Quantized electric-flux-tube solutions to Yang-Mills theory

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We suggest that long-distance Yang-Mills theory is more conveniently described in terms of electric rather than the customary magnetic vector potentials. On this basis we propose as an effective Lagrangian for this regime the most simple gauge-invariant (under the magnetic rather than electric gauge group) and Lorentz-invariant Lagrangian which yields a $1/q^4$ gluon propagator in the Abelian limit. The resulting classical equations of motion have solutions corresponding to tubes of color electric flux quantized in units of $e/2$ (e is the Yang-Mills coupling constant). To exponential accuracy the electric color energy is contained in a cylinder of finite radius, showing that continuum Yang-Mills theory has excitations which are confined tubes of color electric flux. This is the criterion for electric confinement of color.

I. INTRODUCTION

Yang-Mills theory is most easily expressed in terms of equations for the vector potentials $A_\mu^a(x)$; the Yang-Mills Lagrangian is simple in terms of these potentials and the short-distance properties of the theory are also conveniently described in this way. The long-distance properties, however, are not. The correlation functions of these operators are apparently singular at long distance presumably reflecting, in some complicated and as yet not understood way, the confining properties of Yang-Mills theory. It seems unlikely, therefore, that there is any simple description of the long-distance behavior of Yang-Mills theory in terms of $A_\mu^a(x)$; indeed, substantial progress toward understanding confinement has so far been made only for lattice Yang-Mills theory, where the natural variables are gauge-invariant Wilson loops.

On the other hand, the Yang-Mills vacuum is a relativistic dielectric medium, and the correlation functions for the vector potentials describe the properties of this medium. The Schwinger-Dyson equations are an infinite set of coupled equations relating the correlation functions and determining self-consistently the vacuum dielectric properties. They are of the same structure as the equations giving the dielectric properties of any condensed matter medium. The question we must face, then, is how can we extract information from the relations among these gauge-variant singular functions and how this information can best be used to obtain physical predictions about Yang-Mills theory at long range.

In the past[1] we have studied a truncated version of the Schwinger-Dyson equations obtained by expressing the three-point correlation function Γ in terms of the two-point current correlation function Π (the vacuum polarization) according to the simplest possibility consistent with gauge invariance. This reduced the Schwinger-Dyson equations to a single nonlinear integral equation for Π which had the general structure of the integral equation for the dielectric constant in a many-body system. The vacuum polarization is the zero-field dielectric constant, and the truncation procedure expresses the field-dependent part of the dielectric constant back in terms of the dielectric constant itself.

We showed first numerically[2] and then analytically[3] that this equation had a solution for which the Fourier transform of the dielectric constant $\epsilon(q^2)$ behaved like (q^2/M^2) as $q^2\to 0$, where M^2 is an undetermined renormalization-group-invariant mass scale (these results were confirmed by a more thorough analytical study of this equation by Atkinson and Johnson).[4] In this solution the gluon propagator $D_A(q^2)=1/q^2\epsilon(q^2)$ behaves like (M^2/q^4) as $q^2\to 0$.

Because of this singular low-momentum behavior of the gluon propagator it is not possible to use the Dyson equations beyond the stage we have already studied. This again reflects the problems of using vector potentials to calculate the long-distance properties of Yang-Mills theory. To obtain further information we must go beyond the Dyson equations and abandon the use of vector potentials.

In coordinate space our solution of the Dyson equations means that in the simplest long-distance approximation the Yang-Mills vacuum behaves as a linear dielectric medium with dielectric constant ϵ and magnetic permeability $\mu=1/\epsilon$ where $\epsilon=\partial^2/M^2$. The magnetic $\mathbf{H}$ field and the electric displacement vector $\mathbf{D}$ are then related to the electric color field $\mathbf{E}$ and magnetic color field $\mathbf{B}$ by the equations

$$\mathbf{H}=\frac{\partial^2}{M^2}\mathbf{B},\quad \mathbf{D}=\frac{\partial^2}{M^2}\mathbf{E}\,, \tag{1.1}$$

and the equations of motion are

$$\nabla\times\mathbf{H}=\frac{\partial\mathbf{D}}{\partial t},\quad \nabla\cdot\mathbf{D}=0\,, \tag{1.2}$$

$$\nabla\cdot\mathbf{B}=0,\quad \nabla\times\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t}\,. \tag{1.3}$$

Since Eqs. (1.1), (1.2), and (1.3) are linear the color indices play no role. These equations summarize what we have learned from our solution of the Dyson equations and Ward identities and provide the starting point for constructing a consistent long-distance theory.

We wish to emphasize, however, that this paper does not rest on our study of the truncated Dyson equations and Ward identities. Here our starting point is only that the gluon propagator, for whatever reason, and in some gauge, behaves like $1/q^4$ in the infrared; that is, the two-point proper vertex function, in some gauge, behaves like $\Pi_{\mu\nu}\sim q^2(q^2\delta_{\mu\nu}-q_\mu q_\nu)$. These statements are equivalent to the statement that the dielectric medium in the weak-field limit, in some gauge, is described by Eqs. (1.1), (1.2), and (1.3). Our purpose here is to construct, given the above conditions, a phenomenological long-range Yang-Mills theory. As has already been pointed out[5,6] Eqs. (1.1), (1.2), and (1.3) already describe some of the properties of a confining theory. They yield a long-range linear potential between color electric charges and a vanishing long-range force between color magnetic charges, which is a concrete realization of 't Hooft's theorem that electric and magnetic confinement are mutually exclusive.[7] However since Eqs. (1.1), (1.2), and (1.3) describe an Abelian theory they clearly cannot account for the nonlinear dynamics of long-distance Yang-Mills theory. In particular electric flux is not confined to narrow tubes but spreads out through all space.

In an earlier paper[8] we tried to account for these nonlinear effects by constructing a gauge-invariant effective Lagrangian $\mathscr{L}(A)$ which yielded Eqs. (1.1)–(1.3) in the linear Abelian approximation. The Lagrangian $\mathscr{L}(A)$ was constructed as follows: Eqs. (1.3) are automatically solved by

$$\mathbf{B}=\nabla\times\mathbf{A},\quad \mathbf{E}=-\nabla A_0-\frac{\partial\mathbf{A}}{\partial t} \tag{1.4}$$

and the equations of motion (1.2) are then generated by the Lagrangian

$$\mathscr{L}^{\mathrm{MAX}}(A)=-\tfrac{1}{4}F_{\mu\nu}^{\mathrm{MAX}}\left[\frac{\partial^2}{M^2}\right]F_{\mathrm{MAX}}^{\mu\nu}\,, \tag{1.5}$$

where

$$F_{\mu\nu}^{\mathrm{MAX}}\equiv\partial_\mu A_\nu-\partial_\nu A_\mu\,. \tag{1.6}$$

(The superscript and subscript MAX denote Maxwell.) However the Lagrangian (1.5) is not invariant under the non-Abelian gauge transformation

$$A_\mu\to\Omega^{-1}A_\mu\Omega-\frac{i}{e}\Omega^{-1}\partial_\mu\Omega\,. \tag{1.7}$$

The simplest Lagrangian which is invariant under the gauge transformation (1.7) and reduces to (1.5) in the Abelian limit is[9]

$$\mathscr{L}(A)=-\tfrac{1}{4}F^{\mu\nu}\frac{\mathscr{D}^2(A)}{M^2}F_{\mu\nu}\,, \tag{1.8}$$

where

$$F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu-ie[A_\mu,A_\nu] \tag{1.9}$$

and

$$\mathscr{D}_\mu(A)F=\partial_\mu F-ie[A_\mu,F]\,. \tag{1.10}$$

The A_μ as usual are matrices in color space, $A_\mu=\sum T_a A_\mu^a$, where the T_a are the generators of the color gauge group. In the Abelian theory all matrices A_μ are proportional to a single generator. We normalize all matrices so that $2\mathrm{Tr}(T_aT_b)=\delta_{ab}$.

However, the Lagrangian (1.8) does not give any[5] results beyond those already obtained from its Abelian limit. In particular the nonlinear terms do not prevent the spreading of electric flux lines. We now believe that the reason the Lagrangian (1.8) does not adequately describe long-distance Yang-Mills theory is that it was constructed assuming that the long-distance Lagrangian was a simple function of the vector potential A_μ. We have already seen that an expansion in powers of A_μ is not appropriate at long distances due to the singular nature of the propagator. In Sec. III we will obtain an alternate long-distance Lagrangian which also yields Eqs. (1.1), (1.2), and (1.3) in the Abelian limit, but unlike $\mathscr{L}(A)$ should be capable of describing long-distance Yang-Mills theory. To do this we will first write another Lagrangian for the Abelian theory which is completely equivalent to Eq. (1.5) but will suggest the alternate generalization.

