SIAM J. MATH. ANAL.
Vol. 18, No. 3, May 1987

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HOMOCLINIC ORBITS IN SLOWLY VARYING OSCILLATORS*

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Abstract. We obtain existence and bifurcation theorems for homoclinic orbits in three-dimensional flows that are perturbations of families of planar Hamiltonian systems. The perturbations may or may not depend explicitly on time. We show how the results on periodic orbits of the preceding paper are related to the present homoclinic results, and apply them to a periodically forced Duffing equation with weak feedback.

Key words. bifurcation, Hamiltonian system, homoclinic orbit, perturbation theory, Melnikov method

AMS(MOS) subject classifications. 34CXX, 58F14, 70KXX

1. Introduction. In the preceding paper we developed perturbation methods based on ideas of Melnikov [1963] that permit us to approximate Poincaré maps for autonomous and periodically forced slowly varying oscillators, the flows of which are close to those of families of planar Hamiltonian systems. We obtained existence, stability and bifurcation results for periodic orbits in such systems. In the present paper we extend these results to deal with homoclinic orbits and show how the periodic results are related to them.

In § 2 we outline the geometry of the phase space and we describe basic perturbation results. The computational tools and existence and bifurcation theorems are developed in §§ 3 and 4, and the relationship between periodic and homoclinic orbits is discussed in § 5. The example and conclusions follow in §§ 6 and 7.

2. Structure of the phase space. As in Wiggins and Holmes [1987] we will consider systems of the form

$$\left.\begin{aligned} \dot{x} &= f_1(x, y, z) + \varepsilon g_1(x, y, z, t, \boldsymbol{\mu}) \\ \dot{y} &= f_2(x, y, z) + \varepsilon g_2(x, y, z, t; \boldsymbol{\mu}) \\ \dot{z} &= \varepsilon g_3(x, y, z, t; \boldsymbol{\mu}) \end{aligned}\right\} \quad \text{or} \quad \dot{\mathbf{q}} = \mathbf{f}(\mathbf{q}) + \varepsilon \mathbf{g}_{\boldsymbol{\mu}}(\mathbf{q}, t), \tag{2.1}$$

with $0 < \varepsilon \ll 1$, $\mathbf{f}$ and $\mathbf{g}$ sufficiently smooth (C^r, $r \geqq 2$), $\mathbf{g}$ periodic in t with period T and $\boldsymbol{\mu} \in R^k$ a vector of parameters. We will write $\mathbf{g}(\mathbf{q}, t; \boldsymbol{\mu}) = \mathbf{g}_{\boldsymbol{\mu}}(\mathbf{q}, t)$ and frequently drop the explicit dependence on $\boldsymbol{\mu}$. We make the following assumptions on the unperturbed system:

(A1) For $\varepsilon = 0$, (2.1) reduces to a one-parameter family of planar Hamiltonian systems with Hamiltonian $H(x, y, z)$:

$$\begin{aligned} \dot{x} &= f_1(x, y, z) = \frac{\partial H}{\partial y}, \\ \dot{y} &= f_2(x, y, z) = -\frac{\partial H}{\partial x}, \\ \dot{z} &= 0. \end{aligned} \tag{2.2}$$

* Received by the editors June 1, 1985; accepted for publication (in revised form) April 22, 1986. This research was supported in part by the Air Force Office of Scientific Research under AFSOR 84-0051.

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(A2) For each value of z in some open interval $J \subseteq \mathbb{R}$ the "planar" system (2.2) possesses a homoclinic orbit to a hyperbolic saddle point. Thus, when viewed in the full three-dimensional phase space, system (2.2) possesses a normally hyperbolic invariant one-dimensional manifold, $\mathcal{N}$, given by the union of saddle points of the one-parameter family of planar systems. $\mathcal{N}$ has two-dimensional stable and unstable manifolds (denoted by $W^s(\mathcal{N})$, $W^u(\mathcal{N})$, respectively), such that their intersection $W^s(\mathcal{N}) \cap W^u(\mathcal{N}) \stackrel{\text{def}}{=} \Gamma$ is made up of the union of the homoclinic orbits of the one-parameter family of planar systems. Henceforth we assume that $\mathcal{N}$ is connected; if not, the theory is applied separately to each connected component of $\mathcal{N}$.

(A3) The interior of Γ contains a two-parameter family of periodic orbits, which we denote by $\mathbf{q}_0^{\alpha,z}(t-\theta)$ for $z \in J$ and $\alpha \in L(z)$, where for each $z \in J$, $L(z)$ is an open interval in R. We denote $L(z)$ by $(\alpha(z), \alpha_0(z))$ and assume that $\lim_{\alpha \to \alpha_0} T(\alpha, z) = \infty$, where $T(\alpha, z)$ denotes the period of $\mathbf{q}_0^{\alpha,z}(t-\theta)$ and that $T(\alpha, z)$ is a differentiable function of α and z with $dT(\alpha, z)/d\alpha \neq 0$ for $(\alpha, z) \in (L(z), J)$.

Note that the assumptions of Wiggins and Holmes [1987] are included in the above. As before we suspend (2.1) over the space $R^3 \times S^1$ where $S^1 = R/T$ is the circle of length T by defining the function $\phi(t) = t$, mod T and then by T-periodicity of the g_i we have

$$\begin{aligned} \dot{x} &= f_1(x, y, z) + \varepsilon g_1(x, y, z, \phi; \boldsymbol{\mu}), \\ \dot{y} &= f_2(x, y, z) + \varepsilon g_2(x, y, z, \phi; \boldsymbol{\mu}), \\ \dot{z} &= \varepsilon g_3(x, y, z, \phi; \boldsymbol{\mu}), \\ \dot{\phi} &= 1. \end{aligned} \qquad (x, y, z, \phi) \in R^3 \times S^1, \tag{2.3}$$

Again we note that this suspension makes sense even when the g_i are independent of ϕ. At $\varepsilon = 0$, for the suspended system we denote the normally hyperbolic invariant set by $\mathcal{M} \equiv (\mathcal{N}, \phi) = \mathcal{N} \times S^1$. See Fig. 1.

For computations it is convenient to have $\mathcal{M}$ in an explicit form. Recall from assumption (A2) that $\mathcal{N}$ is a one-manifold of equilibrium points for the unperturbed system such that on each $z =$ constant plane the equilibrium point of the associated planar system is hyperbolic. Since we have assumed that the unperturbed vector field is Hamiltonian, a simple computation of the eigenvalues of the linearized vector field at this point shows that $\partial(f_1, f_2)/\partial(x, y) < 0$. Thus, by the implicit function theorem,

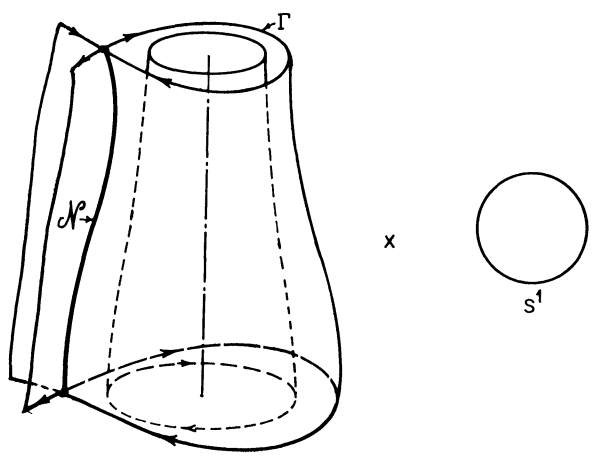


FIG. 1. *The unperturbed phase space.*

$\mathcal{M}$ can be represented as a graph over the z variables:

$$\mathcal{M}=\left\{(\gamma(z),\phi)\,\middle|\,\gamma(z)=(x(z),y(z),z), f_1(x,y,z)=f_2(x,y,z)=0,\ \left.\frac{\partial(f_1,f_2)}{\partial(x,y)}\right|_{\gamma(z)}<0,\ \phi\in S^1,\ z\in J\right\}. \tag{2.4}$$

The following results give us information about the perturbed phase space.

PROPOSITION 2.1. *There exists $\varepsilon_0>0$ such that for $0<\varepsilon<\varepsilon_0\ll 1$ there exists a normally hyperbolic invariant one-manifold*

$$\mathcal{M}_\varepsilon=\{(\gamma(z,\phi;\varepsilon),\phi)=(\gamma(z)+\mathcal{O}(\varepsilon),\phi)\,|\,\phi\in S^1,\ z\in J\}, \tag{2.5}$$

where $\gamma(z,\phi;\varepsilon)$ is a C^r function of z and ε. Moreover, $\mathcal{M}_\varepsilon$ has local stable and unstable manifolds. $W^s_{\text{loc}}(\mathcal{M}_\varepsilon)$, $W^u_{\text{loc}}(\mathcal{M}_\varepsilon)$, which are C^r-close to the local stable and unstable manifolds of $\mathcal{M}$, denoted by $W^s_{\text{loc}}(\mathcal{M})$ and $W^u_{\text{loc}}(\mathcal{M})$, respectively.

