x}\hat{a}^\dagger\hat{a}.$$

Here $\hat{x}$ and $\hat{p}$ are the position and momentum of the oscillator with eigenfrequency ω_m (normalized with respect to their zero-point values). Since the cavity mode usually has a large steady-state amplitude due to a coherent driving from the laser, we can consider perturbations around the steady-state amplitude and linearize the above Hamiltonian. The linearized interaction Hamiltonian is $\hat{H}_{\text{int}} = \hbar\,g\,\bar{a}\,\hat{x}(\hat{a} + \hat{a}^\dagger)$ where $\bar{a}$ is the steady-state amplitude of the cavity mode. With the same procedure as the atom-cavity case, $\hat{\varrho}$ is given by (again the bath has one

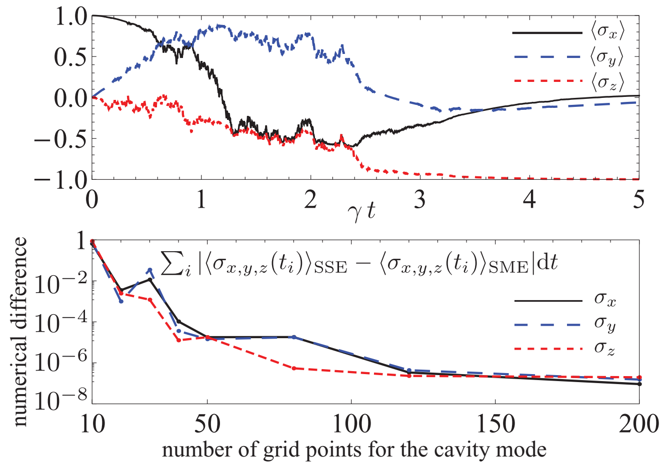

FIG. 3. (Color online) The top panel shows simulation results for the time evolution of $\langle\sigma_x\rangle$, $\langle\sigma_y\rangle$, and $\langle\sigma_z\rangle$ for one realization of dW. The bottom panel is the convergency of the accumulated numerical difference between the SSE and SME simulation results given different number of grid points for the cavity mode.

mode with the subscript k ignored)

$$\hat{\varrho} = [e^{-2i(\Delta - i\gamma)t} f_1\hat{\rho}\hat{\mathcal{A}}^\dagger + \hat{\mathcal{A}}\hat{\rho}]/(1 - |f_1|^2) \tag{12}$$

with $\hat{\mathcal{A}} = e^{-i(\Delta - i\gamma)t}\,[f_0(t) + f_x(t)\hat{x} + f_s(t)\hat{p}]$. These functions f_0, f_1, f_x, and f_s are determined from the consistency condition, and satisfy coupled Riccati equations

$$\begin{aligned} df_0 = &\, e^{-i(\Delta - i\gamma)t}[\,i\gamma\mathrm{Tr}\{\hat{\varrho} + \hat{\varrho}^\dagger\}f_1\,dt - i\sqrt{2\gamma}\,f_1 dW] \\ &- i\,g' e^{-2i(\Delta - i\gamma)t}\,f_0 f_s\,dt, \end{aligned} \tag{13}$$

$$\dot{f}_x = g' + \omega_m f_s - i\,g' e^{-2i(\Delta - i\gamma)t}(f_1 + f_x f_s), \tag{14}$$

$$\dot{f}_s = -\omega_m f_x - i\,g' e^{-2i(\Delta - i\gamma)t}\,f_s^2\,, \tag{15}$$

$$\dot{f}_1 = g' f_s - i\,g' e^{-2i(\Delta - i\gamma)t}\,f_1 f_s\,, \tag{16}$$

where $g' \equiv g\,\bar{a}\,e^{i(\Delta - i\gamma)t}$. These equations can be solved numerically. Similarly, if we average the SME over dW, we will obtain the corresponding non-Markovian master equation. It describes quantum Brownian motion of a harmonic oscillator coupled to a non-Markovian bath with dissipation.