II. USE OF ELECTRIC VECTOR POTENTIALS C_μ TO DESCRIBE THE ABELIAN THEORY

We can solve Eqs. (1.2) by writing

$$\mathbf{D}=-\nabla\times\mathbf{C},\quad \mathbf{H}=-\nabla C_0-\frac{\partial\mathbf{C}}{\partial t}\,. \tag{2.1}$$

We call $C^\mu=(C_0,\mathbf{C})$ the electric vector potential and define

$$G_{\mu\nu}^{\mathrm{MAX}}\equiv\partial_\mu C_\nu-\partial_\nu C_\mu\,. \tag{2.2}$$

Then Eqs. (1.1) can be written

$$(-\partial^2/M^2)\tilde{F}_{\mu\nu}=G_{\mu\nu}^{\mathrm{MAX}}\,, \tag{2.3}$$

where

$$\tilde{F}_{\mu\nu}=\tfrac{1}{2}\epsilon_{\mu\nu\lambda\sigma}F^{\lambda\sigma} \tag{2.4a}$$

and

$$F_{0k}\equiv E_k,\quad F_{ij}\equiv-\epsilon_{ijk}B_k\,. \tag{2.4b}$$

Equation (2.3) determines $\mathbf{E}$ and $\mathbf{B}$ in terms of C_μ, while Eqs. (1.3) are obtained as the equations of motion generated by the Lagrangian

$$\mathscr{L}^{\mathrm{MAX}}(C)=\tfrac{1}{4}G_{\mu\nu}^{\mathrm{MAX}}(-M^2/\partial^2)G_{\mathrm{MAX}}^{\mu\nu}\,. \tag{2.5}$$

This equivalence of the Lagrangians $\mathscr{L}^{\text{MAX}}(A)$, Eq. (1.5), and $\mathscr{L}^{\text{MAX}}(C)$, Eq. (2.5), was pointed out by Nair and Rosenzweig[10] and reflects the fact that sourceless electrodynamics in a linear relativistic dielectric medium can be described either in terms of magnetic vector potentials A_μ or electric vector potentials C_μ. $\mathscr{L}^{\text{MAX}}(C)$ is obtained by making the replacements $A_\mu \to C_\mu$, $\epsilon \to \mu = 1/\epsilon$ in $\mathscr{L}^{\text{MAX}}(A)$. The propagator $D_C(q^2)$ for the C_μ field is

$$D_C(q^2) = \frac{1}{q^2 \mu(q^2)} . \tag{2.6}$$

As $q^2 \to 0$, $\mu(q^2) \to M^2/q^2$ and hence $D_C(q^2) \to 1/M^2$; i.e., the C_μ propagator develops a mass if the A_μ propagator behaves like $1/q^4$ as $q^2 \to 0$.

Including the first correction to the $q^2 \to 0$ limit of ϵ, we obtain

$$\epsilon(q^2) \underset{q^2 \to 0}{\sim} q^2/M^2[1 + q^2/(fM)^2] \tag{2.7a}$$

or

$$\begin{aligned} \mu(q^2) &\underset{q^2 \to 0}{\sim} M^2/q^2[1 - q^2/(fM)^2] \\ &= M^2/q^2 - 1/f^2 , \end{aligned} \tag{2.7b}$$

The truncated Schwinger-Dyson equation determines only the leading behavior of ϵ as $q^2 \to 0$ and hence the constant f^2 in Eq. (2.7a) is not fixed.[11] Inserting Eq. (2.7b) into Eq. (2.6) gives

$$D_C(q^2) \to \frac{1}{M^2 - q^2/f^2} . \tag{2.8}$$

which shows that fM can be interpreted as the mass in the C_μ propagator. As pointed out by Nair and Rosenzweig[10] this indicates that a dielectric medium in which $\epsilon(q^2) \to q^2/M^2$ as $q^2 \to 0$ possesses some of the features of a magnetic superconductor, since the C_μ propagator develops a mass just as the A_μ propagator develops a mass in an electric superconductor (the Meissner effect).

Including the q^4 term in ϵ produces additional terms in Eqs. (1.1), (1.2), and (1.3). In the C_μ language the modified equations are obtained by replacing $-M^2/\partial^2$ by $-M^2/\partial^2 - 1/f^2$ in $\mathscr{L}^{\max}(C_\mu)$. Equation (2.5) is then replaced by

$$\mathscr{L}^{\text{MAX}}(C_\mu) = \tfrac{1}{4} G^{\text{MAX}}_{\mu\nu}(-M^2/\partial^2 - 1/f^2) G^{\mu\nu}_{\text{MAX}} . \tag{2.9}$$

It is convenient to choose C_μ and $\widetilde{F}_{\mu\nu}$ as independent variables in order to eliminate the $1/\partial^2$ factor in $\mathscr{L}^{\text{MAX}}(C_\mu)$. To do this we replace Eq. (2.9) by

$$\begin{aligned} &\mathscr{L}^{\text{MAX}}(C_\mu, \widetilde{F}^{\mu\nu}) \\ &\quad = \frac{1}{4}\left[2 G^{\text{MAX}}_{\mu\nu} \widetilde{F}^{\mu\nu} + \widetilde{F}^{\mu\nu} \frac{\partial^2}{M^2} \widetilde{F}_{\mu\nu} - \frac{1}{f^2} G^{\text{MAX}}_{\mu\nu} G^{\mu\nu}_{\text{MAX}} \right] . \end{aligned} \tag{2.10}$$

Then Eq. (2.3) becomes an equation of motion obtained by varying $\widetilde{F}_{\mu\nu}$ in Eq. (2.10). Varying C_μ in Eq. (2.10) yields

$$\partial_\mu \widetilde{F}^{\mu\nu} = \frac{1}{f^2} \partial_\mu G^{\mu\nu}_{\text{MAX}} . \tag{2.11}$$

The $(1/f^2)\partial_\mu G^{\mu\nu}_{\text{MAX}}$ term in Eq. (2.11) is the change [in Eq. (1.2)] due to the q^4 term in ϵ. The physical implications of Eqs. (1.1), (1.2), and (1.3) (discussed earlier) remain valid in the presence of this next leading term in ϵ.

III. LONG-DISTANCE LAGRANGIAN FOR YANG-MILLS THEORY EXPRESSED IN TERMS OF ELECTRIC VECTOR POTENTIALS

In order to proceed further we use the results of Mandelstam[12] who showed that starting with Yang-Mills theory,

$$\mathscr{L}^{\text{YM}}(A_\mu) = -2\,\text{Tr}\tfrac{1}{4}(F^{\mu\nu}F_{\mu\nu}) , \tag{3.1}$$

one can construct electric vector potential operators C_μ such that the theory is invariant under the group of transformations

$$C_\mu \to \Omega^{-1} C_\mu \Omega - (i/g)\Omega^{-1}\partial_\mu \Omega , \tag{3.2}$$

where $eg = 4\pi$. The group of transformations (3.2) is called the magnetic color gauge group in contrast to the electric color group of gauge transformation, Eq. (1.7). Mandelstam[12] was able to give a kinematic definition of the operators C_μ, but the explicit form of $\mathscr{L}^{\text{YM}}(C_\mu)$, the Yang-Mills Lagrangian expressed in terms of the C_μ fields, was not determined. It is presumably an extremely complicated function of the C_μ, since $\mathscr{L}^{\text{YM}}(A)$ has a simple dependence upon the magnetic vector potentials A_μ and the definition of the C_μ in terms of the A_μ is very complicated. Of course in an Abelian gauge theory Mandelstam's definitions are equivalent to those given in Sec. II and the explicit dependence of the Lagrangian upon the C_μ fields is known and is not complicated.

The idea of our approach is then the following. Because of asymptotic freedom the Yang-Mills Lagrangian (3.1), augmented by simple corrections, is the effective Lagrangian describing short-distance Yang-Mills theory. Because of Mandelstam's work short-distance Yang-Mills theory could in principle be described in terms of electric vector potentials C_μ, but as indicated above it would be extremely complicated. We will see that at long distance the situation is reversed and that the effective long-distance Lagrangian $\mathscr{L}(C)$ is simple when expressed in terms of electric vector potentials C_μ and hence necessarily complicated when expressed in terms of the A_μ. [In particular the effective Lagrangian (1.8) cannot be adequate to describe the long-distance Yang-Mills theory.]

We will not need Mandelstam's explicit construction of the operators C_μ in order to construct $\mathscr{L}(C)$. We only need that $\mathscr{L}(C)$ is invariant under the transformations (3.2) of the magnetic color gauge group and that in the Abelian limit it generates Eqs. (1.1)—(1.3). Furthermore we assume that $\mathscr{L}(C)$ is the minimal Lagrangian having these properties. We assume $\mathscr{L}(C)$ is minimal because in the linear Abelian theory the C_μ propagator develops a mass and hence the C_μ field falls off rapidly at large distances. Thus higher-order terms in the expansion of $\mathscr{L}(C)$ in the variables C_μ can be neglected at long distances. This contrasts with the higher-order corrections to $\mathscr{L}(A)$ [Eq. (18)] which are necessarily large at long distances because A_μ is singular there.

