J. Fluid Mech. (1986), *vol.* 172, *pp.* 531–546
Printed in Great Britain

The period-doubling of gravity–capillary waves

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(Received 20 June 1985 and in revised form 12 December 1985)

In this paper an attempt is made to explain the period-doubling of wind-generated gravity-capillary waves as observed in the experiment of Choi (1977). It is conjectured that period-doubling is closely related to the phenomenon of second-harmonic resonance. In order to obtain a simple dynamical model, results of McGoldrick (1970) and Simmons (1969) are extended to include the effect of wind input and shear in the current. For pure gravity–capillary waves (no wind, no current) the condition for energy transfer from the second harmonic to the fundamental wave of Chen & Saffman (1979) is recovered. We also discuss the effect of wind and we find that wind input gives rise to a very sudden period-doubling. Qualitative agreement with experiment is obtained.

1. Introduction

Currently there is much interest in the evolution of gravity–capillary waves because of the promising remote-sensing technique, which can provide information about surface waves with a wavelength of the order of 4–40 cm, i.e. in the gravity–capillary range. Also, when studying the generation of water waves by wind, it is well known that gravity–capillary waves with a wavelength of the order of 1 cm are the first waves to be generated because they have the largest growth rate.

In this paper we shall concentrate on some aspects of the initial evolution of gravity-capillary waves. We first briefly discuss some experimental results obtained by Choi (1977). For a brief account of these results see also Ramamonjiarisoa, Baldy & Choi (1979). Choi investigated the evolution of gravity–capillary waves in the presence of wind. In particular, the dependence of the wave spectrum as a function of fetch was determined and at a certain fetch a rather sudden transition of the peak frequency of the spectrum to half its value was observed. We term this period-doubling. Chen & Saffman (1979) suggested that this period-doubling is related to the phenomenon of second-harmonic resonance. They considered pure gravity–capillary waves only, i.e. the effects of wind input, viscous dissipation and shear in the water current were disregarded. Chen & Saffman found that around the wavenumber $2k = (2g/T)^{\frac{1}{2}}$, where g is the acceleration due to gravity and T is the surface tension divided by the water density, two types of waves are possible, namely a pure wave with wavenumber $2k$ and a combination wave with wavenumber $2k$ and k. However, the combination wave can only occur if the wave height h of the $2k$-wave satisfies the condition

$$h > \frac{4}{3k}\left|\frac{k^2T}{g} - \frac{1}{2}\right|. \tag{1}$$

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This condition holds true for $k^2T/g \approx \frac{1}{2}$, i.e. for small-amplitude waves; and for large-amplitude waves Chen & Saffman (1980) extended this result numerically. All these calculations were performed for steady-state waves.

As we are interested in the dependence of the amplitude or energy of the gravity–capillary waves on fetch a dynamical model is needed. Starting from results of Phillips (1960), McGoldrick (1965) derived the evolution equation for a resonant triad of waves assuming potential flow and no air flow or current. For the special case of second-harmonic resonance McGoldrick showed experimentally that at resonance a wave with wavenumber k cannot exist and that energy is transferred from the fundamental with wavenumber k to its second harmonic. The measurements showed the spatial evolution of the amplitudes of the interacting modes from their generation at the wavemaker to their extinction through viscous dissipation, in good agreement with his theoretical results.

We, on the other hand, are interested in the opposite case where gravity–capillary waves are being generated near the second harmonic by wind. (This possibility was also suggested by Simmons (1969) who derived the equations for the resonant triads using Whitham's variational principle.) The question then is if and under what conditions, i.e. what value of the fetch, a transfer of energy occurs from the second harmonic to the fundamental mode (period-doubling). In order to answer this question we extended the results of McGoldrick (1970) and Simmons (1969) to include effects of shear in the airflow (giving growth of the gravity–capillary waves) and effects of shear in the surface drift current in the evolution equations for the case of second-harmonic resonance.

We study some of the properties of this simple dynamical model for period-doubling. For pure gravity–capillary waves and no time dependence of the amplitudes of the second-harmonic and the fundamental mode, the fetch dependence of the amplitudes can be obtained in terms of elliptic functions. The condition for period-doubling (1) is rediscovered. Next, we shall discuss the effect of wind by studying the stability of a spatially growing second-harmonic to a small-amplitude fundamental mode.† The period-doubling condition (1) is affected by the addition of an air flow and shear in the current as both are as important as nonlinearity. If the period-doubling condition is satisfied wind input through nonlinearity gives rise to a bi-exponential growth of the amplitude of the fundamental mode. A very rapid transfer of energy from the second harmonic to the fundamental mode is therefore possible, qualitatively in agreement with the observations of Choi (1977).

Qualitative comparison of our theoretical results with the observations of Choi (1977) gives good agreement. However, to check the range of validity of this dynamical model for period-doubling a comparison with experiments, performed at different wind speeds, has to be made as Choi (1977) reported only results for one wind speed.

2. Choi's experiment

Choi (1977) investigated the generation of gravity–capillary waves by wind in the $\frac{1}{5}$-scale wind tunnel described by Favre & Coantic (1974). The wind-wave tank, schematically depicted in figure 1, is 52 cm wide, 8.65 m in length and 54 cm in height,

† Although perhaps it would be more appropriate to call the wave with the most energy the fundamental we shall, in agreement with McGoldrick (1965) and Simmons (1969), call the wave with the smallest wavenumber the fundamental and the wave with twice its wavenumber the second harmonic.

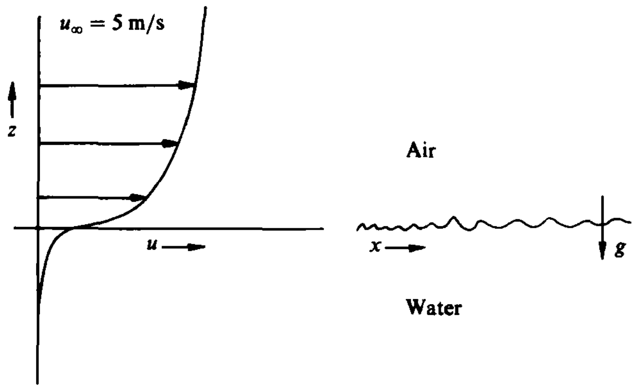

FIGURE 1. Schematic of Choi's experiment.