Remark. $\mathcal{M}_\varepsilon$ is an invariant manifold in the weaker sense in that solutions may leave $\mathcal{M}_\varepsilon$ by virtue of their z values crossing the boundary of J. This will occur on a time scale $\mathcal{O}(1/\varepsilon)$ since motion along $\mathcal{M}_\varepsilon$ has a speed $\mathcal{O}(\varepsilon)$.

Proof. The existence of $\mathcal{M}_\varepsilon$, $W^s_{\text{loc}}(\mathcal{M}_\varepsilon)$, and $W^u_{\text{loc}}(\mathcal{M}_\varepsilon)$ follows from the persistence of normally hyperbolic invariant sets and their stable and unstable manifolds (see Hirsch, Pugh and Shub [1977] or Fenichel [1971]), with some slight technical modifications. The usual theorem requires $\mathcal{M}$ to be compact and boundaryless. However, there are two ways to get around these requirements. One, due to Robinson [1983], involves mapping $\mathcal{M}$ into a compact space (e.g. a sphere) and smoothly extending the vector field to a neighborhood of $\mathcal{M}$ via the use of bump functions, the conclusions then follow from the Hirsch, Pugh and Shub [1977] theory. The second method is due to Kopell [1985] and involves the use of the invariant manifold theory of Fenichel [1971]. Briefly, the vector field on the boundary of $\mathcal{M}$ is zero, $\mathcal{M}$ is then perturbed in a neighborhood of its boundary via a bump function in such a manner that it becomes "overflowing" (resp. "underflowing") invariant (see Fenichel [1971] for precise definitions). The existence of the perturbed manifold and its local unstable (resp. stable) manifold then follows from the Fenichel invariant manifold theorem. Furthermore, the modification of the vector field near the boundary of $\mathcal{M}$ does not affect the dynamics of our original system in the sense that, although now $\mathcal{M}_\varepsilon$ and $W^{s,u}_{\text{loc}}(\mathcal{M}_\varepsilon)$ may depend on the specific modification, asymptotic expansions of these manifolds agree to all orders for arbitrary modifications (Kopell [1985]). (Note: the situation is the same as that which arises in applications of center manifold theory, where the nonuniqueness of the center manifold does not effect recurrent motions.) □

On $\mathcal{M}$ all points are fixed, there is no motion. However, on $\mathcal{M}_\varepsilon$ this need not be the case. The following result gives us information concerning the flow on $\mathcal{M}_\varepsilon$.

PROPOSITION 2.2. *Let $\overline{g_3(\gamma(z))}=1/T\int_0^T g_3(\gamma(z),\phi)\,d\phi$ and suppose there exists $z_0\in J$ such that $\overline{g_3(\gamma(z_0))}=0$, $(d/dz)\overline{g_3(\gamma(z_0))}\neq 0$. Then $(\gamma(z_0,\phi,\varepsilon),\phi)=(\gamma(z_0)+\mathcal{O}(\varepsilon),\phi)$ is a hyperbolic periodic orbit on $\mathcal{M}_\varepsilon$ with period T.*

Proof. This is a straightforward application of the averaging theorem (see Hale [1969] or Guckenheimer and Holmes [1983]) restricted to $\mathcal{M}_\varepsilon$. □

Remark. If g_3 is not explicitly time dependent, then $g_3\equiv\overline{g_3}$ and averaging is unnecessary, so the proof goes through without appeal to the averaging theorem.

In order to visualize the situation we take the following cross section to the flow induced by (2.3):

$$\Sigma^{t_0}=\{(x,y,z,\phi)\in R^3\times S^1\,|\,\phi=t_0\in[0,T)\}. \tag{2.6}$$

If there exists a hyperbolic periodic orbit on $\mathcal{M}_\varepsilon$ there are two possible situations (see Fig. 2).

In these pictures the solid lines are to be interpreted as initial conditions for solutions of the perturbed equation, while the dotted lines can be thought of as actual solutions of the unperturbed equation, since the unperturbed equation is autonomous.

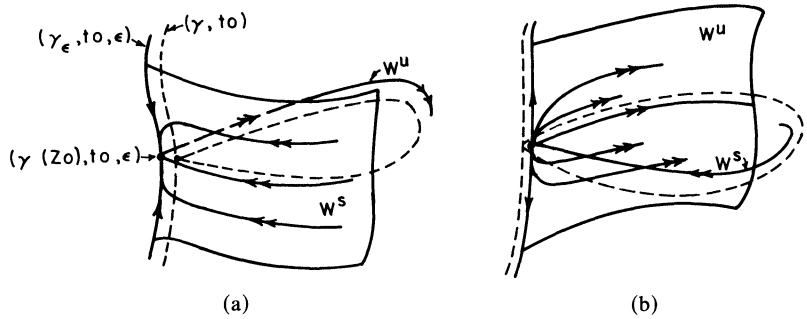


FIG. 2. *The perturbed manifolds.* (a) $\gamma(z_0)+\mathcal{O}(\varepsilon)$ *has* 1-*d unstable and* 2-*d stable manifolds.* (b) $\gamma(z_0)+\mathcal{O}(\varepsilon)$ *has* 1-*d stable and* 2-*d unstable manifolds.*

The following perturbation results allows us to approximate certain solutions in the stable and unstable manifolds of $(\gamma(z_0)+\mathcal{O}(\varepsilon), \phi)$ for arbitrarily long time intervals. This is necessary since we wish to find homoclinic, rather than periodic, orbits.

PROPOSITION 2.3. *Suppose there exists* $z_0 \in J$ *such that* $(\gamma(z_0, \phi; \varepsilon), \phi) = (\gamma(z_0)+\mathcal{O}(\varepsilon), \phi)$ *is a hyperbolic periodic orbit on* $\mathcal{M}_\varepsilon$. *Then, for each* ε *sufficiently small, there exists* $C>0$, $K=\{z\colon z_0 - C\varepsilon < z < z_0 + C\varepsilon\}$ *and solutions* $\mathbf{q}^s_\varepsilon(t, \theta)$, $\mathbf{q}^u_\varepsilon(t, \theta)$ *lying in the stable and unstable manifolds of* $(\gamma(z_0, \phi; \varepsilon), \phi)$ *with the following representations valid in the indicated time intervals*:

Case (a) $\dim W^s[(\gamma(z_0, \phi; \varepsilon), \phi)] = 3$, $\quad \dim W^u[(\gamma(z_0, \phi; \varepsilon), \phi)] = 2$:

$$\mathbf{q}^s_\varepsilon(t, \theta) = \mathbf{q}_0(t-\theta) + \varepsilon \mathbf{q}^s_1(t, \theta) + \mathcal{O}(\varepsilon^2), \qquad z^s_\varepsilon(t_0, \theta) \in K, \quad t \in [t_0, \infty).$$

$$\mathbf{q}^u_\varepsilon(t, \theta) = \mathbf{q}_0(t-\theta) + \varepsilon \mathbf{q}^u_1(t, \theta) + \mathcal{O}(\varepsilon^2), \qquad t \in (-\infty, t_0];$$

Case (b) $\dim W^s[(\gamma(z_0, \phi; \varepsilon), \phi)] = 2$, $\quad \dim W^u[(\gamma(z_0, \phi; \varepsilon), \phi)] = 3$:

$$\mathbf{q}^s_\varepsilon(t, \theta) = \mathbf{q}_0(t-\theta) + \varepsilon \mathbf{q}^s_1(t, \theta) + \mathcal{O}(\varepsilon^2), \qquad t \in [t_0, \infty),$$

$$\mathbf{q}^u_\varepsilon(t, \theta) = \mathbf{q}_0(t-\theta) + \varepsilon \mathbf{q}^u_1(t, \theta) + \mathcal{O}(\varepsilon^2), \qquad z^u_\varepsilon(t_0, \theta) \in K, \quad t \in (-\infty, t_0],$$

where $\mathbf{q}_0(t-\theta)$ *is the solution of the unperturbed equation that connects the point* $\gamma(z_0)$ *on* $\mathcal{M}$ *to itself, i.e., the homoclinic orbit on the* $z = z_0$ *plane.*

Proof. See Wiggins [1985]. □

Remarks. (1) $\mathbf{q}^s_1(t, \theta)$ and $\mathbf{q}^u_1(t, \theta)$ may be obtained by solving the first variational equations

$$\dot{\mathbf{q}}^s_1 = \mathbf{Df}(\mathbf{q}_0)\mathbf{q}^s_1 + \mathbf{g}(\mathbf{q}_0, t), \qquad t \in [t_0, \infty),$$

$$\dot{\mathbf{q}}^u_1 = \mathbf{Df}(\mathbf{q}_0)\mathbf{q}^u_1 + \mathbf{g}(\mathbf{q}_0, t), \qquad t \in (-\infty, t_0].$$

(2) We note that in the (suspended) unperturbed system, $\dim W^s[\gamma(z_0), \phi] = \dim W^u[(\gamma(z_0), \phi)] = 2$ and that for the perturbed system the dimensions of $W^s[(\gamma(z_0)+\mathcal{O}(\varepsilon), \phi)]$ or $W^u[(\gamma(z_0)+\mathcal{O}(\varepsilon), \phi)]$ may increase by one (Cases (a) and (b)). So in order to uniformly approximate solutions in the unperturbed manifolds by

solutions in the unperturbed manifolds for arbitrarily long time intervals, these solutions must initially start out close together. This is the reason for the requirements $z_\varepsilon^{s,u}(t_0, \theta) \in K$ in Cases (a) and (b).