We then seek the minimal Lagrangian which is invariant under transformations (3.2) of the magnetic color gauge group and reduces to $\mathscr{L}^{\rm MAX}(C_\mu,\tilde{F}^{\mu\nu})$, Eq. (2.10), when all the fields are taken to be Abelian. This can be done by making the following replacement in Eq. (2.10):

$$G_{\mu\nu}^{\rm MAX}\to G_{\mu\nu}\equiv\partial_\mu C_\nu-\partial_\nu C_\mu-ig[C_\mu,C_\nu]\ , \tag{3.3}$$

$$\partial_\mu F\to\mathscr{D}_\mu(C)F\equiv\partial_\mu F-ig[C_\mu,F]\ . \tag{3.4}$$

This substitution yields the Lagrangian

$$\mathscr{L}=\frac{2\,{\rm Tr}}{4}\left[2G^{\mu\nu}\tilde{F}_{\mu\nu}+\tilde{F}_{\mu\nu}\frac{\mathscr{D}^2(C)}{M^2}\tilde{F}^{\mu\nu}-\frac{1}{f^2}G^{\mu\nu}G_{\mu\nu}\right]\ . \tag{3.5}$$

$\mathscr{L}$ is clearly invariant under the transformations (3.2) provided the tensor field $\tilde{F}_{\mu\nu}$ transforms under the magnetic color gauge group according to the rule

$$\tilde{F}_{\mu\nu}\to\Omega^{-1}\tilde{F}_{\mu\nu}\Omega\ . \tag{3.6}$$

The Lagrangian (3.5), aside from an additional term, will be the Lagrangian describing long-distance Yang-Mills theory. Before we obtain the final form for $\mathscr{L}$ we again emphasize that the Lagrangian (3.5) is not equivalent to any Lagrangian which has a simple structure in terms of the magnetic potentials A_μ. In particular, it differs fundamentally from the Lagrangian (1.8) even though both reduce in the Abelian limit to (2.10) and (1.5), respectively, which are equivalent.

More generally the Lagrangian (3.5) is essentially different from any non-Abelian Lagrangian which can be expressed simply in terms of magnetic vector potentials. Thus if it, or any other Lagrangian which has a simple structure in terms of the electric vector potentials C_μ, correctly describes long-distance Yang-Mills theory, then any simple approach based entirely upon the use of magnetic vector potentials A_μ is likely to be inadequate. The remark applies to the previous work of the present authors based on the Lagrangian (1.8), and perhaps as well to semiclassical methods in which certain simple configurations of the magnetic vector potential A_μ are assumed to dominate the functional integral.

It is convenient to make some scale changes in Eq. (3.5). We let

$$\mathbf{x}\to\mathbf{x}/fM,\quad C^\mu\to\frac{fM}{g}C^\mu,\quad \tilde{F}^{\mu\nu}\to\frac{M^2}{g}\tilde{F}^{\mu\nu}\ . \tag{3.7}$$

The new distance and fields are all dimensionless, and the fields are renormalization-group invariant. Then Eq. (3.5) becomes

$$\mathscr{L}=2\,{\rm Tr}\tfrac{1}{4}(fM)^2\left[\frac{M}{g}\right]^2[\,2G^{\mu\nu}\tilde{F}_{\mu\nu}+\tilde{F}_{\mu\nu}\mathscr{D}^2(C)\tilde{F}^{\mu\nu}-G^{\mu\nu}G_{\mu\nu}]\ , \tag{3.8}$$

where

$$G_{\mu\nu}=\partial_\mu C_\nu-\partial_\nu C_\mu-i[C_\mu,C_\nu]\ . \tag{3.9}$$

The Lagrangian (3.8) is the minimal gauge-invariant Lagrangian which reduces to Eq. (2.10) when the fields C_μ and $\tilde{F}^{\mu\nu}$ are taken to be Abelian.

We can add to $\mathscr{L}$ a gauge-invariant function of $\tilde{F}_{\mu\nu}$ alone, $-(fM)^2(M/g)^2W(\tilde{F}_{\mu\nu})$, provided W vanishes when $\tilde{F}_{\mu\nu}$ is chosen to be Abelian. The addition of any such W is consistent with gauge invariance and the Abelian limit [Eq. (2.10)].[13] This freedom in the choice of W reflects the effects of short distances on the long-distance Lagrangian. The function W plays the role of a non-Abelian Higgs potential and only certain general features of W will be important. An example of a possible form for W is

$$W(\tilde{F}_{\mu\nu})=\lambda_1 2\,{\rm Tr}[\tilde{F}_{\mu\nu},\tilde{F}_{\alpha\beta}]^2+\lambda_2 2\,{\rm Tr}[\tilde{F}_{\mu\nu},[\tilde{F}_{\alpha\beta},\tilde{F}_{\lambda\sigma}]]^2+\lambda_3\{2\,{\rm Tr}\tilde{F}_{\mu\nu}[\tilde{F}_{\nu\alpha},\tilde{F}_{\alpha\mu}]\}^2\ , \tag{3.10}$$

where λ_1, λ_2, and λ_3 are constants. We will see that only certain ranges of λ_i are possible and that the parameters λ_i determine the strength of the gluon condensate.

The quantity fM is the basic length scale in the theory; we shall from now on denote it by

$$M_f\equiv fM\ .$$

The combination $M/g=4\pi eM$ is the other renormalization-group-invariant mass which naturally appears. It is convenient, for simplicity in writing, to just call this M:

$$M/g\to M\ .$$

In this new notation we then have

$$M_f=fgM\ . \tag{3.11}$$

Adding a term $-W(\tilde{F}_{\mu\nu})$ to Eq. (3.9), we obtain our final expression for the action $S=\int d\mathbf{x}\,\mathscr{L}$ describing long-distance Yang-Mills theory:

$$S=\int d\mathbf{x}\,M^2M_f{}^2\{2\,{\rm Tr}\tfrac{1}{4}[\,2G^{\mu\nu}\tilde{F}_{\mu\nu}+\tilde{F}^{\mu\nu}\mathscr{D}^2(C)\tilde{F}_{\mu\nu}-G^{\mu\nu}G_{\mu\nu}]-W(\tilde{F}_{\mu\nu})\}\ . \tag{3.12}$$

Varying $\tilde{F}_{\mu\nu}$ in S then gives the equations of motion

$$G_{\mu\nu}=-\mathscr{D}^2(C)\tilde{F}_{\mu\nu}+\frac{\delta W}{\delta\tilde{F}^{\mu\nu}}\ , \tag{3.13}$$

while varying C_μ yields the equations of motion

$$\mathscr{D}_\mu(C)G^{\mu\nu}=\mathscr{D}_\mu(C)\tilde{F}^{\mu\nu}+(i/2)[\mathscr{D}_\nu(C)\tilde{F}^{\alpha\beta},\tilde{F}_{\alpha\beta}]\ . \tag{3.14}$$

Note that the function W enters only in Eq. (3.13), which is the non-Abelian generalization of Eqs. (2.3) giving the relation of $\mathbf{B}$ and $\mathbf{E}$ to the potentials C_μ via the dielectric properties of the vacuum. For simplicity we will from now on use the symbol $\mathscr{D}_\mu$ to denote $\mathscr{D}_\mu(C)$.

We next write the equations of motion (3.12) and (3.13) in terms of the fields $\mathbf{D}$, $\mathbf{H}$, $\mathbf{E}$, and $\mathbf{B}$. We define $\mathbf{D}$ and $\mathbf{H}$ in terms of C_μ by the equations

$$H_k=G_{0k},\quad \epsilon_{ijk}D^k=G_{ij}\ , \tag{3.15}$$

where $G_{\mu\nu}$ is defined by Eq. (3.9). Equation (3.15) is just the non-Abelian generalization of Eq. (2.1). [Note comparing (2.4b) and (3.15) we see that under the replacement $F_{\mu\nu}\to G_{\mu\nu}$;$\mathbf{E}\to\mathbf{H}$, $\mathbf{B}\to-\mathbf{D}$.]

Likewise Eqs. (2.4a) and (2.4b) suggest the following definitions of $\mathbf{E}$ and $\mathbf{B}$:

$$E_k=-\tfrac{1}{2}\epsilon_{kij}\widetilde{F}_{ij},\quad B_k=-\widetilde{F}_{0k}\ . \tag{3.16}$$

Note that in the non-Abelian theory the fields $\mathbf{E}$ and $\mathbf{B}$ defined by Eq. (3.16) are not the same as the fields $\mathbf{E}$ and $\mathbf{B}$ defined by Eq. (2.4b) with $F_{\mu\nu}$ given by Eq. (1.9). This is because $F_{\mu\nu}$ transforms according to the electric color gauge group while $\widetilde{F}_{\mu\nu}$ transforms according to the magnetic color gauge group. Thus, as pointed out by Mandelstam, in the non-Abelian theory an unknown color matrix relates the left- and right-hand sides of Eq. (2.4a). Only in the Abelian theory is $\widetilde{F}_{\mu\nu}$ the dual of $F_{\mu\nu}$. In any case we use Eq. (3.16) to define $\mathbf{E}$ and $\mathbf{B}$ since $\widetilde{F}_{\mu\nu}$ is the field appearing in our Lagrangian.