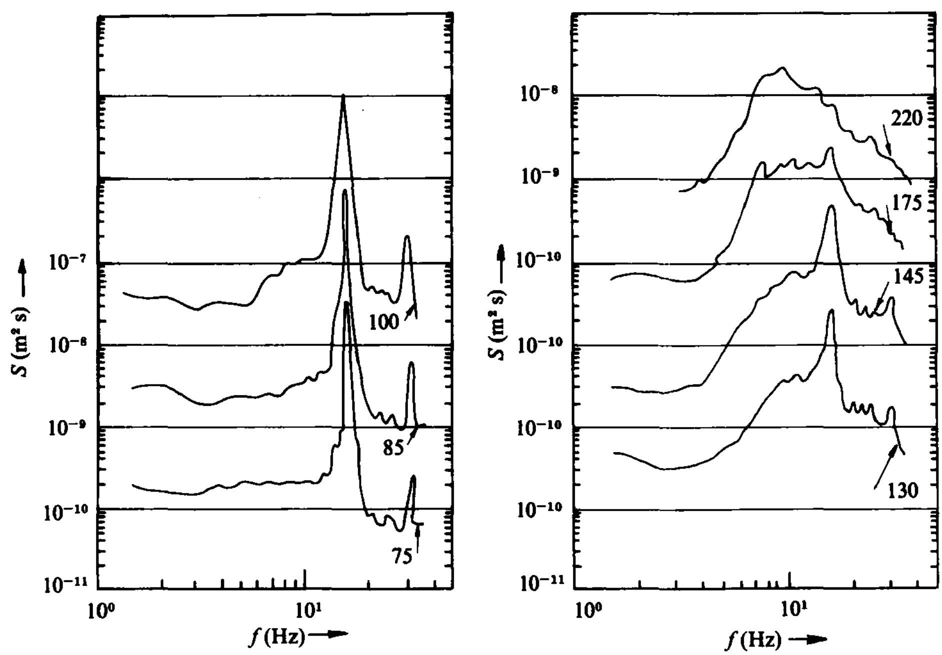

FIGURE 2. Evolution of frequency spectrum as function of fetch.

containing water of 26 cm depth. At the beginning of this wind-wave tank a rigid plate, 1.60 m long and 52 cm wide, was placed in order to (*a*) suppress parasitic disturbances in the air and water, and (*b*) make sure that the wavelets were being generated by a fully turbulent wind. The experiment was performed at a wind speed of 5 m/s.

About one minute after switching on the air flow a steady state was achieved. Choi measured among other things the air velocity as a function of height and fetch, the water current as a function of depth and fetch and also the fetch dependence of the frequency spectrum $S(f)$ of the gravity–capillary waves was determined (figure 2). The wave variance $E = \int \mathrm{d}f S(f)$ as a function of fetch is given in figure 3.

The first wavelets were observed at a fetch of 70 cm. These wind-generated wavelets have a very narrow spectrum with a peak frequency at 16.7 Hz. The waves

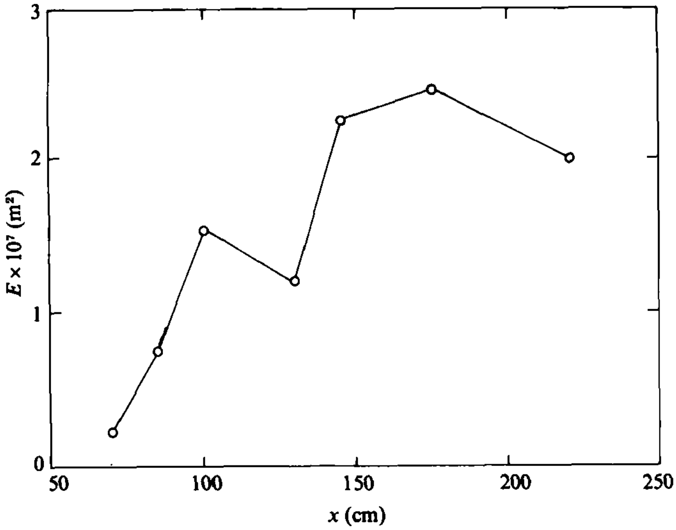

FIGURE 3. The fetch dependence of the total wave energy density.

seem to grow up to a fetch $x = 100$ cm according to the linear theory of the generation of waves by wind (Kawai 1979) although around $x = 85$ cm already some wave energy at half the peak frequency ($f \approx 8$ Hz) is present. Between $x = 100$ cm and 150 cm the spectrum broadens considerably, accompanied by a decrease in wave variance (cf. figure 3). This decrease in wave variance was attributed by Choi to the fact that in this region advection of wave energy in the transverse direction (i.e. perpendicular to the wind direction) becomes important, i.e. the gravity–capillary waves start to occupy the whole water surface (see also our discussion). Between $x = 150$ cm and 220 cm there is a considerable shift in the peak frequency of the spectrum to half its initial value. We note, however, that this process already starts at $x \approx 85$ cm.

It was conjectured by Chen & Saffman (1979) that this period-doubling phenomenon is related to second-harmonic resonance. Second-harmonic resonance occurs if the frequency of the free wave at $2k$ (a free wave is a wave that obeys a dispersion relation $\omega = \omega(k)$) is twice the frequency of the free wave at wavenumber k, i.e.

$$\omega(2k) = 2\omega(k). \tag{2}$$

Whenever this condition is met there is a strong nonlinear interaction between the second harmonic of the free wave at wavenumber k and the free wave at $2k$. Of course, in general it is unlikely that the wind generates the wave with wavenumber $2k$ that satisfies condition (2) so there will be a frequency mismatch. In the case of a small but finite mismatch in the frequency the nonlinear interaction is nearly resonant so that the energy transfer from the second-harmonic to the fundamental wave is less effective.

3. A simple, one-dimensional model for period-doubling

In this section we present approximate evolution equations for gravity–capillary waves in the presence of wind satisfying the conditions for second-harmonic resonance. Also, the effect of small but finite frequency mismatch is considered. This simple, one-dimensional model is just a combination of results of McGoldrick (1970)

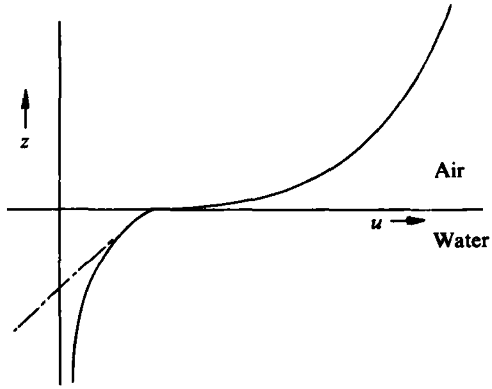

FIGURE 4. Flow profiles in air and water. Also, the linear current profile is shown (dashed–dotted line).

and Simmons (1969), for second-harmonic resonance, and of Kawai (1979) and van Gastel, Janssen & Komen (1985) for the effect of a shear flow in air and water. The only novel feature is that in the coupling coefficients, measuring the strength of nonlinearity, the effect of shear in the current is taken into account.

A formal method to obtain the evolution equations for gravity–capillary waves in the presence of a shear flow in air and water is the following: we assume incompressible flow governed by the Navier–Stokes equations, the only body force being the gravitational force. The boundary conditions express the vanishing of the wave-induced disturbances at large height and depth and the continuity of velocity and normal and tangential stress at the interface of air and water. We include the effect of surface tension because the wavelength of the waves is small. The equilibrium consists of a flat interface and plane parallel flows in air and water that depend on height (see figure 4).