(3) For the stable manifold in Case (a) and the unstable manifold in Case (b), the theorem does not tell us explicitly which solution in the manifolds we are on, only that all solutions with initial z values in K are approximated uniformly by a corresponding solution to the unperturbed equation. Consequently we are no longer able to follow individual solutions in these manifolds during their time evolutions.

In § 5 we will be concerned with periodic orbits limiting on Γ. In this situation we need to approximate perturbed solutions arbitrarily close to Γ by unperturbed periodic orbits so we need some kind of control on the flow on $\mathcal{M}_\varepsilon$.

PROPOSITION 2.4. *Let $(\gamma(z_0)+\mathcal{O}(\varepsilon), \phi)$ be a hyperbolic periodic orbit on $\mathcal{M}_\varepsilon$ and let $\mathbf{q}^{\alpha,z_0}(t-\theta)$ be a periodic orbit of the unperturbed system with period $T(\alpha, z_0)$. Then there exists a perturbed orbit $\mathbf{q}_\varepsilon^{\alpha,z_0}(t, \theta)$, not necessarily periodic, which can be expressed as*

$$\mathbf{q}_\varepsilon^{\alpha,z_0}(t, \theta) = \mathbf{q}_0^{\alpha,z_0}(t-\theta) + \varepsilon \mathbf{q}_1^{\alpha,z_0}(t, \theta) + \mathcal{O}(\varepsilon^2)$$

uniformly in $t \in [t_0, t_0 + T(\alpha, z_0)]$, for ε sufficiently small and all $\alpha \in L(z_0)$.

Proof. See Wiggins [1985]. □

We will remark that Proposition 2.4 only allows us to approximate perturbed orbits by unperturbed orbits for one passage through a neighborhood of $\mathcal{M}_\varepsilon$. This is due to the fact that orbits take arbitrarily long to pass through the neighborhood and therefore the slightest error may be magnified greatly over the long time of passage. Consequently for periodic orbits near Γ we are limited to the study of resonant orbits satisfying $mT = nT(\alpha, z_0)$, $n = 1$. However, since we can pick $T(\alpha, z_0)$ at will, and by (A3) $T \to \infty$ as $\alpha \to \alpha_0$, m can be arbitrarily large.

3. Existence of homoclinic orbits. We now turn our attention to the homoclinic manifold Γ. By Proposition 2.3, in order to approximate to $\mathcal{O}(\varepsilon)$ orbits in the stable and unstable manifolds of $\mathcal{M}_\varepsilon$ by orbits in the stable and unstable manifolds of $\mathcal{M}$ it is necessary that there exist a point $z_0 \in J$ such that $(\gamma(z_0, \phi; \varepsilon), \phi)$ is a hyperbolic periodic orbit on $\mathcal{M}_\varepsilon$. Now a hyperbolic periodic orbit on $\mathcal{M}_\varepsilon$ will have either a three-dimensional stable manifold and a two-dimensional unstable manifold or vice versa. Thus in the four-dimensional phase space we expect the intersection to be generically one-dimensional. In measuring distances between manifolds of solutions in phase space it is only necessary to explore the directions transverse to the manifolds, so the number of measurements necessary in order to determine whether or not the manifolds intersect should be equal to the minimum codimension of the manifolds. In our case that number is one and we expect a single (scalar) measurement to suffice.

Now on the cross section Σ^0 the hyperbolic periodic orbit for the flow, $(\gamma(z_0, \phi; \varepsilon), \phi)$, corresponds to a hyperbolic fixed point, $\gamma(z_0)+\mathcal{O}(\varepsilon)$, for the Poincaré map P_ε, which has either a two-dimensional stable manifold and a one-dimensional unstable manifold (Case (a) of Proposition 2.3) or a one-dimensional stable manifold and a two-dimensional unstable manifold (Case (b) of Proposition 2.3). For definiteness, in the following we assume that Case (a) holds, since the argument and conclusions for Case (b) are identical.

We develop a measure for the distance between the stable manifold $W^s(\gamma(z_0)+\mathcal{O}(\varepsilon))$ and the unstable manifold $W^u(\gamma(z_0)+\mathcal{O}(\varepsilon))$ on the cross-section Σ^0. Let $\alpha_0 = \alpha(z_0)$ denote the value of α on the $z = z_0$ level that corresponds to the unperturbed homoclinic orbit on that z-level, and denote this orbit $\mathbf{q}_0(t-\theta)$, where we have dropped the explicit (α, z) dependence for ease of notation.

At the point $\mathbf{q}_0(-\theta)$ on the cross-section Σ^0 we consider the plane Π normal to the vector $\mathbf{f}(\mathbf{q}_0(-\theta))$. There exists a unique point $\mathbf{q}_\varepsilon^u(-\theta)$ in $W^u(\gamma(z_0)+\mathcal{O}(\varepsilon))\cap\Pi$ which is "closest" to $\gamma(z_0)+\mathcal{O}(\varepsilon)$ in the sense of elapsed time for a solution leaving the neighborhood of $\gamma(z_0)+\mathcal{O}(\varepsilon)$. Similarly, there exists a curve on the plane Π, namely the intersection $W^s(\gamma(z_0)+\mathcal{O}(\varepsilon))\cap\Pi$, which is closest to $\gamma(z_0)+\mathcal{O}(\varepsilon)$ in the sense of elapsed time. We choose the unique point $\mathbf{q}_\varepsilon^s(0,\theta)$ on this curve such that $\mathbf{q}_\varepsilon^u(0,\theta)-\mathbf{q}_\varepsilon^s(0,\theta)$ is parallel to $(-f_2(\mathbf{q}_0(-\theta)), f_1(\mathbf{q}_0(-\theta)), 0)$. Thus we require $z_1^s(0,\theta)=z_1^u(0,\theta)$. We are guaranteed that such a choice of points can be made for each θ by Proposition 2.3 which says that the local perturbed manifolds are C^r ε-close to the local unperturbed manifolds in a neighborhood of $(\gamma(z_0)+O(\varepsilon), \phi)$. Thus their tangent spaces are ε-close. Outside of this neighborhood, solutions remain ε-close to unperturbed solutions for finite times, hence their maximum movement in the z-direction is $\mathcal{O}(\varepsilon)$. See Fig. 3.

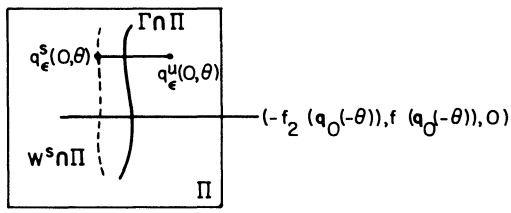


FIG. 3. *Intersections of the stable and unstable manifolds with* Π.

Clearly $|\mathbf{q}_\varepsilon^u(0,\theta)-\mathbf{q}_\varepsilon^s(0,\theta)|$ is a measure of the distance between $W^s(\gamma(z_0)+\mathcal{O}(\varepsilon))$ and $W^u(\gamma(z_0)+\mathcal{O}(\varepsilon))$. However, for easier computation and in order to account for the relative orientations between $W^s(\gamma(z_0)+\mathcal{O}(\varepsilon))$ and $W^u(\gamma(z_0)+\mathcal{O}(\varepsilon))$, we prefer to use the following distance measurement:

$$\begin{aligned} d(\alpha_0,\theta,z_0) &= \frac{(\partial H/\partial x(\mathbf{q}_0(-\theta)), \partial H/\partial y(\mathbf{q}_0(-\theta)), 0)\cdot(\mathbf{q}_\varepsilon^u(0,\theta)-\mathbf{q}_\varepsilon^s(0,\theta))}{\|\mathbf{f}(\mathbf{q}_0(-\theta))\|} \\ &= \frac{\varepsilon[(\partial H/\partial x(\mathbf{q}_0(-\theta)), \partial H/\partial y(\mathbf{q}_0(-\theta)), 0)\cdot(\mathbf{q}_1^u(0,\theta)-\mathbf{q}_1^s(0,\theta))]}{\|\mathbf{f}(\mathbf{q}_0(-\theta))\|}+\mathcal{O}(\varepsilon^2) \qquad (3.1) \\ &\overset{\text{def}}{=} \varepsilon\frac{M(\theta)}{\|\mathbf{f}(\mathbf{q}_0(-\theta))\|}+\mathcal{O}(\varepsilon^2) \end{aligned}$$

where "$\cdot$" is the usual vector dot product, $\|\cdot\|$ is the Euclidean norm, and $M(\theta)$ is defined to be the homoclinic *Melnikov function.*

We now develop a computable expression for $M(\theta)$. Recall that geometrically $M(\theta)$ is the lowest order term in an asymptotic expansion for the distance between the stable and unstable manifolds of a hyperbolic fixed point of a Poincaré map. We shall derive and solve a simple differential equation for a time dependent version of M, as in the standard planar Melnikov calculation.