Let us write Eqs. (3.15), (3.13), and (3.14) in three-dimensional form: the $0k$ component of Eq. (3.15) is

$$\mathbf{H}=-\mathscr{D}C_0-\partial_0\mathbf{C}\ , \tag{3.17}$$

where the ij component of Eq. (3.15) is

$$\mathbf{D}=-\nabla\times\mathbf{C}-(i/2)[\mathbf{C}\times\mathbf{C}]\ . \tag{3.18}$$

The $0k$ component of Eq. (3.13) is

$$\mathbf{H}=\mathscr{D}^2\mathbf{B}+\frac{\partial W}{\partial\mathbf{B}}\ , \tag{3.19}$$

where the ij component of Eq. (3.13) is

$$\mathbf{D}=\mathscr{D}^2\mathbf{E}-\frac{\partial W}{\partial\mathbf{E}}\ . \tag{3.20}$$

In the above equations

$$\mathscr{D}F\equiv\nabla F+i[\mathbf{C},F]\ . \tag{3.21}$$

The 0 component of Eq. (3.14) is

$$\mathscr{D}\cdot\mathbf{H}=\{-\mathscr{D}\cdot\mathbf{B}+i[\mathscr{D}^0\mathbf{E},\cdot\mathbf{E}]-i[\mathscr{D}^0\mathbf{B},\cdot\mathbf{B}]\}\ , \tag{3.22}$$

while the k component of Eq. (3.14) is

$$-\mathscr{D}_0\mathbf{H}-\mathscr{D}\times\mathbf{D}=\mathscr{D}_0\mathbf{B}+\mathscr{D}\times\mathbf{E}-i[\mathscr{D}\mathbf{E},\cdot\mathbf{E}]+i[\mathscr{D}\mathbf{B},\cdot\mathbf{B}]\ . \tag{3.23}$$

Finally the Hamiltonian density derived from the Lagrangian (3.12) is

$$\mathscr{H}=M^2M_f{}^2\left[2\mathrm{Tr}\left\{\frac{H^2}{2}+\frac{\mathbf{D}^2}{2}+\frac{(\mathscr{D}\mathbf{E})^2}{2}+\frac{(\mathscr{D}\mathbf{B})^2}{2}+\frac{(\mathscr{D}_0\mathbf{B})^2}{2}-\tfrac{3}{2}(\mathscr{D}_0\mathbf{E})^2\right\}+W-\mathbf{E}\cdot\frac{\delta W}{\delta\mathbf{E}}\right]. \tag{3.24}$$

IV. ELECTRIC-FLUX-TUBE SOLUTIONS OF THE FIELD EQUATIONS

We first note that since C_μ and $\widetilde{F}_{\mu\nu}$ are vector and tensor fields, respectively, Lorentz invariance requires that the vacuum described by the classical solutions (3.13) and (3.14) have $\widetilde{F}_{\mu\nu}=C_\mu=0$, and hence is the trivial vacuum. Determination of the true vacuum requires studying Eq. (3.13) and (3.14) beyond the classical approximation so that vacuum expectation values of Lorentz- and gauge-invariant operators like $F^{\mu\nu}_{\mathrm{OP}}F^{\mathrm{OP}}_{\mu\nu}$ can be calculated. However, if we look for solutions to Eqs. (3.13) and (3.14) corresponding to nonvacuum solutions such as electric flux tubes or glueballs, then $\widetilde{F}_{\mu\nu}$ or C_μ can take on nonvanishing values because the orientation of the excitation selects a preferred direction in space (e.g., the axis of the electric flux tube). Furthermore the value of $\widetilde{F}_{\mu\nu}$ at large distances from the disturbance should reflect the properties of the true vacuum. We can thus obtain information about the vacuum expectation values by finding the large-distance behavior of $\widetilde{F}_{\mu\nu}$ for a nonvacuum solution of the classical equations (3.13) and (3.14). This is analogous to the situation in an isotropic (Heisenberg) ferromagnet for which the expectation value of any component of the magnetic field $\mathbf{B}$ vanishes since the ferromagnet can have any orientation. Only $\langle\mathbf{B}^2\rangle\neq0$. However in the presence of a disturbance (no matter how weak) which selects at a direction in space (the z axis), $\langle B_z\rangle\neq0$. Furthermore

$$\langle\mathbf{B}^2\rangle_{\substack{\text{unoriented}\\ \text{ferromagnet}}}\simeq\langle B_z\rangle^2_{\substack{\text{oriented}\\ \text{ferromagnet}}}\ .$$

In a similar way we should have

$$\langle\widetilde{F}^{\mathrm{OP}}_{\mu\nu}\widetilde{F}^{\mu\nu}_{\mathrm{OP}}\rangle_{\substack{\text{Yang-Mills}\\ \text{vacuum}}}\simeq\widetilde{F}_{\mu\nu}\widetilde{F}^{\mu\nu}\ ,$$

where $\widetilde{F}_{\mu\nu}$ is the large-distance limit of a nonvacuum solution of Eqs. (3.13) and (3.14).

The above comments (in more general terms) indicate that tensor "Higgs" fields can be used to produce spontaneous symmetry breakdown. Scalar Higgs fields ϕ are usually used since any tensor must have a vanishing vacuum expectation value and hence cannot produce spontaneous symmetry breaking in the classical approximation. However, tensor fields can have nonvanishing expectation values for nonvacuum solutions of the classical equations and the large-distance values of these solutions are related to the vacuum properties of the theory. A nonvanishing large-distance value of a tensor field is analogous to the large-distance value of a scalar Higgs field ϕ, corresponding to the nonvacuum solution of a set of classical equations involving ϕ The difference in the case of scalar fields is that this limiting value of ϕ is also a solution of the vacuum equations.

Our electric-flux-tube solutions will give a concrete realization of the use of tensor fields to produce spontaneous symmetry breaking. This idea is of course not restricted to Eqs. (3.13) and (3.14), or to QCD. It might also be relevant for weak and electromagnetic interactions or for grand unified theories. The fact that the vacuum of any gauge theory is a dielectric medium means that, aside from the fundamental non-Abelian gauge field and field tensor, there naturally appears a second tensor containing the $\mathbf{D}$ and $\mathbf{H}$ vectors. Because of the complicated dielectric properties of the vacuum, this tensor cannot be explicitly determined in terms of the original gauge fields, and hence must appear as an independent dynamical variable in the effective Lagrangian. The new tensor field can provide the mechanism for spontaneous symmetry breakdown of this Lagrangian. The fields $\tilde{F}_{\mu\nu}$ appearing in the action Eq. (3.12) provide a concrete realization of this. Of course the general case does not refer specifically to a color electric or color magnetic gauge group. Which one is relevant depends upon the dynamics. In the particular example of the Lagrangian (3.12) the dynamical variables correspond to the set $C_\mu, \tilde{F}^{\mu\nu}$.

Let us now show how these ideas apply in the problem of determining a static electric-flux-tube solution of Eqs. (3.13) and (3.14). We choose the z axis along the direction of the flux tube and introduce cylindrical coordinates r, ϕ, and z. Then at large distances from the flux tube, C_μ and $\tilde{F}_{\mu\nu}$ should approach a static solution of the equations

$$G_{\mu\nu}=0, \quad \mathscr{D}_\alpha \tilde{F}_{\mu\nu}=0, \quad \frac{\delta W}{\delta \tilde{F}_{\mu\nu}}=0 , \tag{4.1}$$

which are the particular solution of Eqs. (3.13) and (3.14) corresponding to a vacuum at large distances. We are looking for solutions of Eq. (4.1) which are not topologically equivalent to the trivial solution $C_\mu=\tilde{F}_{\mu\nu}=0$.

We will study the problem in terms of the three-dimensional variables $\mathbf{E}$, $\mathbf{B}$, $\mathbf{C}$, and C_0. In terms of these variables the action Eq. (3.12) takes the form

$$S=\int d\mathbf{x}\, M^2 M_f{}^2 \left[2\mathrm{Tr} \left[-\mathbf{D}\cdot\mathbf{E}-\frac{\mathbf{D}^2}{2}+\frac{\mathbf{E}\cdot\mathscr{D}^2\mathbf{E}}{2}+\mathbf{B}\cdot\mathbf{H} + \frac{\mathbf{H}^2}{2}-\frac{\mathbf{B}\cdot\mathscr{D}^2\mathbf{B}}{2} \right] - W(\mathbf{E},\mathbf{B}) \right] , \tag{4.2}$$

where the color magnetic $\mathbf{H}$ field and the displacement vector $\mathbf{D}$ are defined in terms of $\mathbf{C}$ and C_0 by Eqs. (3.17) and (3.18). We begin by looking for solutions where $\mathbf{B}=\mathbf{H}=0$. If $C_0=0$, then from Eq. (3.17) it follows that $\mathbf{H}=0$. (We are interested in static solutions.) Then Eqs. (3.19) and (3.22) are identically satisfied and Eqs. (3.18), (3.20), and (3.23) determine the vector potential $\mathbf{C}$ and the electric field $\mathbf{E}$. The action then reduces to

$$S=\int d\mathbf{x}\, M^2 M_f{}^2 \left[2\mathrm{Tr} \left[-\mathbf{D}\cdot\mathbf{E}-\frac{\mathbf{D}^2}{2}-\frac{\mathbf{E}\cdot\mathscr{D}^2\mathbf{E}}{2} \right] - W(\mathbf{E}) \right] . \tag{4.3}$$

The requirements

$$\frac{\delta S}{\delta \mathbf{E}}=0, \quad \frac{\delta S}{\delta \mathbf{C}}=0 \tag{4.4}$$

then yield Eqs. (3.20) and (3.23).