It is assumed that the turbulence in the air gives rise to a linear–logarithmic profile as sketched in figure 4. The effect of the turbulence on the waves is however neglected (quasi-laminar approximation). The water motion is assumed to be laminar. The water current will be however time dependent, as was for example noted by Kawai (1979). The timescale over which the current varies appreciably is assumed to be much longer than a typical period of the gravity–capillary waves so that for our purposes the current can be regarded as time independent. The waves are regarded as a small perturbation of the equilibrium so that the wave steepness $\epsilon = kA$ (where A is the amplitude of the surface elevation) is small. In addition, the ratio of air density to water density $r = \rho_a/\rho_w$ is small and we assume that the Reynolds numbers in air and water, $u_{*a}/k\nu_a$ and $u_{*w}/k\nu_w$ respectively (where u_* is the friction velocity and ν is the viscosity and the subscripts a and w denote air and water respectively) are large. This suggests that one can solve the problem of second-harmonic resonance in the presence of wind by a perturbation method. In fact, for infinitesimal waves ($kA \rightarrow 0$), van Gastel *et al.* (1985) solved the resulting eigenvalue problem (e.g. the Orr–Sommerfeld equations plus boundary conditions) in a perturbative manner. The phase speed and growth rate of the gravity–capillary waves was in good agreement with the numerical results of Kawai (1979).

Here we extend this approach by allowing for finite-amplitude effects which we assume to be of the same order as the growth of the waves by wind input.

At lowest order, i.e. $\epsilon \rightarrow 0$, $r \rightarrow 0$ and no viscosity, the problem to study is one of

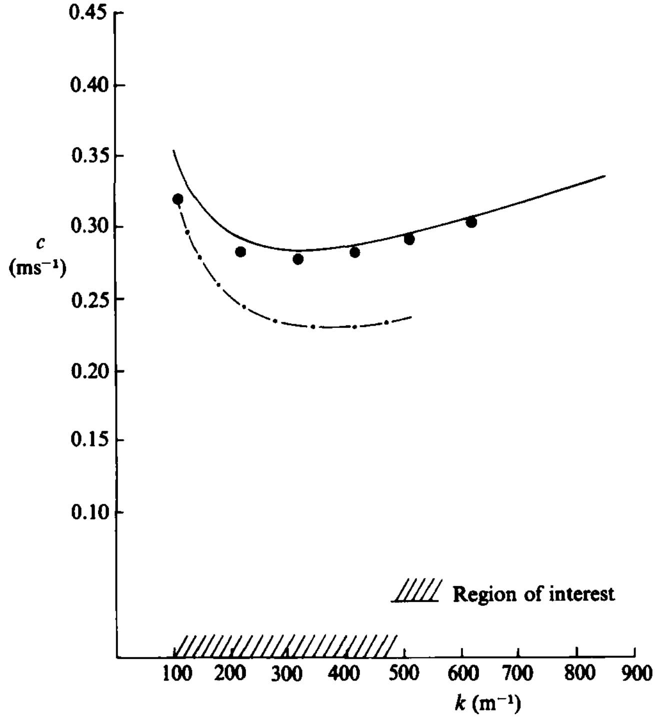

FIGURE 5. The phase velocity c as a function of wavenumber for $\lambda = 2.64\ \mathrm{cm}^{-1}$. ——, 'exact' (Kawai 1979; van Gastel *et al.* 1985); ●, equation (7) with linear shear current. For comparison we have also shown the phase velocity for pure gravity-capillary waves (–·–·–).

gravity–capillary waves on a shear current. As a result one has to solve Rayleigh's equation for the perturbed stream function subject to the boundary condition that the pressure be continuous at $z = 0$ (see the Appendix for details). A good approximation for the current profile is (van Gastel *et al.* 1985)

$$U_{\mathrm{w}} = U_0 \exp \lambda z \quad (z < 0), \tag{3}$$

where λ follows from the requirement that the tangential stress be continuous at the interface, or

$$\rho_{\mathrm{a}} \nu_{\mathrm{a}} U'_{\mathrm{a}} = \rho_{\mathrm{w}} \nu_{\mathrm{w}} U'_{\mathrm{w}} \quad (z = 0). \tag{4}$$

Here, a prime denotes differentiation of an equilibrium quantity with respect to z. Since the left-hand side of (4) is just the stress $\tau_{\mathrm{a}} = \rho_{\mathrm{a}} u^2_{*\mathrm{a}}$ in the boundary layer one obtains

$$\lambda = \frac{r u^2_{*\mathrm{a}}}{\nu_{\mathrm{w}} U_0}. \tag{5}$$

For the exponential current profile (3) one may solve the Rayleigh equation in terms of hypergeometric functions, but the resulting dispersion relation can only be solved numerically. A great simplification is achieved if one takes, instead of (3), the linear current profile

$$U_{\mathrm{w}} = U_0(1 + \lambda z). \tag{6}$$

This profile was also used by Choi (1977). Then, at lowest order, the dispersion relation becomes

$$W = \frac{1}{2}\frac{\lambda U_0}{k} \pm \left[\left(\frac{\lambda U_0}{2k}\right)^2 + \frac{g}{k} + kT\right]^{\frac{1}{2}}, \tag{7}$$

where $W = U_0 - c$, and c is the phase speed ω_0/k. The linear-shear-current approximation gives a good approximation to the exact result for waves that are not too long; see figure 5 where we have compared (7) with the exact result as obtained by van Gastel *et al.* (1985). Choi (1977) found a good agreement between (7) and his experimental results (see also Ramamonjiarisoa *et al.* 1978).

To next order it is assumed that the effects of finite amplitude, wind and viscosity are equally important. Let the complex amplitude of the surface elevation of the fundamental wave (k_1) be denoted by A_1 and that of its second harmonic by A_2. We choose the wavenumber k_1 such that the condition of second-harmonic resonance is satisfied, i.e. $\omega(2k_1) = 2\omega(k_1)$, where ω follows from the dispersion relation (7). Note that if the phase speed as a function of wavenumber k has a minimum then there always is second-harmonic resonance as (2) implies $c(k_1) = c(2k_1)$ (cf. figure 5). Then, for the linear shear current profile, the evolution equations for the amplitudes A_1 and A_2 become

$$\left.\begin{aligned}\left(\frac{\partial}{\partial t}+V_{g1}\frac{\partial}{\partial x}\right)A_1 &= \mathrm{i}\alpha A_1^* A_2+\gamma_1 A_1,\\ \left(\frac{\partial}{\partial t}+V_{g2}\frac{\partial}{\partial x}\right)A_2 &= \mathrm{i}\beta A_1^2+\gamma_2 A_2,\end{aligned}\right\} \tag{8}$$

where $V_{gi} = \partial\omega/\partial k_i$ is the group velocity of the ith mode and

$$\alpha = \frac{k_1^2 W_1}{1-\frac{1}{2}\mu_1}G; \quad \beta = \frac{1}{2}\frac{k_1^2 W_1}{(1-\frac{1}{4}\mu_1)}G.$$

Here, $G = 1-2\mu_1+2\mu_1^2$ and $\mu_1 = U_w'(0)/k_1 W_1$ measures the effect of shear in the current. The growth rates γ_1 and γ_2 represent the effects of shear in the air flow and viscous dissipation in water. For a linear–logarithmic wind profile γ_1 and γ_2 are given by the numerical results of Kawai (1979); see also van Gastel *et al.* (1985) where growth of the waves is determined through a perturbation analysis. An example of a growth-rate curve γ as function of the wavenumber is presented in figure 6.