General system-bath coupling. The previous models are exactly solvable due to special features of the system operator that the bath is coupled to. For a general system-bath coupling, there is no transparent route leading to a closed-form solution of $\hat{\varrho}_k$. However, if the coupling is weak, namely $g_k < \gamma_k$, we can systematically solve it with perturbation method by obtaining a hierarchy of equations at different orders of g_k/γ_k. Here, we show the leading-order result for $\hat{\varrho}_k$, which is given by

$$\hat{\varrho}_k = \sum_{k'}\int_0^t d\tau[e^{-i\mathbf{M}\tau}]_{kk'}\,g_{k'}\hat{L}(-\tau)\hat{\rho}_s, \tag{17}$$

where $\hat{L}(-\tau) = e^{-i\hat{H}_s\tau/\hbar}\,\hat{L}\,e^{i\hat{H}_s\tau/\hbar}$ under the free evolution and the matrix element $\mathbf{M}_{kk'} = \omega_k\,\delta_{kk'} - i\sqrt{\gamma_k\gamma_{k'}}$ with $\delta_{kk'}$ being the Kronecker delta. Basically, $\hat{\varrho}_k$ is equal to $\hat{L}\,\hat{\rho}_s$ convoluted with the Green's function of the bath. In other words, we are effectively coupled to a dynamical quantity of the system that is shaped by the bath—a quantum filter. One can therefore engineer the bath to measure desired observables of the system (e.g., a QND observable) as illustrated in the following two examples.

The first example is measuring mechanical energy quantization considered in Refs. [20–23], aiming at unequivocally demonstrating the quantumness of a macroscopic mechanical oscillator. In the proposed experiment, the position of a mechanical oscillator is quadratically coupled to a cavity mode, namely,

$$\hat{H}_{\rm int} = \hbar\, g\, \hat{x}^2(\hat{a} + \hat{a}^\dagger). \tag{18}$$

If the cavity bandwidth γ is less than the mechanical frequency ω_m, we expect a direct probe of the slowly-varying part of $\hat{x}^2$, which is proportional to the QND variable—energy or equivalently the phonon number $\hat{N}$. Indeed, from $\hat{x}(-\tau) = \hat{x}\cos\omega_m\tau - \hat{p}\sin\omega_m\tau$,

$$\hat{\varrho} = g\int_0^t d\tau\, e^{-\gamma\tau}\hat{x}^2(-\tau)\hat{\rho} \approx \frac{g}{\gamma}\hat{N}\hat{\rho}, \tag{19}$$

where we have ignored terms proportional to $e^{-\gamma t}$, as the characteristic measurement time scale is γ^{-1}. The leading-order SME for the oscillator reads [cf. Eq. (4)]

$$\begin{aligned} d\hat{\rho} = &-i[\omega_m \hat{N}, \hat{\rho}]\, dt - g_{\rm eff}[\hat{X}^2, [\hat{N}, \hat{\rho}]]\, dt \\ &+ \sqrt{2g_{\rm eff}}\,[\{\hat{N}, \hat{\rho}\} - 2\langle\hat{N}\rangle\hat{\rho}]dW \end{aligned} \tag{20}$$

with $g_{\rm eff} = g^2/\gamma$. Note that such a measurement is not an exact QND measurement, because we have $[\hat{X}^2, [\hat{N}, \hat{\rho}]]$ instead of the usual Lindblad term: $[\hat{N},[\hat{N},\hat{\rho}]]$. This term describes a two-phonon process that induces quantum jumps. However, after numerically solving this SME, we find that it does not have significant effects, and a QND measurement can indeed be effectively realized. This is in accord with the argument by Martin and Zurek [20]: The two-photon process happens at $2\,\omega_m$, which is strongly suppressed due to a small cavity bandwidth γ.

The second example is measuring the QND observable of a free mass, the momentum $\hat{p}$. This is of particular interest in quantum-limited force measurement with mechanical probes (e.g., detecting gravitational waves [24]).

By monitoring the momentum change, one can detect the force signal without quantum back action, enabling to surpass the standard quantum limit (SQL) [25]. To achieve this, we can couple the position $\hat{x}$ of the free mass with two coupled cavity modes $\hat{a}_1$ and $\hat{a}_2$, of which the interaction Hamiltonian is given by

$$\hat{H}_{\rm int} = \hbar\,\omega_s(\hat{a}_1\hat{a}_2^\dagger + \hat{a}_1^\dagger\hat{a}_2) + \hbar\, g\,\hat{x}(\hat{a}_1 + \hat{a}_1^\dagger), \tag{21}$$

where ω_s is the coupling constant between two cavity modes. The cavity mode $\hat{a}_1$ is coupled to the external probe field. From Eq. (17), we derived that

$$\hat{\varrho} = 2g\int_0^t d\tau\, e^{-\gamma\tau}\cos\left(\frac{\omega_s\tau}{2}\right)\hat{x}(-\tau)\hat{\rho} \approx \frac{4g}{\omega_s^2}\,\dot{\hat{x}}(0)\,\hat{\rho}, \tag{22}$$

where we have used the stationary-phase approximation by assuming $\omega_s \gg \gamma$, and also ignored terms proportional to $e^{-\gamma t}$. The effective observable is therefore equal to the momentum, as $\hat{p} = m\dot{\hat{x}}(0)$. Indeed, such a coupled-cavity scheme has been proposed as the so-called speed meter for advanced gravitational-wave detectors [26].