The following simple scaling argument shows that there are no solutions to Eqs. (4.4) if $W(E)=0$.

Suppose $\mathbf{E}(\mathbf{x})$ and $\mathbf{C}(x)$ are a solution to Eq. (4.4). Let

$$\mathbf{E}_a(\mathbf{x})=\mathbf{E}(\mathbf{x}/a), \quad \mathbf{C}_a(\mathbf{x})=a^{-1}\mathbf{C}(\mathbf{x}/a) \tag{4.5}$$

then

$$\mathbf{D}_a=\frac{1}{a^2}\mathbf{D}(\mathbf{x}/a) .$$

Let S_a be the value of the action (4.3) evaluated with fields given by Eq. (4.5); then Eqs. (4.4) imply $dS/da\,|_{a=1}=0$. Now since $d=2$ (cylindrical symmetry), $d\mathbf{x}=a^2 d(\mathbf{x}/a)$ and the first and third terms in (4.3) are independent of a. The a-dependent terms in S are then

$$S=M^2M_f{}^2\left[-a^2\int d\mathbf{x}\, 2\,\mathrm{Tr}\frac{D^2}{2} - a^2\int d\mathbf{x}\, W(\mathbf{E}) \right] .$$

Then the equation $dS/da\,|_{a=1}=0$ yields

$$\int d\mathbf{x}\, 2\,\mathrm{Tr}-\frac{D^2}{2}=\int d\mathbf{x}\, W(\mathbf{E}) . \tag{4.6}$$

Hence if $W(\mathbf{E})=0$, then $\mathbf{D}=0$, and it is then easy to show that $\mathbf{E}=0$ and $\mathbf{C}=0$. If W does not vanish, then we conclude from Eq. (4.6) only that

$$\int d\mathbf{x}\, W(\mathbf{E})>0 . \tag{4.7}$$

In this case one needs to use the explicit form (3.10) of $W(\mathbf{E})$ to demonstrate that there are no pure electric solutions with cylindrical symmetry. However Eq. (4.7) already indicates there will be problems, because of the following: As $r\to\infty$, $\mathbf{E}$ approaches $\mathbf{E}_\infty$, determined by Eq. (4.1). That is, $\mathscr{D}\mathbf{E}=0$, $\delta W/\delta\mathbf{E}=0$, for $\mathbf{E}=\mathbf{E}_\infty$. Furthermore the constant term in $W(\mathbf{E})$ is chosen so that the integral Eq. (4.7) converges, i.e., $W(\mathbf{E}_\infty)=0$. If for simplicity we neglect the multicomponent nature of the field $\mathbf{E}$, we have

$$W(\mathbf{E})\to\left[\frac{\delta^2 W}{\delta \mathbf{E}^2}\right]_{\mathbf{E}_\infty}(\mathbf{E}-\mathbf{E}_\infty)^2$$

as $r\to\infty$. Assuming that the large-distance contribution to the integral (4.7) is by itself negative we conclude

$$\left[\frac{\delta^2 W}{\delta \mathbf{E}^2}\right]_{\mathbf{E}=\mathbf{E}_\infty}>0 . \tag{4.8}$$

At large distances we can linearize Eq. (3.20) in the difference $(\mathbf{E}-\mathbf{E}_\infty)$. We can then show using Eq. (4.8) that these equations have no solutions for which $(\mathbf{E}-\mathbf{E}_\infty)$ vanish as $r\to\infty$. If the sign of Eq. (4.8) had been positive then there would be solutions which vanish exponentially as $r\to\infty$ as is necessary in order to have a confined electric flux tube.

Thus there must be a nonvanishing magnetic field $\mathbf{B}$ in order to have a confined electric flux tube. Let us look for static solutions of Eqs. (3.19), (3.20), (3.12), and (3.22) for the case that the gauge group is SU(2). We make the following ansatz for the electric and magnetic fields:

$$\mathbf{E}=E\hat{e}_z T_3, \quad \mathbf{B}=\mathbf{B}_1 T_1+\mathbf{B}_2 T_2 , \tag{4.9}$$

where E is a function of the cylindrical coordinate r. The electric field then lies along the z axis in ordinary space and the 3 axis in color space, while the magnetic field lies in the 1-2 plane in color space. We choose the vector potential to have the structure

$$\mathbf{C}=C\hat{e}_\phi T_3, \quad C_0=C_0 T_1 , \tag{4.10}$$

where we use the same notation C_0 for the coefficient of the matrix C_0 along T_1. The vector potential lies along the ϕ direction in ordinary space and along the 3 direction in color space. The **H** and **D** fields determined by Eqs. (3.17) and (3.18) are then given by

$$\mathbf{H}=-\nabla C_0 T_1 + CC_0\hat{e}_\phi T_2, \quad \mathbf{D}=-(\nabla\times C\hat{e}_\phi)T_3 . \tag{4.11}$$

The **H** and **D** vectors then have the same nonvanishing color components as **B** and **E**. We take C_0 and C to be functions only of r. Then **D** will also lie along the z axis, while **H** will have components in the r and ϕ directions. This suggests that we take

$$\mathbf{B}_1=B_1\hat{e}_r, \quad \mathbf{B}_2=B_2\hat{e}_\phi , \tag{4.12}$$

where B_1 and B_2 are functions of r.

With the above ansatz Eq. (3.20) has only a z component in space and a 3 component in color space and takes the form

$$-\frac{1}{r}\frac{d}{dr}(rC)=-(\nabla^2+C_0^2)E-\frac{\delta W}{\delta E} , \tag{4.13}$$

where

$$\nabla^2 E\equiv\frac{1}{r}\frac{d}{dr}\left[r\frac{dE}{dr}\right] .$$

Equation (3.19) lies in the 1-2 plane in color space and yields the two equations

$$-\frac{dC_0}{dr}=-\tilde{\nabla}^2 B_1+\frac{2CB_2}{r}+C^2B_1+\frac{\delta W}{\delta B_1} \tag{4.14}$$

and

$$CC_0=-\tilde{\nabla}^2 B_2+\frac{2CB_1}{r}+C^2B_2-C_0^2B_2+\frac{\delta W}{\delta B_2} , \tag{4.15}$$

where

$$\tilde{\nabla}^2\equiv\nabla^2-\frac{1}{r^2} . \tag{4.16}$$

Inserting the ansatz into our expression (3.10) for W yields the result

$$\begin{aligned} W=&\frac{-x}{2}[E^2(B_1^2+B_2^2)-B_1^2B_2^2] \\ &-\frac{z}{2}[E^2(B_1^4+B_2^4)-B_1^2B_2^2(B_1^2+B_2^2) \\ &-E^4(B_1^2+B_2^2)]-\frac{w}{2}(E^2B_1^2B_2^2) , \end{aligned} \tag{4.17}$$

where $x=-16\lambda_1$, $z=-32\lambda_2$, and $w=72\lambda_3$ from which the explicit form of Eqs. (4.13), (4.14), and (4.15) are readily obtained. It can be shown for the ansatz, Eq. (4a), that Eq. (4.17) gives the most general sixth-order expression for W. In this case all the sixth-order terms in W not included in Eq. (3.10) also have the form, Eq. (4.17).