The derivation of the coupling coefficients α and β is given in the Appendix. As a check on the expressions for α and β we note that in the case of no shear ($\mu_1 = 0$) the coupling coefficients are independent of the magnitude of the current. This is just what one would expect as the nonlinear interaction is invariant under Galilei transformations.

It is also of interest to remark that the nonlinear interactions in (8) conserve wave energy and momentum. Apart from a constant the energy of a mode is given by†

$$E_k = \frac{\omega}{g}\frac{\partial}{\partial\omega}D|A_k|^2,$$

where $D = kW^2 - U_0' W - c_0^2 k$, $c_0^2 = g/k + kT$ and $D = 0$ gives the dispersion relation (7). Then, one obtains

$$E_k = -2\frac{\omega}{g}W(1-\tfrac{1}{2}\mu)|A_k|^2, \quad \mu = \frac{U_0'}{kW}$$

and it it a simple matter to show that the total energy $E = \sum_k E_k$ is conserved by the nonlinear interaction. A similar assertion holds for the wave momentum $M_k = E_k/c(k)$.

We finally note that in the limit of no current one recovers the results of McGoldrick and Simmons on second-harmonic resonance.

Equations (8) describe the evolution of gravity–capillary waves at or near

† This follows from Whitham's (1974) averaged variational principle, where $L = D|A_k|^2 + O(|A_k|^4)$ and $E_k = \omega(\partial L/\partial\omega) - L$.

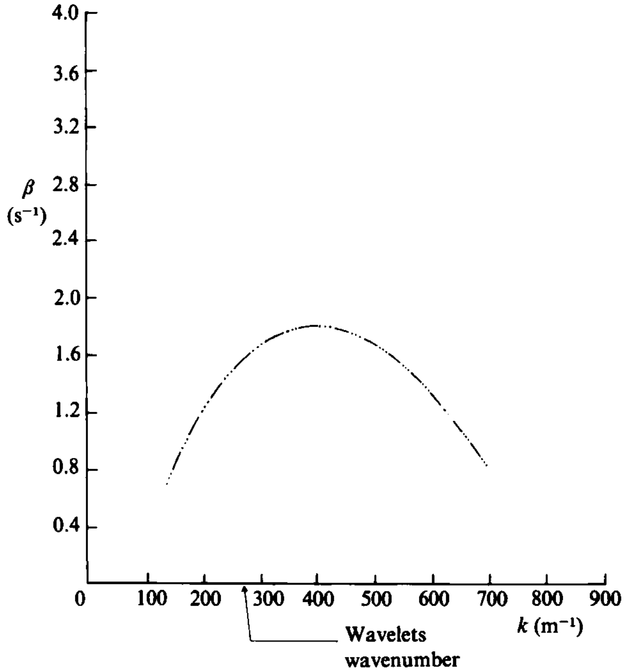

FIGURE 6. The growth rate of the energy of the waves $\beta = 2\gamma$ as a function of wavenumber for $u_{*a} = 21.4$ cm/s, $U_0 = 9.8$ cm/s. We have also indicated the wavenumber where the initial wavelets in Choi's experiment are being generated.

second-harmonic resonance. As the amplitudes A_1 and A_2 are assumed to be slowly varying in time and space (see the Appendix), the frequency mismatch is only allowed to be of the order of the wave steepness ϵ. To incorporate such a frequency mismatch explicitly in our simple model for period-doubling we consider the boundary-value problem that at $x = 0$ one excites a wave with frequency slightly off its resonant value, or

$$\omega = \omega_1(1+\delta), \tag{9}$$

where δ is the frequency mismatch.

Then, the boundary conditions for A_1 and A_2 become

$$\left.\begin{aligned} A_1(0,t) &= \hat{A}_{10}\exp[-\mathrm{i}\delta\omega_1 t], \\ A_2(0,t) &= \hat{A}_{20}\exp[-2\mathrm{i}\delta\omega_1 t]. \end{aligned}\right\} \tag{10}$$

It is then tempting to transform A_j ($j = 1, 2$) according to $A_j = \hat{A}_j \exp[\mathrm{i}(\kappa_j x - j\delta\omega_1 t)]$ where $\kappa_j = j\delta\omega_1/V_{gj}$, to obtain for $\hat{A}_j$ the following evolution equations:

$$\left.\begin{aligned} \left(\frac{\partial}{\partial t} + V_{g1}\frac{\partial}{\partial x}\right) A_1 &= \mathrm{i}\alpha A_1^* A_2\,\mathrm{e}^{\mathrm{i}Nx} + \gamma_1 A_1, \\ \left(\frac{\partial}{\partial t} + V_{g2}\frac{\partial}{\partial x}\right) A_2 &= \mathrm{i}\beta A_1^2\,\mathrm{e}^{-\mathrm{i}Nx} + \gamma_2 A_2, \end{aligned}\right\} \tag{11}$$

with boundary conditions $A_1(0,t) = A_{10}$ and $A_2(0,t) = A_{20}$. Here, we have dropped the carets and

$$N = 2\delta\omega_1\left(\frac{1}{V_{g2}} - \frac{1}{V_{g1}}\right)$$

is a measure for the frequency mismatch. For finite frequency mismatch the coupling coefficients between the second harmonic and the fundamental becomes periodic in x thereby making the interaction less effective.

In passing we remark that McGoldrick (1972) also considered the effect of finite N, but he derived (11) from a simplified one-dimensional model. Also, calculations on third-harmonic resonance were reported and qualitative agreement between the theoretical results and measurements on the amplitude response as a function of frequency mismatch was obtained.

In the rest of this paper we shall discuss some of the properties of our simple model for second-harmonic resonance (or period-doubling). We are especially interested in the fetch dependence of A_1 and A_2, i.e. we assume $(\partial/\partial t)A_1 = (\partial/\partial t)A_2 = 0$ in (11).

3.1. *No wind input*

If there is no wind input then $\gamma_1 = \gamma_2 = 0$ and the resulting equations may be solved in terms of elliptic functions. This has already been noted by McGoldrick (1972) who followed Bretherton (1964), so we discuss this case only briefly. Introducing the action density fluxes

$$\phi_i = \frac{V_{gi}}{\alpha_i}|A_i|^2, \quad \alpha_i = (\alpha, \beta), \tag{12}$$

one obtains from (11) in the steady state

$$\left(\frac{\mathrm{d}}{\mathrm{d}x}\phi_1\right)^2 + \mathscr{V}(\phi_1) = 0 \tag{13}$$

where the potential $\mathscr{V}$,

$$\mathscr{V}(\phi_1) = -4\left\{\frac{\alpha^2\beta}{V_{g1}^2 V_{g2}}(\phi_0 - \phi_1)\phi_1^2 - (L - \tfrac{1}{2}N\phi_1)^2\right\}, \tag{14}$$

depends on two conserved quantities, namely the total action density flux ϕ_0

$$\phi_0 = \phi_1 + \phi_2 = \text{const.} \tag{15}$$

and a quantity related to 'angular momentum',

$$L = \tfrac{1}{2}N\phi_1 + \mathrm{Im}\,(\mathrm{i}A_2 A_1^{*2}\,\mathrm{e}^{\mathrm{i}Nx}) = \text{const.}$$