Letting

$$\begin{aligned} \Delta(t,\theta) &= f_1(\mathbf{q}_0(-\theta))(y_1^u(t,\theta)-y_1^s(t,\theta))-f_2(\mathbf{q}_0(t-\theta))(x_1^u(t,\theta)-x_1^s(t,\theta)) \qquad (3.2) \\ &\overset{\text{def}}{=} \Delta^u(t,\theta)-\Delta^s(t,\theta) \end{aligned}$$

we compute

$$\begin{aligned}\dot{\Delta}^u(t,\theta)=&\left(\frac{\partial f_1}{\partial x}(\mathbf{q}_0(t-\theta))+\frac{\partial f_2}{\partial y}(\mathbf{q}_0(t-\theta))\right)\Delta^u(t,\theta)\\&+f_1(\mathbf{q}_0(t-\theta))g_2(\mathbf{q}_0(t-\theta,t))-f_2(\mathbf{q}_0(t-\theta))g_1(\mathbf{q}_0(t-\theta,t))\\&+\left[f_1(\mathbf{q}_0(t-\theta))\frac{\partial f_2}{\partial z}(\mathbf{q}_0(t-\theta))\right.\\&\qquad\left.-f_2(\mathbf{q}_0(t-\theta))\frac{\partial f_1}{\partial z}(\mathbf{q}_0(t-\theta))\right]z_1^u(t,\theta).\end{aligned}\tag{3.3}$$

Henceforth we suppress the arguments of the f_i, g_i and their partial derivatives. We have assumed the unperturbed vector field to be Hamiltonian, so that $(\partial f_1/\partial x)+(\partial f_2/\partial y)=0$, and (3.3) becomes

$$\dot{\Delta}^u(t,\theta)=f_1g_2-f_2g_1+\left(f_1\frac{\partial f_2}{\partial z}-f_2\frac{\partial f_1}{\partial z}\right)z_1^u(t,\theta),\tag{3.4}$$

where $z_1^u(t,\theta)$ is obtained by solving the z component of the first variational equation:

$$\dot{z}_1^u(t,\theta)=g_3(\mathbf{q}_0(t-\theta),t),\qquad t\in(-\infty,0].\tag{3.5}$$

Equation (3.4) can be integrated immediately to give

$$\Delta^u(0,\theta)-\Delta^u(-\infty,0)=\int_{-\infty}^{0}\left[f_1g_2-f_2g_1+\left(f_1\frac{\partial f_2}{\partial z}-f_2\frac{\partial f_1}{\partial z}\right)z_1^u(t,\theta)\right]dt.\tag{3.6}$$

Similarly, we obtain an expression for $\dot{\Delta}^s(t,\theta)$, which leads to

$$\Delta^s(\infty,\theta)-\Delta^s(0,\theta)=\int_{0}^{\infty}\left[(f_1g_2-f_2g_1)+\left(f_1\frac{\partial f_2}{\partial z}-f_2\frac{\partial f_1}{\partial z}\right)z_1^s(t,\theta)\right]dt.\tag{3.7}$$

Now $\Delta^s(\infty,\theta)$ and $\Delta^u(-\infty,\theta)$ are both zero since $\mathbf{q}_1^{u,s}(t,\theta)$ is bounded for all time (Proposition 2.3) and the unperturbed vector field goes to zero exponentially fast as $\gamma(z_0)$ is approached. Similarly, the improper integrals converge and we have

$$\begin{aligned}M(\theta)=&\int_{-\infty}^{-\infty}(f_1g_2-f_2g_1)\,dt+\int_{-\infty}^{0}\left(f_1\frac{\partial f_2}{\partial z}-f_2\frac{\partial f_1}{\partial z}\right)z_1^u(t,\theta)\,dt\\&+\int_{0}^{\infty}\left(f_1\frac{\partial f_2}{\partial z}-f_2\frac{\partial f_1}{\partial z}\right)z_1^s(t,\theta)\,dt.\end{aligned}\tag{3.8}$$

Since we assume that the unperturbed vector field is Hamiltonian, the reader can easily verify that

$$f_1\frac{\partial f_2}{\partial z}-f_2\frac{\partial f_1}{\partial z}=\frac{\partial H}{\partial y}\frac{\partial^2 H}{\partial x\,\partial z}+\frac{\partial H}{\partial x}\frac{\partial^2 H}{\partial y\,\partial x}=-\frac{d}{dt}\left(\frac{\partial H}{\partial z}\right)+\frac{\partial^2 H}{\partial z^2}\dot{z},\tag{3.9}$$

and that

$$\left(f_1\frac{\partial f_2}{\partial z}-f_2\frac{\partial f_1}{\partial z}\right)(\mathbf{q}_0(t-\theta))=-\frac{d}{dt}\left(\frac{\partial H}{\partial z}(\mathbf{q}_0(t-\theta))\right),\tag{3.10}$$

since $z=$ constant on an unperturbed orbit. Using this fact and integrating by parts

once, we find

$$\int_{-\infty}^{0}\left(f_1\frac{\partial f_2}{\partial z}-f_2\frac{\partial f_1}{\partial z}\right)(\mathbf{q}_0(t-\theta),t)z_1^u(t,\theta)\,dt$$

$$(3.11)\qquad =\frac{\partial H}{\partial z}(\mathbf{q}_0(-\infty))z_1^u(-\infty,\theta)-\frac{\partial H}{\partial z}(\mathbf{q}_0(-\theta))z_1^u(0,\theta)$$

$$+\int_{-\infty}^{0}\frac{\partial H}{\partial z}(\mathbf{q}_0(t-\theta))g_3(\mathbf{q}_0(t-\theta),t)\,dt,$$

and similarly

$$\int_{0}^{\infty}\left(f_1\frac{\partial f_2}{\partial z}-f_2\frac{\partial f_1}{\partial z}\right)(\mathbf{q}_0(t-\theta))z_1^s(t,\theta)\,dt$$

$$(3.12)\qquad =\frac{\partial H}{\partial z}(\mathbf{q}_0(-\theta))z_1^s(0,\theta)-\frac{\partial H}{\partial z}(\mathbf{q}_0(\infty))z_1^s(\infty,\theta)$$

$$+\int_{0}^{\infty}\frac{\partial H}{\partial z}(\mathbf{q}_0(t-\theta))g_3(\mathbf{q}_0(t-\theta),t),\,dt.$$

Thus we have

$$(3.13)\qquad \begin{aligned}M(\theta)=&\int_{-\infty}^{+\infty}\left(f_1g_2-f_2g_1+\frac{\partial H}{\partial z}g_3\right)(\mathbf{q}_0(t-\theta),t)\,dt+\frac{\partial H}{\partial z}(\mathbf{q}_0(-\infty))z_1^u(-\infty,\theta)\\&-\frac{\partial H}{\partial z}(\mathbf{q}_0(\infty))z_1^s(\infty,\theta)+\frac{\partial H}{\partial z}(\mathbf{q}_0(-\theta))(z_1^s(0,\theta)-z_1^u(0,\theta)).\end{aligned}$$

We note that $(\partial H/\partial z)(\mathbf{q}_0(-\infty))=(\partial H/\partial z)(\mathbf{q}_0(\infty))$, since the unperturbed orbit approaches $\gamma(z_0)$ for $t\to\pm\infty$, and that $z_1^u(-\infty,\theta)$ and $z_1^s(\infty,\theta)$ converge to the saddle point on the section Σ^0. See Robinson [1985] or Wiggins [1985] for a discussion of this limit process. It follows that

$$(3.14)\qquad \frac{\partial H}{\partial z}(q(-\infty))z_1^u(-\infty,\theta)-\frac{\partial H}{\partial z}(q(\infty))z_1^s(\infty,\theta)=0,$$

and by our original choice of $q_\varepsilon^s(0,\theta)$ and $q_\varepsilon^u(0,\theta)$ we have $z_1^s(0,\theta)-z_1^u(0,\theta)=0$; thus we arrive at an expression for the Melnikov function:

$$(3.15)\qquad M(\theta)=\int_{-\infty}^{\infty}\left(f_1g_2-f_2g_1+\frac{\partial H}{\partial z}g_3\right)(\mathbf{q}_0(t-\theta),t)\,dt.$$

Finally, using $f_1=\partial H/\partial y$, $f_2=-\partial H/\partial x$, and transforming $t\to t+\theta$, (3.15) can be rewritten in the compact form

$$(3.16)\qquad M(\theta)=\int_{-\infty}^{\infty}(\nabla H\cdot g)(\mathbf{q}_0(t),t+\theta)\,dt.$$

Now on the cross-section Σ^0, the distance between the stable and unstable manifolds of $\gamma(z_0)+\mathcal{O}(\varepsilon)$ is measured by $d(\theta)=\varepsilon(M(\theta)/\|f(\mathbf{q}_0(-\theta))\|)+\mathcal{O}(\varepsilon^2)$, so if $M(\theta)$ has a simple zero at $\hat\theta$ ($M(\hat\theta)=0$, $(dM/d\theta)(\hat\theta)\neq 0$), then by the implicit function theorem, $d(\theta)$ also has a zero near $\hat\theta$. We have now proved the following.