Equation (3.22) lies entirely along the 1 axis in color space and has the form

$$(-\nabla^2 C_0+C^2C_0)=-\frac{1}{r}\frac{d}{dr}(rB_1)-CB_2+C_0(E^2-B_2^2) . \tag{4.18}$$

Equation (3.23) lies along the ϕ direction in ordinary space and the 3 direction in color space and has the form

$$-\tilde{\nabla}^2 C-C_0^2C=C_0B_2-\frac{\partial E}{\partial r}-C(B_1^2+B_2^2)-\frac{2B_1B_2}{r} . \tag{4.19}$$

Thus we see our ansatz is self-consistent and Eqs. (3.19), (3.20), (3.22), and (3.23) reduce to the five coupled equations (4.13)—(4.15), (4.18), and (4.19) for the five functions C, C_0, E, B_1, and B_2. The action Eq. (4.2) expressed in terms of these variables takes the form

$$\begin{aligned} S=\int d\mathbf{x}\, M^2M_f^2\Bigg[&\frac{E}{r}\frac{d}{dr}(rC)+\tfrac{1}{2}C\tilde{\nabla}^2C-\tfrac{1}{2}E\nabla^2E+\tfrac{1}{2}E^2C_0^2-B_1\frac{dC_0}{dr}+CC_0-\tfrac{1}{2}C_0(\nabla^2-C^2)C_0 \\ &+\tfrac{1}{2}B_1(\tilde{\nabla}^2-C^2)B_1+\tfrac{1}{2}B_2(\tilde{\nabla}^2-C^2+C_0^2)B_2-\frac{2CB_1B_2}{r}-W\Bigg] , \end{aligned} \tag{4.20}$$

while the Hamiltonian density $\mathscr{H}$ Eq. (3.24) becomes

$$\begin{aligned} \mathscr{H}=M^2M_f^2\Bigg[\frac{1}{2}\Bigg[&C_0(-\nabla^2+C^2)C_0+C(-\nabla^2)C+B_1(-\tilde{\nabla}^2+C^2)B_1+B_2(-\tilde{\nabla}^2+C^2+C_0^2)B_2 \\ &+\frac{4CB_1B_2}{r}+E(-\nabla^2-3C_0^2)E\Bigg]+W-\frac{EdW}{dE}\Bigg] . \end{aligned} \tag{4.21}$$

W is given in Eq. (4.17), and finally

$$W-\frac{EdW}{dE}=\frac{x}{2}[E^2(B_1^2+B_2^2)+B_1^2B_2^2]+\frac{z}{2}[3E^4(B_1^2+B_2^2)-E^2(B_1^4+B_2^4)-B_1^2B_2^2(B_1^2+B_2^2)]+\frac{w}{2}B_1^2B_2^2E^2 .$$

V. THE QUANTIZATION OF ELECTRIC FLUX

We will begin by finding the large-distance behavior of the solutions to Eqs. (3.19)—(3.23). This behavior is determined by the vacuum equations (4.1). In the three-dimensional notation Eqs. (4.1) are

$$\mathbf{D}=\mathbf{H}=0 , \tag{5.1}$$

$$\frac{\delta W}{\delta \mathbf{E}}=\frac{\delta W}{\delta \mathbf{B}}=0\ , \tag{5.2}$$

$$\mathscr{D}\mathbf{B}=\mathscr{D}\mathbf{E}=\mathscr{D}_0\mathbf{E}=\mathscr{D}_0\mathbf{B}=0\ . \tag{5.3}$$

First we look for solutions of Eq. (5.3). We restrict ourselves for simplicity to fields having the structure of our special ansatz [Eq. (4.10)] for which the vector potential $\mathbf{C}$ lies in a single color direction. Only these solutions of Eq. (5.3) are relevant to the problem of solving Eqs. (4.13)–(4.19). However it is also of interest to solve Eqs. (5.3) without making this ansatz. We can then find solutions of Eq. (5.3) in which the vector potential $\mathbf{C}$ does not point along a single direction in color space. These potentials will give the large-distance behavior of Eqs. (3.19)–(3.22) in a general gauge. We will later discuss this more general solution.

Inserting the ansatz (4.9) and (4.10) into the equations $\mathscr{D}_0\mathbf{B}=\mathscr{D}_0\mathbf{E}=0$ yields the equations

$$C_0\mathbf{B}_2=C_0\mathbf{E}=0\ . \tag{5.4}$$

Equation (5.4) can be satisfied by taking

$$C_0=0 \quad \text{as } r\to\infty\ . \tag{5.5}$$

The other possibility $\mathbf{E}=\mathbf{B}_2=0$ will conflict with Eq. (5.2). Next the equation $\mathscr{D}\mathbf{E}=0$ reduces to $\nabla\mathbf{E}=0$. Hence

$$E(r)\to\mathscr{E} \quad \text{as } r\to\infty\ , \tag{5.6}$$

where $\mathscr{E}$ is a constant. Finally the equation $\mathscr{D}\mathbf{B}=0$ yields

$$\begin{aligned}&\nabla\mathbf{B}_1+C\hat{e}_\phi\mathbf{B}_2=0\ ,\\&\nabla\mathbf{B}_2-C\hat{e}_\phi\mathbf{B}_1=0\ .\end{aligned} \tag{5.7}$$

Using Eqs. (4.12) for $\mathbf{B}_1$ and $\mathbf{B}_2$ we obtain from Eqs. (5.7), the following equations determining the large-distance behavior of B_1 and B_2:

$$\begin{aligned}&\frac{B_1}{r}+CB_2=0,\quad \frac{-B_2}{r}-CB_1=0 \quad \text{as } r\to\infty\ ,\\&\frac{dB_1}{dr}=0,\quad \frac{dB_2}{dr}=0\ .\end{aligned} \tag{5.8}$$

There are three solutions of Eqs. (5.8):

$$\begin{aligned}&\text{(a) } B_1=-B_2=b,\quad C=+1/r\ ,\\&\text{(b) } B_1=B_2=b,\quad C=-1/r\ ,\end{aligned} \tag{5.9}$$

or

$$\text{(c) } B_1=B_2=0,\quad C=0\ ,$$

where b is a constant. The first two solutions are physically equivalent. They correspond to a solution for which the vector potential at large distance satisfies

$$\int_{S_1} d\mathbf{l}\cdot\mathbf{C}=\pm 2\pi T_3\ . \tag{5.10}$$

The integral in Eq. (5.10) is over a large circle S_1 surrounding the z axis. Expressing Eq. (5.10) in terms of the original unscaled potential $\mathbf{C}_{\mathrm{us}}$ [see Eq. (3.7)] gives

$$\int_{S_1} d\mathbf{l}\cdot\mathbf{C}_{\mathrm{us}}=\pm\frac{2\pi}{g}T_3=\pm\frac{e}{2}T_3\ , \tag{5.11}$$

since $eg=4\pi$. The solutions (5.9a) and (5.9b) therefore correspond to an electric flux tube containing a quantum $e/2$ of electric flux. Expressed in terms of the unscaled color electric displacement vector $\mathbf{D}_{us}$ Eq. (5.11) becomes

$$\begin{aligned}\int_S \mathbf{D}_{\mathrm{us}}\cdot d\mathbf{S}&=-\int(\nabla\times\mathbf{C}_{\mathrm{us}})\cdot d\mathbf{S}\\&=-\int\mathbf{C}_{\mathrm{us}}\cdot d\mathbf{l}=\mp\frac{e}{2}T_3\ .\end{aligned} \tag{5.12}$$

The integral in (5.12) is over the plane perpendicular to the electric flux tube. To obtain this result the solution of Eqs. (4.13)–(4.19) must be regular everywhere. We will show later in this section that this is the case.

The third solution (c) of Eqs. (5.8) corresponds to the trivial perturbative vacuum. This solution will not satisfy the condition (5.2) because of the presence of the potential W. Equation (5.2) determines the value of the constants $\mathscr{E}$ and b in Eqs. (5.6) and (5.9).

Finally we note that the solution (5.5) and (5.9) automatically satisfies Eq. (5.1). Alternatively Eq.(5.5) follows from (5.1) and (5.9).

It is convenient to express the solution (a) in terms of the vector

$$\mathbf{B}_+\equiv\mathbf{B}_1+i\mathbf{B}_2\ . \tag{5.13}$$

From Eqs. (4.12) and (5.9a) we then have

$$\mathbf{B}_+=b(\hat{e}_r-i\hat{e}_\phi)=be^{i\phi}(\hat{e}_x-i\hat{e}_y)\ . \tag{5.14}$$

Next note that from the solution (5.9) of Eqs. (5.3) we can construct others by performing a gauge transformation about the three-axis in color space. The transformed vector potentials $\mathbf{C}'$ and electric field $\mathbf{E}'$ will still lie along the 3 axis. We choose the rotation angle $\chi(\phi)$ in color space to be a function only of ϕ. Then the transformed vector potential will also lie in the ϕ direction. We can write

$$\mathbf{E}'=E'\hat{e}_zT_3,\quad \mathbf{C}'=C'\hat{e}_\phi T_3\ . \tag{5.15}$$

Denote the one and two components of the transformed color magnetic fields by $\mathbf{B}_1'$ and $\mathbf{B}_2'$, respectively. $\mathbf{B}_1'$ and $\mathbf{B}_2'$ will in general have both r and ϕ components. The gauge transformation expressed in terms of the variables $\mathbf{B}_+'=\mathbf{B}_1'+i\mathbf{B}_2'$, C' and E' is

$$\mathbf{B}_+'=\mathbf{B}_+e^{i\chi(\phi)},\quad C'=C+\frac{1}{r}\frac{d\chi}{d\phi},\quad E'=E\ . \tag{5.16}$$

Using (5.9), (5.15), and (5.16) we then find the following family of solutions of Eqs. (5.3) as $r\to\infty$.:

$$\begin{aligned}&\mathbf{B}_+'=b\,e^{i\alpha(\phi)}(\hat{e}_x-i\hat{e}_y)\ ,\\&C'=\frac{1}{r}\frac{d\alpha(\phi)}{d\phi}\ ,\\&E'=\mathscr{E}\ ,\end{aligned} \tag{5.17}$$

where $\alpha(\phi)=\phi+\chi(\phi)$.