As we are interested in understanding Choi's experiment where one initially observes a large second harmonic and a small fundamental wave, we take as boundary conditions

$$x = 0: \quad A_1 = A_{10}, \quad A_2 = A_{20}, \quad A_{10} \ll A_{20}, \tag{16}$$

and for simplicity we choose $L = \frac{1}{2}N\phi_1(0)$. Then, the potential $\mathscr{V}(\phi_1)$ may be approximated by

$$\mathscr{V}(\phi_1) = -\phi_1(\phi_+ - \phi_1)(\phi_1 - \phi_-), \tag{17}$$

where

$$\phi_\pm = \tfrac{1}{2}[\phi_0 - \Delta \pm \{(\phi_0 - \Delta)^2 + 8\Delta\phi_1(0)\}^{\frac{1}{2}}],$$

and

$$\Delta = \frac{N^2 V_{g1}^2 V_{g2}}{4(\alpha^2\beta)}.$$

The behaviour of the solution ϕ_1 as a function of x may be inferred from the form of the potential $\mathscr{V}$, of course realizing that $\mathscr{V}(\phi_1) \leqslant 0$ in order that $\mathrm{d}\phi_1/\mathrm{d}x$ be real. The form of the potential is given in figure 7 and from this figure one infers that there are two cases to be distinguished. If $\phi_0 < \Delta$, then nothing interesting happens as ϕ_1 remains of the order of its initial value. On the other hand if $\phi_0 > \Delta$ one finds periodic solutions with a maximum amplitude $\phi_{1\max}$ given by

$$\phi_{1\max} = \phi_+ \approx \phi_0 - \Delta. \tag{18}$$

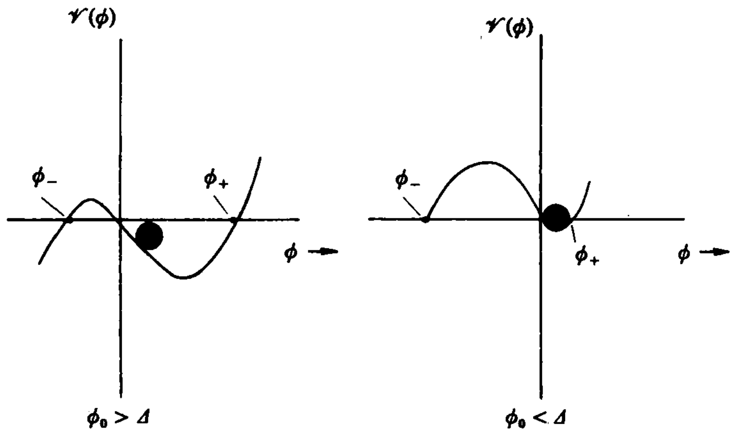

FIGURE 7. The potential $\mathscr{V}(\phi)$ of equation (17) for the stable ($\phi_0 < \Delta$) and unstable case ($\phi_0 > \Delta$).

Hence, if the condition

$$\phi_0 > \Delta, \quad \text{or} \quad A_{20} \geqslant \left|\frac{1}{2}\frac{V_{g1}}{\alpha}N\right|, \tag{19}$$

is satisfied the amplitude of the fundamental wave may become appreciable. For pure gravity–capillary waves, condition (19) simply corresponds to the bifurcation condition (1) given by Chen & Saffman. To see this, we note that by (16) $\phi_0 \approx \phi_2(0)$ and that the wave height $h = 4A_{20}$. Thus, the amplitude of the second harmonic must be sufficiently large to overcome the stabilizing effect of finite frequency mismatch. And if so, then an appreciable amplitude of the fundamental wave is to be found, especially if $\phi_0 \gg \Delta$ as then almost all the energy of the second harmonic is transferred to the fundamental. In the latter event we expect a significant change in the position of the peak of the spectrum to half its initial value (period-doubling).

3.2. *Effect of wind input*

For wind blowing over gravity–capillary waves the appropriate set of equations becomes

$$\left.\begin{aligned} V_{g1}\frac{\partial}{\partial x}A_1 &= \mathrm{i}\alpha A_1^* A_2\,\mathrm{e}^{\mathrm{i}Nx}+\gamma_1 A_1, \\ V_{g2}\frac{\partial}{\partial x}A_2 &= \mathrm{i}\beta A_1^2\,\mathrm{e}^{-\mathrm{i}Nx}+\gamma_2 A_2, \end{aligned}\right\} \tag{20}$$

where, again, $N = 2\delta\omega_1(1/V_{g2}-1/V_{g1})$. We are interested in the effect of wind input on the condition for period-doubling (19). To that end, keeping in mind Choi's experiment, we study the linear stability of the solution

$$A_1^{(0)} = 0, \quad A_2^{(0)} = A_{20}\exp\frac{\gamma_2}{V_{g2}}x, \tag{21}$$

so we consider the stability of a growing second harmonic as a result of wind. Linearizing around the solution (21) one obtains the following second-order differential equation for the perturbation $A_1^{(1)}$:

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}A_1^{(1)}-\frac{\mathrm{d}}{\mathrm{d}x}A_1^{(1)}\left[\mathrm{i}N+\frac{2\gamma_1}{V_{g1}}+\frac{\gamma_2}{V_{g2}}\right]+A_1^{(1)}\left[\mathrm{i}\frac{N\gamma_1}{V_{g1}}+\frac{\gamma_1}{V_{\gamma1}}\left(\frac{\gamma_1}{V_{g1}}+\frac{\gamma_2}{V_{g2}}\right)-f\right] = 0, \tag{22}$$

where
$$f = \left[\alpha A_{20} \exp\left(\frac{\gamma_2}{V_{g2}} x\right) \Big/ V_{g1}\right]^2.$$

This equation can be solved in terms of Bessel functions Z_ν,

$$A_1^{(1)} = \exp\left(\frac{\gamma_1}{V_{g1}} x\right) f^{\frac{1}{2}\alpha} Z_\nu(\beta f^{\frac{1}{2}}), \tag{23}$$

where $\beta = \mathrm{i} V_{g2}/\gamma_2$, $\alpha = \frac{1}{2}(1+\mathrm{i}NV_{g2}/\gamma_2)$ and $\nu = \pm\alpha$. This is still a rather complicated expression for the fetch dependence of the amplitude of the fundamental wave. There is, however, one special case where the solution is simple, namely when there is no frequency mismatch, $N = 0$. Then, $\nu = \alpha = \frac{1}{2}$ and for special boundary conditions the solution reads

$$A_1^{(1)} = A_{10} \exp\left\{\frac{\gamma_1}{V_{g1}} x + \frac{\alpha A_{20}}{V_{g1}} \frac{V_{g2}}{\gamma_2}\left(\exp\frac{\gamma_2}{V_{g2}} x - 1\right)\right\} \quad (N = 0). \tag{24}$$

Apart from exponential growth due to wind, we note that the nonlinear coupling between the fundamental and its second harmonic gives rise to bi-exponential growth. This suggests a very sudden transition of the position of the peak of the spectrum to half its initial value.