THEOREM 3.1. *Suppose $M(\theta)$ has at least one simple zero; then for ε sufficiently small, near this point $W^s(\gamma(z_0)+\mathcal{O}(\varepsilon))$ and $W^u(\gamma(z_0)+\mathcal{O}(\varepsilon))$ intersect transversely. On the other hand, if $M(\theta)$ is bounded away from zero for all θ then $W^s(\gamma(z_0)+\mathcal{O}(\varepsilon))\cap W^u(\gamma(z_0)+\mathcal{O}(\varepsilon))=\varnothing$.*

This result is important since it allows us to test for transverse homoclinic points in the Poincaré map of specific differential equations. Thus, by the Smale–Birkhoff homoclinic theorem, we know that some iterate of the Poincaré map, $(P_\varepsilon)^N$, has an invariant hyperbolic set near such a point; i.e., a Smale horseshoe with its attendant chaotic dynamics (Guckenheimer and Holmes [1983, Chapter 5]).

Remark. In the formulation (3.14) $f_1g_2-f_2g_1 \stackrel{\text{def}}{=} (f\wedge g)_{1,2}$ is the usual planar Melnikov function, while $(\partial H/\partial z)g_3$ is the additional contribution due to the slow variation of z. We note that analogous expressions involving such extra terms occur in multidegree of freedom Hamiltonian examples, of Koiller [1984], Holmes and Marsden [1983] and Holmes [1986]. Also see Robinson [1983] and Gruendler [1985]. We remark that, if g is a time-independent perturbation, then, as formulation (3.16) makes clear, $M=\int_{-\infty}^{\infty}(\nabla H\cdot g)(\mathbf{q}_0(t))\,dt$ is θ-independent and thus simple zeros cannot occur and transversal intersections cannot be found. This is not surprising, since in that case we have an *autonomous* three-dimensional vector field and $\gamma(z_0)+\mathcal{O}(\varepsilon)$ is a fixed point with a one-dimensional unstable manifold and a two-dimensional stable manifold (or vice versa). If such manifolds intersect, they necessarily do so along a solution curve and thus the intersection cannot be transversal (cf. Guckenheimer and Holmes [1983, § 1.8]). However, if g depends upon parameters, then such "autonomous" homoclinic orbits can occur naturally as a parameter varies (see Theorem 4.2 below).

4. Bifurcations. In this section we give two theorems relevant to the case where the slowly varying oscillator depends upon a parameter $\mu\in R$.

THEOREM 4.1. (*Nonautonomous*). *Consider system* (2.1) *depending on a scalar parameter* $\mu\in K$, *where* K *is some open interval in* R. *Suppose there exists a point* (θ_0,μ_0) *such that*

(a) $M(\theta_0,\mu_0)=0,$

(b) $\left.\dfrac{\partial M}{\partial\theta}\right|_{(\theta_0,\mu_0)}=0.$

(c) $\left.\dfrac{\partial^2 M}{\partial\theta^2}\right|_{(\theta_0,\mu_0)}\neq 0,$

(d) $\left.\dfrac{\partial M}{\partial\mu}\right|_{(\theta_0,\mu_0)}\neq 0.$

Then, for $\varepsilon\neq 0$ *sufficiently small, near* μ_0 *there is a bifurcation value* $\hat{\mu}$ *at which quadratic homoclinic tangencies occur.*

As we have noted, if g is time-independent then M is necessarily θ-independent. Hence hypothesis (c) of Theorem 4.1 cannot be satisfied, and Theorem 3.1 cannot be applied. However, in this case we have the following.

THEOREM 4.2. (*Autonomous*). *Consider system* (2.1), *where* $\mathbf{g}(\mathbf{q};\mu)$ *is time independent but depends on a scalar parameter* $\mu\in K\subseteq\mathbb{R}$. *Suppose there exists a point* $\mu_0\in K$ *such that*

(a) $M(\mu_0)=0,$

(b) $\left.\dfrac{\partial M}{\partial\mu}\right|_{\mu=\mu_0}\neq 0.$

Then, for $\varepsilon\neq 0$ *sufficiently small, near* μ_0 *there is a bifurcation value* μ *at which* (*nontransverse*) *homoclinic orbits occur.*

Proofs. These two results are proved by straightforward Taylor series expansion of M about the point (θ_0, μ_0) (resp. μ_0) and application of the implicit function theorem. See Guckenheimer and Holmes [1983, § 4.5]. □

We remark that in the autonomous case it does not immediately follow that homoclinic orbits imply horseshoes, although that conclusion does follow for certain types of saddle-point with complex eigenvalues and in some cases with real eigenvalues (Silnikov [1965], [1967], [1970], Devaney [1976], Holmes [1980], Sparrow [1982]).

5. Interaction of periodic and homoclinic orbits. In the preceding paper (Wiggins and Holmes [1987]) we studied the two parameter family of periodic orbits inside Γ which remain bounded away from Γ, i.e., their periods were uniformly bounded above by some constant.

Now we will relate the periodic Melnikov theory to the homoclinic results of the present paper. From Wiggins and Holmes [1987, (2.7) and (3.11)-(3.19)], we have, omitting the arguments $\mathbf{q}_0^{\alpha,z}(t)$

$$\begin{aligned}\mathbf{M}^{m/n}(I,\theta,z) &= \left[\frac{1}{\Omega(1,Z)}\left\{\int_0^{mT}(\nabla H\cdot\mathbf{g})\,dt - \frac{\partial H}{\partial z}\bigg|_I \int_0^{mT} g_3\,dt\right\}, \int_0^{mT} g_3\,dt\right] \\ &\overset{\text{def}}{=} (M_1^{m/n}, M_3^{m/n}).\end{aligned} \tag{5.1}$$

In that paper (5.1) was obtained by a computation involving action angle variables; however, a cartesian (x, y, z) computation analogous to that of § 3 above yields precisely the same expression. To see this, refer to (3.13) of the present paper, use periodicity of $\mathbf{q}_0^{\alpha,z}(t)$, change the limits of integration to $-mT/2 \to mT/2$ and observe that $\Omega(I,z) = \|\mathbf{f}(\mathbf{q}_0^{\alpha,z}(-\theta))\|$. Finally, in dealing with periodic orbits the term $(\partial H/\partial z)(\mathbf{q}_0(-\infty))z_1^u(-\infty,\theta) - (\partial H/\partial z)(\mathbf{q}_0(\infty))z_1^s(\infty,\theta)$ becomes

$$\frac{\partial H}{\partial z}\bigg|_{I=\text{const}}\left(z_1\left(\frac{-mT}{2},\theta\right) - z_1\left(\frac{mT}{2},\theta\right)\right)\bigg| = \frac{-\partial H}{\partial z}\bigg|_I \int_{-mT/2}^{mT/2} g_3\,dt,$$

as required, and the other boundary term of (3.13) vanishes.

We now show that the limit of $\mathbf{M}^{m/1}$ exists on the homoclinic manifold and give an interpretation of that limit in terms of the dynamics of the slowly varying oscillators. It is clear that the homoclinic manifold cannot be approached in an arbitrary manner, since not all solutions in the perturbed manifolds can be uniformly approximated by solutions in the unperturbed manifolds for arbitrarily long time intervals: by Propositions 2.3 and 2.4 this can be done only for perturbed solutions with initial z values in $O(\varepsilon)$ neighborhood of a point $z_0 \in J$ such that $(\gamma(z_0) + O(\varepsilon), \phi)$ is a hyperbolic periodic orbit (or fixed point in the autonomous case) on M_ε. However, we have the following result.

PROPOSITION 5.1. *Suppose that there exists a point $z_0 \in J$ such that $(\gamma(z_0) + \mathcal{O}(\varepsilon), \phi)$ is a hyperbolic periodic orbit on $\mathcal{M}_\varepsilon$. Then*

$$(1) \qquad \lim_{\substack{m\to\infty \\ z\to z_0}} M_3^{m/1} = \int_{-\infty}^{\infty} g_3(q_0(t), t+\theta)\,dt = 0,$$

$$(2) \qquad \lim_{\substack{m\to\infty \\ z\to z_0}} M_1^{m/1} = \frac{1}{\|\mathbf{f}(\mathbf{q}_0(-\theta))\|}\int_{-\infty}^{\infty} (\nabla H\cdot\mathbf{g})(\mathbf{q}_0(t), t+\theta)\,dt = M(\theta),$$

where the integrands are evaluated on the unperturbed homoclinic orbit with $z = z_0$.