Combining Eqs. (5.16) and (5.17) and dropping the prime notation we can write Eq. (5.17) as

$$\mathbf{B}_1+i\mathbf{B}_2=b\,e^{i\alpha(\phi)}(\hat{e}_x-i\hat{e}_y)\ ,$$
$$\mathbf{C}=\frac{\hat{e}_\phi T_3}{r}\frac{d\alpha}{d\phi}\ , \tag{5.18}$$
$$\mathbf{E}=\mathscr{E}\,\hat{e}_z T_3\ .$$

Equations (5.18) satisfy Eqs. (5.3) for any value of $\alpha(\phi)$. Furthermore, substitution of Eq. (5.18) in the expression (3.10) for W yields an expression which is independent of α. Hence Eq. (5.18) also satisfies Eq. (5.2) for any value of α. The expressions given by Eqs. (5.18) are a one-parameter family of solutions for the vacuum. These solutions for $\mathbf{E}$ and $\mathbf{B}$ are characterized by the fact that they are invariant under a rotation by an angle θ about the 3 axis in color space combined with a rotation by angle $-\theta$ about the z axis in ordinary space. This is because under a rotation by θ about the z axis

$$\hat{e}_x-i\hat{e}_y\to e^{-i\theta}(\hat{e}_x-i\hat{e}_y),\quad \hat{e}_z\to\hat{e}_z\ .$$

The $e^{-i\theta}$ factor then compensates a color rotation by an angle θ. This combined invariance reflects the cylindrical symmetry of the vacuum solution. Equation (5.18) is the most general solution in which $\mathbf{C}$ lies along the 3 axis in color space.

Now let us return to the problem of finding flux-tube solutions to Eqs. (3.19)—(3.22). As $r\to\infty$ for fixed ϕ, the solution must approach a vacuum solution. It therefore must be of the form of (5.18) for some value of α. As we move around the circle surrounding the flux tube from $\phi=0$ to $\phi=2\pi$, the corresponding vacuum solution characterized by $\alpha(\phi)$ varies: Thus every solution, Eqs. (3.19)—(3.22), determines a map of the circle S_1 into group U(1) of phases $e^{i\alpha}$. Since the map must be single valued we must have

$$e^{i\alpha(\phi)}=e^{i\alpha(\phi+2\pi)}\ ,$$

i.e.,

$$\alpha(\phi+2\pi)=\alpha(\phi)+2\pi n\ ,$$

or

$$\frac{1}{2\pi}\int_0^{2\pi}d\phi\frac{d\alpha(\phi)}{d\phi}=n\ , \tag{5.19}$$

where n is an integer.

The solutions of Eqs. (3.19)—(3.22) then break up into classes of maps $S_1\to$U(1) such that the members of each class can be continuously deformed into each other. Each class is characterized by an integer n which determined how many times the map $S_1\to$U(1) winds around the circle. Consider those asymptotic solutions (5.18) belonging to the class n. Then, from Eqs. (5.18) and (5.19),

$$\int\mathbf{C}\cdot d\mathbf{l}=\int d\phi\frac{d\alpha(\phi)}{d\phi}T_3=\frac{n}{2}T_3\ . \tag{5.20}$$

Thus these solutions correspond to a flux tube having n units of quantized flux. Furthermore from (5.18) and (5.19) we see that as ϕ varies from 0 to 2π, the asymptotic solution for $\mathbf{B}_+$ covers the vacuum solution n times. The solution with $n=0$ is continuously deformable into the vacuum solution while the solution (5.14) belongs to the class $n=1$. Thus if $b\neq 0$, the solution of Eqs. (4.13)—(4.19) cannot be deformed into the trivial vacuum.

Geometrically, when $n=1$, the radial and tangential components of $\mathbf{B}$ remain radial and tangential as they rotate around the cylinder axis, so that after going all the way around, $\mathbf{B}_1$ and $\mathbf{B}_2$ have rotated exactly once (see Fig. 1). For $n>1$, the $\mathbf{B}$ vector rotates n times during one revolution around the axis; thus at an intermediate stage in the rotation what started out as radial and tangential components do not remain so[14] (see Fig. 2).

We can also solve Eqs. (5.3) with vectors whose components in cylindrical coordinates depend only upon r, but whose color and spatial structure are otherwise arbitrary. In this case we find a further solution of Eq. (5.3) in which

$$\mathbf{C}\to\frac{\hat{e}_\phi}{r}(A_1T_1+A_2T_2+A_3T_3)\ , \tag{5.21}$$

where

$$A_1{}^2+A_2{}^2+A_3{}^2=1\ .$$

In this solution the limiting value of $\mathbf{B}$ also has color components in the 3 color direction. Equation (5.21) has the structure of a gauge-rotated $n=1$ flux tube.

Finally we note the relation of these results to the vortices of quantized magnetic flux obtained by Nielsen and Olesen[15] in the Abelian Higgs model. The non-Abelian electric vector potential $\mathbf{C}$ plays the role of the Abelian vector potential $\mathbf{A}$ in that model, while the magnetic field vector $\mathbf{B}$ plays the role of the charged scalar Higgs field.

VI. SOLUTION OF EQS. (4.13)—(4.19)

We now discuss the solution to Eqs. (4.13)—(4.19). First we determine the constants $\mathscr{E}$ and b. Equations (5.2) reduce to the two conditions

$$\left.\frac{\delta W}{\delta B_1}\right|_{\substack{B_1=B_2=b\\ E=\mathscr{E}}}=0,\quad \left.\frac{\delta W}{\delta E}\right|_{\substack{B_1=B_2=b\\ E=\mathscr{E}}}=0\ . \tag{6.1}$$

Using Eq. (4.17) for W we obtain the following equations determining b and $\mathscr{E}$ in terms of the parameters x, z, and w in W:

$$\begin{gathered}2x\mathscr{E}-4z\mathscr{E}^3+2z\mathscr{E}b^2+w\mathscr{E}b^2=0\ ,\\ x(\mathscr{E}^2-b^2)-z(\mathscr{E}^4+3b^4-2b^2\mathscr{E}^2)+wb^2\mathscr{E}^2=0\ .\end{gathered} \tag{6.2}$$

Note $\mathscr{E}=0$ and $b^2=-x/3z$ is always one solution for any value of x, z, and w.

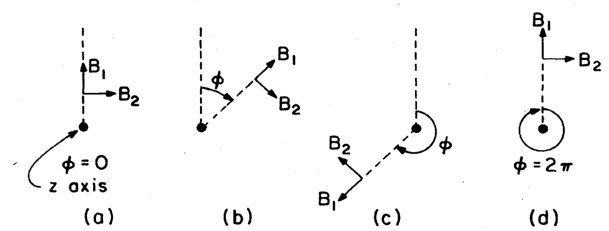

FIG. 1. The color magnetic fields $\mathbf{B}_1$ and $\mathbf{B}_2$ at large distances from an $n=1$ electric flux tube.

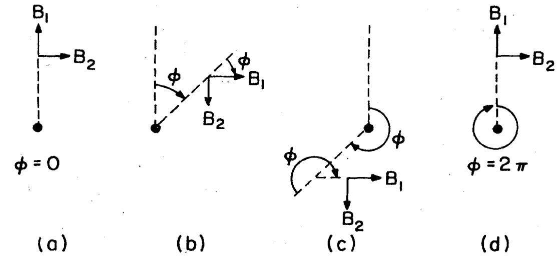

FIG. 2. The color magnetic fields $\mathbf{B}_1$ and $\mathbf{B}_2$ at large distances from an $n=2$ electric flux tube.