The case of finite frequency mismatch ($N \neq 0$) is harder to analyse, although some qualitative insight can be gained for small growth rate γ_2, i.e. $\gamma_2/V_{g2}N \ll 1$. Thus, the order ν of the Bessel function Z_ν is large and one can use some of its asymptotic properties. One may then distinguish three regions. With $z \equiv \beta f^{\frac{1}{2}}$ these regions are $|z| \ll |\nu|$, $|z| \approx |\nu|$ and $|z| \gg |\nu|$. Notice that in the limit $\gamma_2 \to 0$ the condition $|z| = |\nu|$ just corresponds to the period-doubling condition (19) for the case without wind input.

The behaviour of the solution in these three regions is as follows. At the beginning of the wave tank ($|z| \ll |\nu|$), apart from growth by wind, nothing happens. In the turning-point region $|z| \approx |\nu|$ an asymptotic representation for Z_ν, (23), valid for large ν, is

$$Z_\nu(z) \sim \frac{w}{\sqrt{3}} \exp\{\mathrm{i}[\tfrac{1}{6}\pi - \nu(w - \tfrac{1}{3}w^3 - \arctan w)]\}\, H_{\frac{1}{3}}^{(2)}(\tfrac{1}{3}\nu w^3) + O\left(\frac{1}{|\nu|}\right), \tag{25}$$

where $w = [(z^2/\nu^2) - 1]^{\frac{1}{2}}$ and $H_{\frac{1}{3}}^{(2)}$ is a Hankel function (cf. Gradshteyn & Ryzhik 1965, p. 964). We are interested in the transition from an exponentially damped to an exponentially growing solution. We anticipate that this will occur for $|w| = O(1)$. As ν is large one can then use the following asymptotic expression for $H_{\frac{1}{3}}^{(2)}$:

$$H_{\frac{1}{3}}^{(2)}(y) \sim \left(\frac{2}{\pi y}\right)^{\frac{1}{2}} \mathrm{e}^{-\mathrm{i}(y - \frac{5}{12}\pi)}, \quad y \gg \tfrac{1}{3}, \tag{26}$$

where $y = \frac{1}{3}\nu w^3$. The transition point then follows from the requirement that $\mathrm{Im}\,(y_t) = 0$, or in the limit of small γ_2,

$$A_{2t} = \frac{1}{2}\left|\frac{NV_{g1}}{\alpha}\right|\left(1 + \left|\frac{\gamma_2}{\sqrt{3}NV_{g2}}\right|\right), \tag{27}$$

where $A_{2t} = A_{20} \exp(\gamma_2 x_t/V_{g2})$. Equation (27) shows the effect of wind input on the period-doubling condition (compare with (19)). Note that it is possible to use (26) as at the turning point $Re(y) = O(\nu) = O(V_{g2}/\gamma_2 N)$ is large since $\gamma_2 \to 0$. Finally, in the region $|z| \gg |\nu|$ one can use the asymptotic expression for Z_ν and the solution $A_1^{(1)}$ becomes ($x \to \infty$, ν fixed)

$$A_1^{(1)} \sim \exp\left\{\left(\frac{\gamma_1}{V_{g1}} - \tfrac{1}{2}\mathrm{i}N\right) x + \frac{\alpha A_{20}}{V_{g1}} \frac{V_{g2}}{\gamma_2}\left(\exp\frac{\gamma_2}{V_{g2}} x - 1\right)\right\}; \tag{28}$$

hence for large x one again has bi-exponential growth. Clearly, in this last region nonlinear effects will become important and may reduce the amplitude of the second harmonic. There is, however, a strong indication of a sudden transition of the peak frequency of the spectrum to half its initial value.

To summarize our results on the effect of wind input we note that the condition for period-doubling is affected by the inclusion of growth by wind (see (27)), and the combined effect of nonlinearity and growth by wind gives rise to a very fast growth of the fundamental wave. Apart from this there are also indirect effects on the phenomenon of period-doubling as in the presence of wind a non-uniform current develops. Shear in the current affects the group velocity and the coupling coefficients α and β.

4. Application to Choi's experiment

The parameters we need from Choi's experiment are the following. A wind velocity $U_\infty = 5$ m/s corresponds to a friction velocity $u_{*a} = 20$ cm/s. Using $\rho_a/\rho_w = 1.22\times 10^{-3}$, $\nu_w = 0.0114$, and the measured value of the current at the surface $U_w(0) = 15$ cm/s one finds from (5) that the inverse shear length of the current $\lambda = 2.85$ cm^{-1}. In addition, we take $g = 981$ cm s^{-2} and $T = 74$ dyn/cm.

Let us first consider the condition for second-harmonic resonance in the presence of shear in the current. With the dispersion relation (7) the condition for second-harmonic resonance to occur (i.e. $2\omega(k_1) = \omega(2k_1)$) results in the following quartic equation for the dimensionless wavenumber $l = k_1(T/g)^{\frac{1}{2}}$:

$$(l^2-\tfrac{1}{2})^2 = \frac{3}{4}\frac{\lambda^2}{g}\left(\frac{T}{g}\right)^{\frac{1}{2}} l. \tag{29}$$

In the absence of shear ($\lambda = 0$) one finds $l = 2^{\frac{1}{2}}$ and this is the resonance condition for pure gravity–capillary waves. For small but finite λ the dimensionless wavenumber is approximately given by

$$l \approx 2^{-\frac{1}{2}}(1-2^{-\frac{1}{4}}a^{\frac{1}{2}}), \quad a = \frac{3}{4}\frac{\lambda^2}{g}\left(\frac{T}{g}\right)^{\frac{1}{2}}. \tag{30}$$

This shows that the effect of shear is to decrease the wavenumber k_1 (this can also be inferred from figure 3). For the parameters of Choi's experiment we find by iteration of (29), using (30) as a starting point, $k_1 = 1.31$ cm^{-1}, whereas in the absence of shear one would obtain $k_1 = 2.57$ cm^{-1}, hence the effect of shear is a factor of two. Using $k_1 = 1.31$ cm^{-1}, the frequency of the fundamental $f_1 = 6.68$ Hz and that of the second harmonic $f_2 = 13.36$. As in Choi's experiment waves are being generated at $f = 16.7$ Hz, the frequency mismatch $\delta \equiv f/f_2 - 1 = 0.25$. With $k_1 = 1.31$ one finds that the wavenumber of the second harmonic is 2.62 cm^{-1} and as may be inferred from figure 6 this just corresponds to the maximum of the wave growth curve. In other words, waves with wavenumber around $k = 2.62$ are the first that are being generated so that the stability calculations of §3.2 make sense. In order to evaluate the period-doubling condition (27) we need V_{g1} (= 30.6 cm/s), V_{g2} (= 34 cm/s), α (= -100.0 cm^{-1} s^{-1}) and we take $\gamma_2 = 1$ s^{-1}. The result is that, according to (27), one finds that transition from f_2 to f_1 starts to occur when the wave height $h_2 = 4A_2$ satisfies

$$h_2 \geqslant 0.053 \text{ cm}. \tag{31}$$

Choi measured the wave variance E related to the surface elevation. Then, $E_2 = \frac{1}{8}h_2^2$ so that in terms of wave variance the condition for transition becomes $E_2 \geqslant 3.5 \times 10^{-8}\ \mathrm{m}^2$. According to figure 3 one should therefore expect that transition occurs around a fetch $x = 80$ cm. The detailed spectra of figure 2 showed that waves at half the peak frequency are being generated at a fetch $x = 85$ cm. There is, therefore, a reasonable agreement between this experiment and our simple dynamical model.