Proof. The proof of (1) follows by integrating the z_1 component of the first variational equation and examining the limiting behavior as $t \to \pm\infty$.

The proof of (2) involves a straightforward modification of arguments given in Theorem 4.6.4 of Guckenheimer and Holmes [1983]. □

This proposition shows that the periodic Melnikov functions have a meaning on the homoclinic manifold, although the dynamical interpretations are very different. Inside the homoclinic manifold, a simple zero of $(M_1^{m/1}, M_3^{m/1})$ implies the existence of an isolated periodic orbit of period mT. On the homoclinic manifold, a simple zero of M_3 implies a hyperbolic periodic orbit, $(\gamma(z_0)+\mathcal{O}(\varepsilon), \phi)$, on $\mathcal{M}_\varepsilon$ and a simple zero of M_1 implies a transversal intersection between the stable and unstable manifolds of $(\gamma(z_0)+\mathcal{O}(\varepsilon), \phi)$. However, note that Proposition 5.1 allows us to think of M_1 and M_3 as functions of (α, z, θ) with $\theta \in R$, $z \in J$ and $\alpha \in L(z) = [\alpha(z), \alpha_0(z)]$, where $\alpha_0(z)$ is the value of α which gives an orbit on the homoclinic manifold for that particular z-value.

We end this section by remarking on the case where the system (2.1) is autonomous. In this case the limits of integration for the subharmonic Melnikov vector are $-T(\alpha, z)/2 \to T(\alpha, z)/2$ where $T(\alpha, z)$ is the period of an unperturbed orbit. In showing that the subharmonic Melnikov vector has a meaning on the homoclinic manifold in this case we take limits as $\alpha \to \alpha_0$, $z \to z_0$, where α_0 is the value of α on the homoclinic manifold and z_0 is a z value such that $\gamma(z_0)+\mathcal{O}(\varepsilon)$ is a hyperbolic fixed point on $\mathcal{M}_\varepsilon$.

Now we will show that the hypotheses of the homoclinic bifurcation Theorems 4.1 and 4.2 also imply the existence of nearby families of periodic orbits, which converge to the homoclinic orbits as $\mu \to \hat{\mu}$.

$M^{m/1}(I, z, \mu)$ and $M^{m/1}(I, \theta, z, \mu)$ denote the autonomous and nonautonomous subharmonic Melnikov vectors respectively (we will drop the superscript $m/1$ for notational convenience), unless $(I(\theta), z)$ belongs to the homoclinic orbit, in which case they denote the homoclinic Melnikov functions of Proposition 5.1. In that case, the requirement of a hyperbolic set in $\mathcal{M}_\varepsilon$ fixes $z = z_0$ (via $\bar{g}_3(\gamma(z_0)=0)$ and $I = I_0$ is fixed by the unperturbed homoclinic orbit on the plane $z = z_0$. Thus, in § 4 we merely wrote $M(\mu)$ for $M_1(I_0, z_0, \mu)$ and $M(\theta, \mu)$ for $M_1(I_0, \theta, z_0, \mu)$.

We remark that we have replaced the parameter α with I, the action. This is convenient since we are interested in periodic orbits limiting on homoclinic orbits and is justifiable by Proposition 5.1 and the fact that I represents the *area* bounded by an orbit on a fixed z plane, and this area is defined even for the homoclinic orbit.

Our two main results are the following.

THEOREM 5.2 (*Autonomous*). *Consider the parametrized Melnikov functions* $\mathbf{M}(I, z, \mu)$ *for a parameter* $\mu \in R$. *Suppose there exists* $z_0 = z_0(\mu) \in J$ *such that* $\gamma(z_0(\mu), \mu)+\mathcal{O}(\varepsilon)$ *is a hyperbolic fixed point on* $\mathcal{M}_\varepsilon$ *for each* μ *in an open interval* K *containing a value* μ_0 *and let* $I = I_0$ *be the value of the action corresponding to the homoclinic orbit on the* $z = z_0(\mu)$ *level at which*

(a) $M_1(I_0, z_0, \mu_0) = 0,$

(b) $\dfrac{\partial M_1}{\partial \mu}(I_0, z_0, \mu_0) \neq 0,$

(c) $\dfrac{\partial g_1}{\partial x}(\gamma(z_0, \mu_0)) + \dfrac{\partial g_2}{\partial y}(\gamma(z_0, \mu_0)) \neq 0,$

(d) $\dfrac{\partial g_3}{\partial z}(\gamma(z_0, \mu_0)) \neq 0.$

Then for $\varepsilon \neq 0$ *sufficiently small the solutions of* (2.1) *contain a family* $\Lambda(\mu)$, *of periodic orbits* $(\mu \in K)$, *which converge on the homoclinic orbit with periods approaching infinity as* $\mu \to \hat{\mu} = \mu_0 + \mathcal{O}(\varepsilon)$, *where* $\hat{\mu}$ *is the homoclinic bifurcation value.*

THEOREM 5.3 (*Nonautonomous*). *Consider the parametrized Melnikov functions* $\mathbf{M}(I, \theta, z, \mu)$ *for a parameter* $\mu \in R$. *Suppose there exists* $z_0 = z_0(\mu) \in J$ *such that* $(\gamma(z_0(\mu), \mu) + \mathcal{O}(\varepsilon), \phi)$ *is a hyperbolic periodic orbit on* $\mathcal{M}_\varepsilon$ *for each* μ *in an open interval* K *containing a value* μ_0, *and let* $I = I_0$ *be the value of the action corresponding to the homoclinic orbit on the* $z = z_0(\mu)$ *level such that at the point* $(I_0, \theta_0, z_0(\mu_0), \mu_0)$ *we have*

(a) $$M_1(I_1, \theta_0, z_0(\mu_0), \mu_0) = 0,$$

(b) $$\frac{\partial M_1}{\partial \theta}(I_1, \theta_0, z_0(\mu_0), \mu_0) = 0,$$

(c) $$\frac{\partial^2 M_1}{\partial \theta^2}(I_1, \theta_0, z_0(\mu_0), \mu_0) \neq 0,$$

(d) $$\frac{\partial M_1}{\partial \mu}(I_1, \theta_0, z_0(\mu_0), \mu_0) \neq 0,$$

(e) $$\frac{\partial \bar{g}_3}{\partial z}(\gamma(z_0(\mu_0), \mu_0)) \neq 0.$$

Then, for $\varepsilon \neq 0$ *sufficiently small, the homoclinic bifurcation is a countable limit of subharmonic saddle-node bifurcations to higher and higher periods.*

Proofs. The proofs of Theorems 5.2 and 5.3 involve straightforward, though tedious, calculations with the Melnikov functions. The interested reader is referred to Wiggins [1985] for the details. These results generalize the autonomous planar homoclinic bifurcation theorems of Andronov et al. [1971] and the nonautonomous planar Melnikov [1963] methods of Greenspan and Holmes [1983]. We remark that the theorems can also be proved using the more "geometric" arguments of Silnikov [1965], [1967], [1970], in which a local analysis near the hyperbolic set γ is combined with a near identity global return map (cf. Guckenheimer and Holmes [1983, § 6.5]). □

6. An example. In this section we apply the theory developed above to the equation

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= x - x^3 - z - \varepsilon\delta y, \\ \dot{z} &= \varepsilon(\gamma x - \alpha z + \beta \cos t), \end{aligned} \tag{6.1}$$

which models a single degree of freedom nonlinear oscillator subject to weak linear damping and weak feedback control (Holmes and Moon [1983], Holmes [1983], [1985]). If $\beta = 0$, the system is autonomous and, for sufficiently strong damping ($\varepsilon\delta$) and feedback ($\varepsilon\gamma$) the feedback stabilizes the equilibrium position (0, 0, 0), which is a saddle-point for small $\varepsilon\gamma$. In Holmes [1985] local bifurcation results were obtained for a slight variant of this system and in Holmes [1983] an ad hoc perturbation method was used to argue that transverse homoclinic orbits would occur in the nonautonomous ($\beta \neq 0$) case. (The term $\beta \cos t$ represents a desired response characteristic, which is relayed to the system via the feedback loop). This example therefore illustrates both the autonomous and nonautonomous theories developed above.

The unperturbed Hamiltonian corresponding to (6.1) is

$$H(x, y; z) = \frac{y^2}{2} - \frac{x^2}{2} + \frac{x^4}{4} + xz, \tag{6.2}$$

and straightforward analysis reveals the unperturbed phase space structure sketched in Fig. 4. A hyperbolic manifold $\mathcal{N}$, given by $x=\bar{x}(z)$, $y=0$, where $\bar{x}$ is the intermediate size root of

$$x^3-x+z=0, \tag{6.3}$$

exists for $-2/3\sqrt{3}<z<2/3\sqrt{3}$. For each $z=z_0$ in this range, the Hamiltonian system $H(x, y; z_0)$ restricted to $z=z_0$ has a hyperbolic saddle-point $(\bar{x}(z_0), 0)$ with a "figure 8" double homoclinic loop enclosing two elliptic fixed points (the other two roots of (6.3)). For $|z|>2/3\sqrt{3}$ (6.3) has a single root and H has a single elliptic fixed point.