Conclusion. We have included the quantum measurement process in the general system-bath coupling model, and eliminated the bath modes nonadiabatically, which yields a non-Markovian SME for the system. Conceptually, this shows how non-Markovianity arises in the continuous measurement process. In practice, if the system is indeed all we care about, the non-Markovian dynamics derived here is the most efficient way of obtaining the conditional evolution of the system, both in terms of analytical and numerical complexity. In addition, we can utilize non-Markovianity to effectively measure quantum observables of interest, which is an important topic from both theoretical and experimental perspectives.

Note added. Recently, we noticed that a similar model is considered by Diósi [27].

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[1] Review article by L. Carr, D. DeMille, R. Krems, and J. Ye, New. J. Phys. **11**, 055049 (2009).
[2] Review articles by T. J. Kippenberg, and K. J. Vahala, Science **321**, 1172 (2008); and by F. Marquardt, and S. M. Girvin, Physics **2**, 40 (2009).
[3] R. Feynman and F. Vernon, Ann. Phys. (NY) **24**, 118 (1963).
[4] L. Diósi and W. Strunz, Phys. Lett. A **235**, 569 (1997).
[5] L. Diosi, N. Gisin, and W. T. Strunz, Phys. Rev. A **58**, 1699 (1998).
[6] W. T. Strunz, L. Diosi, and N. Gisin, Phys. Rev. Lett. **82**, 1801 (1999).
[7] L. Diósi, Phys. Rev. Lett. **100**, 080401(2008); **101**, 149902(E) (2008).
[8] H. M. Wiseman and J. M. Gambetta, Phys. Rev. Lett. **101**, 140401 (2008).
[9] Refer to Sec. 5.3.5 in C. W. Gardiner, *Quantum Noise* (Springer-Verlag, Berlin, 1991).
[10] The probe field serves as the Heisenberg cut, essential for unequivocally interpreting measurement results.
[11] A. Imamoğlu, Phys. Rev. A **50**, 3650 (1994).
[12] L. Mazzola, S. Maniscalco, J. Piilo, K.A. Suominen, and B.M. Garraway, Phys. Rev. A **80**, 012104 (2009).
[13] H. J. Carmichael, *An Open Systems Approach to Quantum Optics* (Springer, Berlin, 1993).
[14] H. M. Wiseman and G. J. Milburn, Phys. Rev. A **47**, 642 (1993).
[15] T. Yu, L. Diosi, N. Gisin, and W. T. Strunz, Phys. Rev. A **60**, 91 (1999).

[16] H. P. Breuer and F. Petruccione, *The Theory of Open Quantum Systems* (Oxford University Press, Oxford, 2007).
[17] F. Marquardt, J. P. Chen, A. A. Clerk, S. M. Girvin, Phys. Rev. Lett. **99**, 093902 (2007).
[18] I. Wilson-Rae, N. Nooshi, W. Zwerger, and T. J. Kippenberg, Phys. Rev. Lett. **99**, 093901 (2007).
[19] C. Genes, D. Vitali, P. Tombesi, S. Gigan, and M. Aspelmeyer, Phys. Rev. A **77**, 033804 (2008).
[20] I. Martin and W. H. Zurek, Phys. Rev. Lett. **98**, 120401 (2007).
[21] J. Thompson *et al.*, Nature (London) **452**, 72 (2008).
[22] A. M. Jayich *et al.*, New J. Phys. **10**, 095008 (2008).
[23] H. Miao, S. Danilishin, T. Corbitt, and Y. Chen, Phys. Rev. Lett. **103**, 100402 (2009).
[24] LIGO Scientific Collaboration, Rep. Prog. Phys. **72**, 076901 (2009).
[25] V. B. Braginsky and F. Khalili, *Quantum Measurement* (Cambridge University Press, Cambridge, 1992).
[26] P. Purdue and Y. Chen, Phys. Rev. D **66**, 122004 (2002), and references therein.
[27] L. Diósi, Phys. Rev. A **85**, 034101 (2012).