5. Conclusions

We have obtained a simple, one-dimensional model for the period-doubling of gravity–capillary waves. The model includes the growth of the waves by wind, viscous dissipation and the nonlinear interaction between the second harmonic and the fundamental waves (including the effect of shear in the current). The condition for period-doubling (27) seems to be in reasonable agreement with the observations of Choi. We add to this that dramatic changes from the second harmonic to the fundamental wave are to be expected only if the wave height is much larger than the one given by (27). The transition, however, occurs rather suddenly as growth by wind combined with the nonlinear interaction gives a bi-exponential growth of the fundamental wave. Finally, one might wonder whether the phenomenon of period-doubling as observed by Choi is just a happy coincidence because waves have to be generated by wind with wavenumbers that (nearly) satisfy the condition for second-harmonic resonance. In other words, if the frequency mismatch δ becomes too large the transfer of energy from the second-harmonic to the fundamental wave is quenched. This situation might happen for large wind speeds as the maximum of the wind input shifts to higher wavenumbers whereas the minimum of the phase velocity of the waves shifts to lower wavenumbers (Kawai 1979). However, conditions might then become favourable for third-harmonic resonance. Hence, for large wind speeds one might expect to observe period-tripling instead of period-doubling. It would therefore be interesting to perform experiments at different wind speeds in order to determine the range of validity of our dynamical model for period-doubling and to see whether for large wind speeds period-tripling occurs. We note, however, that third-harmonic resonance is rather weak and therefore is presumably effective over a rather long fetch, hence a long wave tank is needed.

The author benefited from discussions with K. van Gastel and G. J. Komen. He is pleased to acknowledge useful discussions with P. G. Saffman and he thanks him for the kind hospitality at Caltech. This work was supported by the Office of Naval Research under Contract N00014-79-C-0412, NR062-639.

Appendix

In this Appendix we briefly present the derivation of the equations for second-harmonic resonance in the presence of a non-uniform current. In this derivation we neglect viscosity and the effect of air flow and we consider one-dimensional propagation only.

With η the surface elevation, $\boldsymbol{u} = (u, 0, w)$ the velocity, $\boldsymbol{u} = \boldsymbol{\nabla} \times \boldsymbol{\psi}$ with $\boldsymbol{\psi}$ the stream function and p_w the dynamic pressure, the Euler equations plus boundary conditions become

$$\left.\begin{array}{l} z \leqslant \eta \left\{ \begin{array}{l} \left(\dfrac{\partial}{\partial t} - \dfrac{\partial \psi}{\partial z}\dfrac{\partial}{\partial x} + \dfrac{\partial \psi}{\partial x}\dfrac{\partial}{\partial z}\right) \Delta \psi = 0, \quad \Delta = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial z^2}, \\ \dfrac{\partial}{\partial x} p_w = \left(\dfrac{\partial}{\partial t} - \dfrac{\partial \psi}{\partial z}\dfrac{\partial}{\partial x} + \dfrac{\partial \psi}{\partial x}\dfrac{\partial}{\partial z}\right) \dfrac{\partial}{\partial z}\psi, \end{array} \right. \\ z = \eta \left\{ \begin{array}{l} \left(\dfrac{\partial}{\partial t} + u \dfrac{\partial}{\partial x}\right)\eta = w, \\ p_w = g\eta - T\nabla\cdot\left(\dfrac{\nabla\eta}{[1+(\nabla\eta)^2]^{\frac{1}{2}}}\right). \end{array} \right. \end{array}\right\} \tag{A 1, A 2}$$

Expanding the boundary conditions to second order one has

$$\left.\begin{array}{r} \dfrac{\partial}{\partial t}\eta + \left(u + \eta \dfrac{\partial}{\partial z} u\right)\dfrac{\partial}{\partial x}\eta = w + \eta \dfrac{\partial}{\partial z} w, \\ p_w + \eta \dfrac{\partial}{\partial z} p_w = g\eta - T \dfrac{\partial^2}{\partial x^2}\eta, \end{array}\right\} \quad z = 0 \tag{A 3}$$

For $z \to -\infty$ we require that the solutions decay sufficiently rapidly. The steady state is one in which the current U_w depends only on z so that $U_w(z) = -\partial\psi_0/\partial z$. We intend to seek small-amplitude solutions (with wave steepness ϵ) around this steady state. To that end we introduce the expansions

$$\left.\begin{array}{c} \psi = \psi_0 + \epsilon\psi_1 + \epsilon^2\psi_2 + \ldots, \quad p_w = p_1 + \epsilon p_2 + . \\ \eta = 0 + \epsilon\eta_1 + \epsilon^2\eta_2 + \ldots, \\ \dfrac{\partial}{\partial t} = \dfrac{\partial}{\partial \tau_0} + \epsilon \dfrac{\partial}{\partial \tau_1} + \ldots; \quad \dfrac{\partial}{\partial x} = \dfrac{\partial}{\partial x_0} + \epsilon \dfrac{\partial}{\partial x_1} + \ldots \end{array}\right\} \tag{A 4}$$

where $\tau_l = \epsilon^l t$ and $x_l = \epsilon^l x$ $(l = 0, 1, 2, 3)$. Substituting this expansion in (A 1), (A 2) we obtain a hierarchy of equations. Only the first- and second-order approximations are considered in this Appendix.

A.1. *First-order theory*

At first order the equations for the stream function ψ_1 and and the pressure p_1 become

$$\mathrm{L}\psi_1 \equiv \left\{\left(\frac{\partial}{\partial \tau_0} - \psi_0' \frac{\partial}{\partial x_0}\right)\Delta_0 + \psi_0''' \frac{\partial}{\partial x_0}\right\}\psi_1 = 0, \tag{A 5a}$$

$$\frac{\partial}{\partial x_0} p_1 = \left\{\left(\frac{\partial}{\partial \tau_0} - \psi_0' \frac{\partial}{\partial x_0}\right)\frac{\partial}{\partial z} + \psi_0'' \frac{\partial}{\partial x_0}\right\}\psi_1, \tag{A 5b}$$

where $\Delta_0 = (\partial^2/\partial x_0^2) + (\partial^2/\partial z^2)$ and a prime denotes differentiation with respect to z. We only consider two modes, i.e.

$$\psi_1 = \sum_{i=1}^{2} B_i f_i(z) \exp \mathrm{i}\theta_i + \mathrm{c.c.}, \quad \theta_i = k_i x_0 - \omega_1 \tau_0, \tag{A 6}$$

that satisfy the condition for second-harmonic resonance, i.e. $2\theta_1 = \theta_2$. Substitution of (A 6) into (A 5a) gives Rayleigh's equation for $f_i(z)$

$$\left.\begin{array}{c} W_i\left(\dfrac{\mathrm{d}^2}{\mathrm{d}z^2} - k_i^2\right)f_i - W_i'' f_i = 0, \\ f_i(0) = 1, \quad f_i(-\infty) \to 0, \end{array}\right\} \tag{A 7}$$

where $W_i = U_w - c_i$, $U_w = -\psi_0'$ and $c_i = \omega_i/k_i$.