First we apply Propositions 2.1 and 2.2 to this system, and seek periodic orbits for the perturbed flow in the manifold $\mathcal{M}_\varepsilon=(\mathcal{N}, \phi)+\mathcal{O}(\varepsilon)$. This necessitates computation of

$$\begin{aligned}\overline{g_3(\gamma(z))}&=\frac{1}{2\pi}\int_0^{2\pi}(\gamma\bar{x}(z)-\alpha z+\beta\cos t)\,dt\\&=\gamma\bar{x}(z)-\alpha z.\end{aligned} \tag{6.4}$$

We note that the same result obtains for $\beta=0$ or $\beta\neq 0$. We also check that

$$\frac{\partial(f_1, f_2)}{\partial(x, y)}=3x^2-1 \tag{6.5}$$

is strictly negative on $\mathcal{N}$, since the middle root of (6.3) lies in the range $(-1/\sqrt{3}, 1/\sqrt{3})$ for $-2/3\sqrt{3}<z<2/3\sqrt{3}$.

From (6.4), for a zero of $\bar{g}_3$ we require $z=\gamma\bar{x}(z)/\alpha$, and thus, using (6.3), we must solve

$$x^3+\left(\frac{\gamma}{\alpha}-1\right)x=0, \qquad \Rightarrow x=0 \quad \text{or} \quad x=\pm\sqrt{1-\frac{\gamma}{\alpha}}, \qquad \frac{\gamma}{\alpha}<1. \tag{6.6}$$

Also, note that the derivative

$$\frac{d}{dz}\bar{g}_3=\gamma\bar{x}'-\alpha=\frac{\gamma}{1-3\bar{x}^2}-\alpha, \tag{6.7}$$

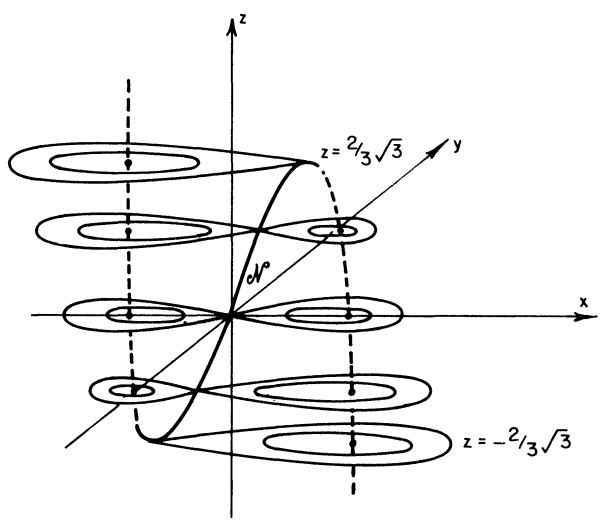


FIG. 4. *The unperturbed phase space of* (6.1).

takes the values $\gamma-\alpha$ for $x=0$ and $2\alpha(\alpha-\gamma)/(3\gamma-2\alpha)$ for $x=\pm\sqrt{1-\gamma/\alpha}$. Thus, for simple zeros we require $\gamma\neq\alpha$ at $x=0$ and $2\alpha/3<\gamma<\alpha$ at $x=\pm\sqrt{1-\gamma/\alpha}$. If these conditions are met, then Proposition 2.2 guarantees that the perturbed Poincaré map has hyperbolic fixed points near $(0,0,0)$ and $(\pm\sqrt{1-\gamma/\alpha},\ 0,\ \pm(\gamma/\alpha)\sqrt{1-\gamma/\alpha})$. Examination of the sign of $d\bar{g}_3/dz$ shows that the dynamics on $\mathcal{M}_\varepsilon$ is as sketched in Fig. 5.

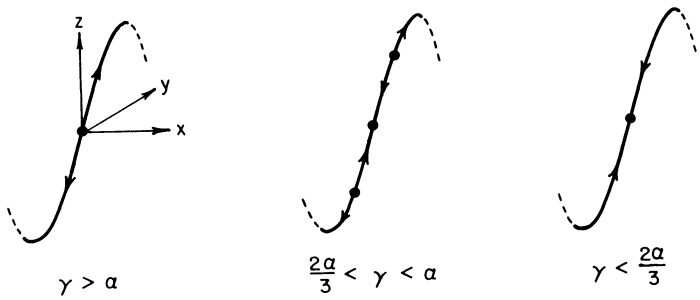


FIG. 5. *Dynamics on $\mathcal{M}_\varepsilon$, $\varepsilon\neq 0$.*

We next apply the perturbation theory of § 3, computing the Melnikov functions for $z=0$ and $z=\pm(\gamma/\alpha)\sqrt{1-\gamma/\alpha}$. The unperturbed solutions on these z-planes are given by

$$\mathbf{q}_0(t)=(x,y,z)(t)=(\pm\sqrt{2}\,\mathrm{sech}\,t,\ \mp\sqrt{2}\,\mathrm{sech}\,t\tanh t,\ 0), \tag{6.8}$$

$$\mathbf{q}_0^+(t)=\left(\frac{2cS+ab}{2bS-a},\ \frac{-2ad^3ST}{(2bS-a)^2},\ \frac{\gamma}{\alpha}\sqrt{1-\frac{\gamma}{\alpha}}\right) \tag{6.9a}$$

and

$$\mathbf{q}_0^-(t)=\left(\frac{2cS-ab}{2bS+a},\ \frac{2ad^3ST}{(2bS+a)^2},\ \frac{\gamma}{\alpha}\sqrt{1-\frac{\gamma}{\alpha}}\right), \tag{6.9b}$$

on the plane $z=\gamma\sqrt{1-\gamma/\alpha}/\alpha$. Here we define

$$\begin{gathered} a=\sqrt{\frac{2\gamma}{\alpha}},\quad b=\sqrt{1-\frac{\gamma}{\alpha}},\quad c=1-\frac{2\gamma}{\alpha},\quad d=\sqrt{\frac{3\gamma}{\alpha}-2}, \\ S=\mathrm{sech}\,(dt)\quad\text{and}\quad T=\tanh(dt). \end{gathered} \tag{6.10}$$

On $z=-\gamma\sqrt{1-\gamma/\alpha}/\alpha$ we have $\bar{\mathbf{q}}_0^+=-\mathbf{q}_0^+$ and $\bar{\mathbf{q}}_0^-=-\mathbf{q}_0^-$. Note that, as $\gamma/\alpha\to 1$, (6.9a, b) $\to$ (6.8).

The Melnikov integral is

$$\int_{-\infty}^{\infty}(\nabla H\cdot g)(\mathbf{q}_0(t),t+\theta)\,dt=\int_{-\infty}^{\infty}(-\delta y^2+\gamma x^2-\alpha xz+\beta x\cos(t+\theta))\,dt. \tag{6.11}$$

In the first case this yields

$$\begin{aligned} M&=\int_{-\infty}^{\infty}(-2\delta\,\mathrm{sech}^2 t\tanh^2 t+2\gamma\,\mathrm{sech}^2 t\pm\sqrt{2}\beta\,\mathrm{sech}\,t\cos(t+\theta))\,dt \\ &=-\frac{4\delta}{3}+4\gamma\pm\sqrt{2}\beta\pi\,\mathrm{sech}\left(\frac{\pi}{2}\right)\cos\theta. \end{aligned} \tag{6.12}$$

In the second case, on the plane $z=(\gamma/\alpha)\sqrt{(1-(\gamma/\alpha))}$, we obtain the expressions

$$
\begin{aligned}
M = &-4\delta\left[\frac{d^4}{3}+b^2d^2+\frac{\gamma bd}{\sqrt{2}\alpha}\left(\sin^{-1}\sqrt{\frac{2\alpha}{\gamma}}b\pm\frac{\pi}{2}\right)\right] \\
&+2\gamma\left[2d+\sqrt{2}b\left(\sin^{-1}\sqrt{\frac{2\alpha}{\gamma}}b\pm\frac{\pi}{2}\right)\right]\mp 2\sqrt{2}\pi\beta\frac{\sinh\left(1/d\,\sin^{-1}\sqrt{d\alpha/\gamma}\right)}{\sinh(\pi/d)}\cos\theta,
\end{aligned}
\tag{6.13}
$$