The boundary conditions to first order read

$$\left.\begin{aligned}\left(\frac{\partial}{\partial \tau_0}+U_{\mathrm{w}}\frac{\partial}{\partial x_0}\right)\eta_1 &= \frac{\partial}{\partial x_0}\psi_1 &\text{(A 8a)}\\ p_1 &= g\eta_1 - T\frac{\partial^2}{\partial x_0^2}\eta_1 &\text{(A 8b)}\end{aligned}\right\}\quad z=0.$$

Writing the surface elevation as

$$\eta_1 = \Sigma\, A_i\, \mathrm{e}^{\mathrm{i}\theta_i} + \mathrm{c.c.} \tag{A 9}$$

one finds from (A 8*a*)

$$A_i = \frac{B_i}{W_i} \tag{A 10}$$

Then, by means of (A 5*b*) and (A 8*b*) one arrives at the dispersion relation

$$f_i'(0)\, W_i^2 - U_0'\, W_i - k_i c_{0i}^2 = 0, \quad z=0, \tag{A 11}$$

where $c_{0i}^2 = g/k_i + k_i T$. In general this is a complicated dispersion relation as f_i' depends on W_i as well. Only in the simple case of a linear current profile a simple dispersion relation results. Then, $U''=0$ and $f_i = \exp k_i z$. The dispersion relation is then given by

$$W_i = \frac{1}{2}\frac{U_{\mathrm{w}}'}{k} \pm \left\{\left(\frac{U_{\mathrm{w}}'}{2k}\right)^2 + c_{0i}^2\right\}^{\frac{1}{2}}. \tag{A 12}$$

This concludes the first-order theory.

A.2. *Second-order theory*

To second order the equation for the stream function ψ_2 becomes ($z<0$)

$$\begin{aligned}\mathrm{L}\psi_2 = &-\frac{\partial}{\partial \tau_0}\Delta_1\psi_1 - \frac{\partial}{\partial \tau_1}\Delta_0\psi_1 + \psi_0'\left(\frac{\partial}{\partial x_0}\Delta_1\psi_1 + \frac{\partial}{\partial x_1}\Delta_0\psi_1\right)\\ &+\frac{\partial}{\partial z_0}\psi_1\frac{\partial}{\partial x_0}\Delta_0\psi_1 - \frac{\partial}{\partial x_0}\psi_1\frac{\partial}{\partial z_0}\Delta_0\psi_1 - \psi_0'''\frac{\partial}{\partial x_1}\psi_1,\end{aligned} \tag{A 13}$$

where $\Delta_0 = (\partial^2/\partial z^2)+(\partial^2/\partial x_0^2)$ and $\Delta_1 = 2(\partial^2/\partial x_0\,\partial x_1)$. The pressure equation to second order reads ($z<0$)

$$\begin{aligned}\frac{\partial}{\partial x_0}p_2 = &-\frac{\partial}{\partial x_1}p_1 + \frac{\partial^2}{\partial \tau_0\,\partial z}\psi_2 + \frac{\partial^2}{\partial \tau_1\,\partial z}\psi_1 + U_{\mathrm{w}}\left[\frac{\partial^2}{\partial x_0\,\partial z}\psi_2 + \frac{\partial^2}{\partial x_1\,\partial z}\psi_1\right]\\ &-\frac{\partial}{\partial z}\psi_1\frac{\partial^2}{\partial x_0\,\partial z}\psi_1 + \frac{\partial}{\partial x_0}\psi_1\frac{\partial^2}{\partial z^2}\psi_1 + \psi_0''\frac{\partial}{\partial x_1}\psi_1 + \psi_0''\frac{\partial}{\partial x_0}\psi_2.\end{aligned} \tag{A 14}$$

The equation for the surface elevation becomes ($z=0$)

$$\begin{aligned}\left(\frac{\partial}{\partial \tau_0}+U_{\mathrm{w}}(0)\frac{\partial}{\partial x_0}\right)\eta_2 = &\frac{\partial}{\partial x_0}\psi_2 + \frac{\partial}{\partial x_1}\psi_1 - \frac{\partial}{\partial \tau_1}\eta_1 - U_{\mathrm{w}}(0)\frac{\partial}{\partial x_1}\eta_1 - U_{\mathrm{w}}'\eta_1\frac{\partial}{\partial x_0}\eta_1\\ &+\frac{\partial}{\partial z}\psi_1\frac{\partial}{\partial x_0}\eta_1 + \eta_1\frac{\partial^2}{\partial z\,\partial x_0}\psi_1,\end{aligned} \tag{A 15}$$

and finally the boundary condition on the pressure becomes ($z=0$)

$$p_2 + \eta_1\frac{\partial}{\partial z}p_1 = g\eta_2 - T\left(\frac{\partial^2}{\partial x_0^2}\eta_2 + 2\frac{\partial^2}{\partial x_0\,\partial x_1}\eta_1\right). \tag{A 16}$$

In order to solve for ψ_2, p_2 and η_2 one substitutes the expressions for the first-order quantities such as ψ_1 and η_1 into (A 13)–(A 16). One then finds that, for example, products of first-order quantities give rise to secularity because we are at second-harmonic resonance. In order to prevent secularity the amplitudes A_1 and A_2 have to satisfy certain evolution equations on the slow time and spatial scale. For example for B_2 one finds

$$\left[\frac{f_2'}{ik_2}W_2+\frac{c_{02}^2}{\mathrm{i}W_2}\right]\frac{\partial}{\partial\tau_1}A_2+\left[\frac{c_2f_2'}{ik_2}W_2+\frac{c_2c_{02}^2}{\mathrm{i}W_2}-2\frac{k_2T}{\mathrm{i}}\right]\frac{\partial}{\partial x_1}A_2+W_2\psi_{22}'(0)$$
$$+A_1^2\left[\frac{k_1}{k_2}W_1^2f_1''-\frac{k_1}{k_2}W_1^2f_1'^2+W_1\frac{\mathrm{d}}{\mathrm{d}z}(W_1f_1'-W_1'f_1)+\frac{c_{02}^2}{W_2}(k_1U_\mathrm{w}'-2k_1W_1f_1')\right]=0,\ z=0,$$

where $\psi_{22}'(0)$ is a rather complicated expression involving integrals of the solution of the Rayleigh equation:

$$\psi_{22}'(0)=\frac{1}{ik}\int_{-\infty}^{0}\mathrm{d}z\,\frac{f_2}{W_2}\left\{-\frac{U_\mathrm{w}''}{W_2}f_2\frac{\partial}{\partial\tau_1}B_2+f_2\frac{\partial}{\partial x_1}B_2\left(2k_2^2W_2-U_\mathrm{w}''\frac{c_2}{W_2}\right)\right.$$
$$\left.+\mathrm{i}k_1B_1^2f_1'f_1\frac{U_\mathrm{w}''}{W_1}-\mathrm{i}k_1B_1^2f_1\frac{\mathrm{d}}{\mathrm{d}_z}\left(\frac{U_\mathrm{w}''}{W_1}f_1\right)\right\}.$$

A great simplification is achieved by assuming a linear current profile and the final result is given in (8).

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