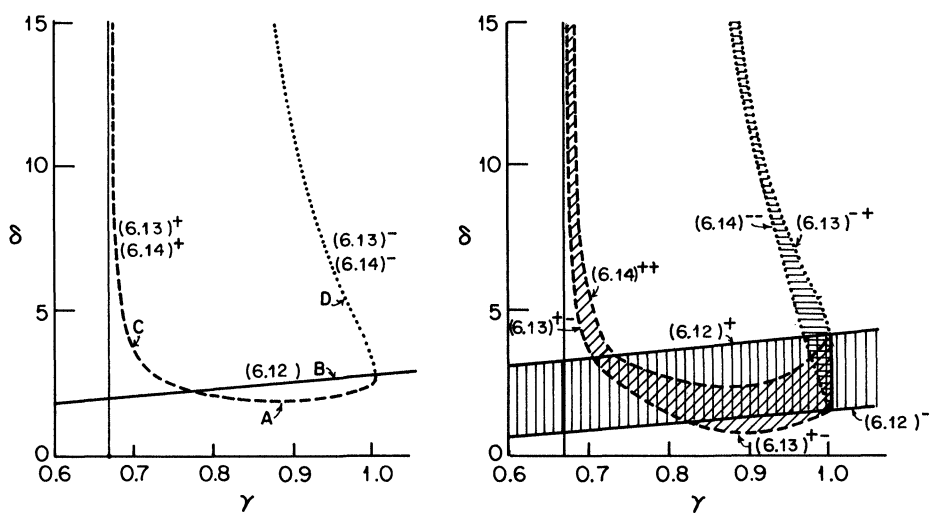
where the upper choice of sign refers to the larger homoclinic loop q_0^+ and the lower choice to the smaller loop, q_0^- (cf. Fig. 5). On $z=-(\gamma/\alpha)\sqrt{1-(\gamma/\alpha)}$, we find

$$
\begin{aligned}
M = &-4\delta\left[\frac{d^4}{3}+b^2d^2+\frac{\gamma bd}{\sqrt{2}\alpha}\left(\sin^{-1}\sqrt{\frac{2\alpha}{\gamma}}b\pm\frac{\pi}{2}\right)\right] \\
&+2\gamma\left[2d+\sqrt{2}b\left[\left(\sin^{-1}\sqrt{\frac{2\alpha}{\gamma}}b\pm\frac{\pi}{2}\right)\right]\pm 2\sqrt{2}\pi\beta\frac{\sinh\left(1/d\,\sin^{-1}\sqrt{d\alpha/\gamma}\right)}{\sinh(\pi/d)}\cos\theta.\right.
\end{aligned}
\tag{6.14}
$$

In all four cases the principal value $0\leqq\sin^{-1}(\cdot)\leqq\pi/2$ is to be taken. We note that, when $\gamma/\alpha=1$, so that $b=0$ and $d=1$, both (6.13) and (6.14) reduce to (6.12).

We present the bifurcation results that follow from these computations and Theorem 4.2, for the autonomous case, ($\beta=0$) in Fig. 6(a), where we show the bifurcation sets $M(\delta,\gamma)=0$ for fixed $\alpha=1$ computed from (6.12)-(6.14) using the definitions of a, b, c, d in (6.10). The linear set (6.12) $\delta=3\gamma$ and the two curves from $(6.13)^{\pm}$ are indicated on the figure. For $\beta=0$ $(6.14)^{\pm}$ gives curves coincident with those of $(6.13)^{\pm}$. Note that as $\gamma/\alpha\to 1^-$ (where the three fixed points on M_ε coalesce) all three curves meet, and also that the curves for the homoclinic orbits near $z=\pm(\gamma/\alpha)(1-(\gamma/\alpha))^{1/2}$ go to infinity as $\gamma\to 2/3^+$ (where the two nontrivial fixed points reach the boundary of M_ε). We remark that a branch of the curve labeled $(6.13)^-$ and $(6.14)^-$ has not been shown in Fig. 6(a) since it assumes δ values outside the range of our graph for γ values of physical interest. Figure 7 gives schematic phase portraits corresponding to parameter values labeled in Fig. 6(a).

In the nonautonomous case ($\beta\neq 0$) we see that the effect of the nonautonomous perturbation β cost is to open each of these curves into a band of width $\mathcal{O}(\beta)$ (see Fig. 6(b)). By Theorem 4.1 we conclude that quadratic homoclinic tangencies occur



(a) (b)

FIG. 6. Homoclinic bifurcation sets: (a) *autonomous*; (b) *nonautonomous.*

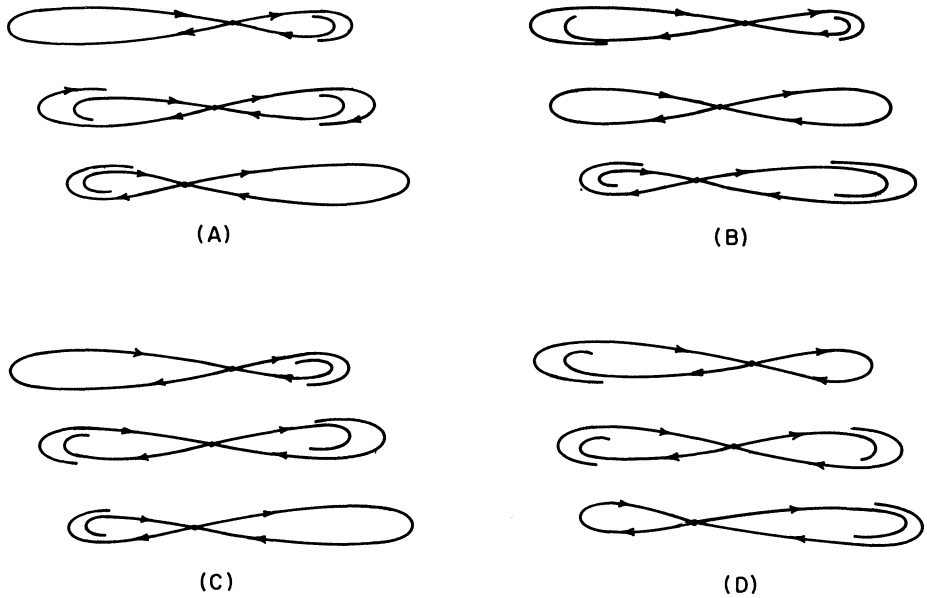


FIG. 7. *Phase portraits at points* A, B, C, D of *Fig.* 6(a).

on the boundaries of these bands with transversal intersections inside. Therefore, from Theorem 5.3 we know that the points on the boundaries of these bands are countable limits of saddle-node bifurcation points of periodic orbits to higher and higher periods. In this case, then, we have deterministic chaos for parameter values in the bands indicated in Fig. 6(b).

We conclude this section by remarking on the situation that occurs when the gain, γ, goes to zero. From (6.1) we see that the z component of the vector field decouples from the x and y components. Thus z can be solved for as an explicit function of time which is asymptotically periodic ($z \sim \varepsilon\beta \sin t + O(\varepsilon^2)$ as $t \to \infty$), and this solution can be substituted into the x and y components, resulting in an equation for a planar forced oscillator. Then one would expect to recover the usual Melnikov function for the equation

$$\dot{x} = y, \qquad \dot{y} = x - x^3 - \varepsilon[\beta \sin t + \delta y] + O(\varepsilon^2)$$

as studied by Greenspan and Holmes [1983], and inspection of (6.12) shows that this is indeed the case. In this respect, we note that the gain γ acts as a destabilizing influence resulting in the effective damping $((4\delta/3) - 4\gamma)$ in (6.12) in comparison with the term $4\delta/3$ in the uncoupled "planar" Duffing equation. Consequently the critical force level for the appearance of transverse homoclinic orbits and chaos is

$$\beta_{\text{crit}} = \frac{((4\delta/3) - 4\gamma)}{\sqrt{2}\pi} \cosh\left(\frac{\pi}{2}\right)$$

rather than

$$\beta_{\text{crit}} = \frac{4\delta}{3\sqrt{2}\pi} \cosh\left(\frac{\pi}{2}\right).$$

These results go some way in explaining the destabilizing effect of gain observed in

numerical integrations of this and similar systems by Moon (see Holmes and Moon [1983]).

7. Conclusions. In this and the preceding paper we have developed a global perturbation theory for slowly varying oscillators that collapse to one parameter families of Hamiltonian systems in the limit $\varepsilon = 0$. As such, they typically possess two parameter (α, z) families of periodic orbits and one parameter (z) families of homoclinic orbits to hyperbolic manifolds of equilibria. The perturbation theory we have developed uses these highly degenerate structures to seek isolated periodic and homoclinic orbits for $\varepsilon \neq 0$, small. We have given existence, stability and codimension one bifurcation theorems for periodic orbits in resonance with an external forcing and an existence theorem for transverse homoclinic orbits in the nonautonomous case and homoclinic bifurcation theorems for both cases. The hypotheses of the theorems can be checked explicitly in examples by computations involving integration around the unperturbed closed orbits. We have illustrated such computations with examples of a nonlinear oscillator subject to weak feedback control and external forcing.

In the interests of providing detailed results and specific applications, we have chosen to limit our analyses to three-dimensional systems, but we remark that the methods generalize in a natural way to systems in which x and y are each n-dimensional and z is m-dimensional: i.e., slowly varying perturbations of m-parameter families of n-degree of freedom Hamiltonians (see Wiggins [1986]).

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