Next we determine the leading correction to the large-distance behavior of C, C_0, E, B_1, and B_2. We define functions g_1, g_2, f, f_1, and f_2 as

$$\begin{aligned} &C(r)=\frac{1}{r}+g_1,\quad C_0(r)=g_2\ ,\\ &E(r)=\mathscr{E}+f\ ,\\ &B_1(r)=b+f_1,\quad B_2(2)=-b+f_2\ . \end{aligned} \tag{6.3}$$

In order to have a confined flux tube the functions g_1, g_2, f, f_1, and f_2 must vanish exponentially at large distances. Equations (4.13)–(4.19) can then be linearized and become five coupled linear equations which can be solved analytically. The solutions are as follows: as $r\to\infty$

$$\begin{aligned} &g_1\sim K_1(\alpha r),\quad g_2\sim K_1(\alpha r),\quad f\sim K_0(\alpha r)\ ,\\ &f_1\sim K_0(\alpha r)+K_2(\alpha r),\quad f_2\sim K_0(\alpha r)-K_2(\alpha r)\ , \end{aligned} \tag{6.4}$$

where $K_n(x)$ is the modified Bessel function which decreases exponentially as $x\to+\infty$. The parameter α is a root of the following equation:

$$\begin{vmatrix} (\alpha^2-2b^2) & -b & \alpha & 0 & 0\\ b & (\alpha^2+\mathscr{E}^2-b^2) & 0 & \alpha & \alpha\\ \alpha & 0 & \alpha^2-\lambda & \eta & 0\\ 0 & \alpha & -\eta & 2(\alpha^2-\lambda_+) & 0\\ 0 & \alpha & 0 & 0 & 2(\alpha^2-\lambda_-) \end{vmatrix}=0\ , \tag{6.5}$$

The constants λ, λ_+, λ_-, and η are functions of the parameters x,z,w:

$$\begin{aligned} &\lambda=-4zb^4-6z\mathscr{E}^2b^2+wb^4\ ,\\ &\lambda_+=6zb^4+2z\mathscr{E}^2b^2-2w\mathscr{E}^2b^2\ ,\\ &\lambda_-=-4z\mathscr{E}^4-2z\mathscr{E}^2b^2+2zb^4\ ,\\ &\eta=-4z\mathscr{E}b^3-2w\mathscr{E}b^3\ , \end{aligned} \tag{6.6}$$

where $\mathscr{E}$ and b are determined in terms of x, z, and w by Eqs. (6.2). Each positive value of α which is a root of Eq. (6.5) generates a solution which dies exponentially. This means that the linearization of Eqs. (4.13)–(4.19) used to generate the large-distance solution is justified and Eq. (6.4) is a valid large-distance solution of these equations.

Equation (6.5) is a fifth-order equation in the variable α^2. Suppose it has five distinct positive roots. Equation (5.24) then gives five linearly independent solutions which have physically acceptable asymptotic behavior. On the other hand, it is easy to show that Eqs. (4.13)–(4.19) have five linearly independent solutions which have the following acceptable behavior at the origin:

$$\begin{aligned} &C_0\sim\text{constant},\quad E\sim\text{constant}\ ,\\ &C\sim r,\quad B_1\sim r,\quad B_2\sim r\ . \end{aligned} \tag{6.7}$$

We then integrate Eqs. (4.13)–(4.19) numerically starting at large r with a linear combination of Eqs. (6.4) and starting at small r with a linear combination of Eqs. (6.7). The constants defining these linear combinations are then determined by requiring that each of the five functions are continuous and have continuous derivatives at some intermediate matching point. Thus we expect that if Eq. (6.5) has five positive roots for α^2, then Eqs. (4.13)–(4.19) have a solution corresponding to a confined electric flux tube. We have explicitly verified this for several choices of the parameters x, z, and w. The solutions are all qualitatively similar and we will discuss one of them in detail later.

On the other hand, if Eq. (6.5) has a negative root for α^2, the solution oscillates at long distances and is not physically acceptable. Then there are no longer five linearly independent long-distance solutions and hence not enough freedom to match the five linearly independent short-distance solutions. Equations (4.13)–(4.19) then have no regular solutions. Next suppose Eq. (6.4) has a pair of complex conjugate solutions for α^2 with positive real part. These roots will yield a long-distance solution which is a product of an oscillating function and an exponentially vanishing function.

The long-distance behavior of the solution should be determined by the masses m_i of the lowest-lying states (glueballs) which couple to the flux tube. Thus at long distances the solution should decrease like a sum of exponentials $e^{-m_i r}$, and should not contain any oscillating factors. For this reason we have chosen the parameters x, z, and w so that all the roots α^2 are real and positive.

Using the above results we can now show that the pure electric problem described by the Lagrangian Eq. (4.3) has no solution. We can obtain the equations for the pure electric problem from Eqs. (4.13)–(4.19) by making the following substitution:

$$\begin{aligned} &E\to E,\quad C\to C,\quad B_1\to iE_\phi\ ,\\ &B_2\to iE_r,\quad C_0\to iC_z\ , \end{aligned} \tag{6.8}$$

where the electric field matrix $\mathbf{E}$ is

$$\mathbf{E}=E\hat{e}_z T_3+E_\phi \hat{e}_\phi T_1+E_r\hat{e}_r T_2 \ . \qquad (6.9)$$

Since

$$B_1\equiv B_r=-\widetilde{F}_{0r},\quad B_2\equiv B_\phi=-\widetilde{F}_{0\phi}\ ,$$

$$E_\phi=-\widetilde{F}_{zr},\quad E_r=\widetilde{F}_{z\phi},\quad E_z=-\widetilde{F}_{r\phi}\ ,$$

we see that the substitution (6.8) is just a replacement of the index zero by the index z in the fields C_μ and $\widetilde{F}_{\alpha\beta}$. Since our solutions are both time and z independent they must be invariant under such a transformation except for changes in sign due to changing a timelike direction into a spacelike direction. These changes of sign are accounted for by the factors i in Eq. (6.8). Equations for color electric fields are then obtained from Eqs. (4.13)—(4.19) by changing a few signs. The determinant which governs the long-distance behavior of these equations is formed by changing a few signs in Eq. (6.5). It can then be shown that for all values of the parameters x, z, and w, there is at least one nonpositive eigenvalue of α^2. We thus conclude that in two dimensions there are no regular pure electric field solutions to Eq. (3.19)—(3.23). This is the proof of the result asserted in Sec. IV. The statement that at least one eigenvalue is nonpositive is the rigorous generalization of Eq. (4.8).

We conclude that in order to have a confined electric flux tube we must have a nonvanishing magnetic field. Furthermore there is no nontrivial solution of Eqs. (4.13)—(4.19) with $b\neq 0$. Hence the magnetic field must be nonvanishing at large distances. Recall that in our dielectric medium described by electric vector potentials $\mathbf{C}$, the magnetic field $\mathbf{B}$ plays the role of $-\mathbf{D}$ in a normal dielectric medium. Therefore at large distances $\mathbf{B}$ is the magnetic dipole moment per volume of magnetic charge density, since $\mathbf{H}\rightarrow 0$ as $r\rightarrow\infty$. The distribution of magnetic charge density in the Yang-Mills vacuum plays the crucial role in producing a confined electric flux tube, just the electrically charged Cooper pairs of the superconducting state are responsible for quantized magnetic vortices.

To solve Eqs. (4.13)—(4.19) we must vary the parameters x, y, and w so that all the roots of Eq. (6.5) are positive. We have not done a thorough investigation of this problem, but have found that when $\mathscr{E}=0$ it is easy to obtain only positive roots. For certain values of x, z, and w it is also possible to have $\mathscr{E}\neq 0$ and all positive roots. However, we have only solved the full equations (4.13)—(4.19), when $\mathscr{E}=0$.

With $\mathscr{E}=0$, the second of Eqs. (6.2) gives

$$b^2=-\frac{x}{3z}\ ,$$

which means that x and z must have the opposite sign. From Eqs. (6.6) we see that if z is positive, both λ_+ and λ_- are positive. Then w must be positive in order to have $\lambda>0$. Thus the expression (4.17) for W possesses the essential feature that all three parameters λ_+, λ_-, and λ can be made positive. In the figures we plot the solutions corresponding to

$$z=0.506,\quad x=-2.61,\quad w=4.63\ .$$

This corresponds to $b=1.3$.

In Fig. 3 we plot the vector potential C and its radial derivative versus the dimensionless distance r. In Fig. 4 we plot the electric field $E(r)$ and the electric displacement vector $D(r)=-(1/r)(d/dr)[rC(r)]$. The confined nature of the electric flux is evident. In Fig. 5 we plot B_1 and B_2. Note that they remain close at all distances before approaching their asymptotic constant value of 1.3. In Fig. 6 we plot $r\mathscr{H}(r)$ where $\mathscr{H}$ is the energy density given by (4.21) and (4.22). The string tension is positive and has the value

$$\kappa=15.5M^2\ .$$

Finally we list in Table I the values of the parameters z, x, and w for four different solutions along with the values of the string tension κ and the magnetic condensate b.

We note that the value of the string tension is not very sensitive to large changes of the parameters z, x, and w.[16]

VII. ROLE OF THE "POTENTIAL" W

In the last section we explicitly solved Eqs. (4.13)—(4.15), (4.18), and (4.19) when W had the form (4.17), for three different choices of the parameters x, z, and w. It can be shown that the expression (4.17) for W is the most general sixth-order form for W. However there is no *a priori* reason why higher-order terms in W should not be included. The question then arises: To what extent do our results depend upon the choice of W? For this reason we review the role of the potential W in our fundamental action Eq. (3.12).

The action (3.12), for any choice of $W(\widetilde{F}_{\mu\nu})$ which vanishes when $\widetilde{F}_{\mu\nu}$ is Abelian, yields a theory which is invariant under non-Abelian transformations of the color magnetic gauge group, and which in the Abelian limit describes a linear dielectric medium with $\epsilon=-\nabla^2$. Furthermore, in the theory described by the action (3.12) there either exist tubes of quantized electric flux [solutions (5.9a) or (5.9b)] or else all fields vanish at large distances as in the perturbative vacuum [solution (5.9c)]. Which of these solutions is realized depends upon the properties of W. The flux-tube solution will be realized provided W has the following properties:

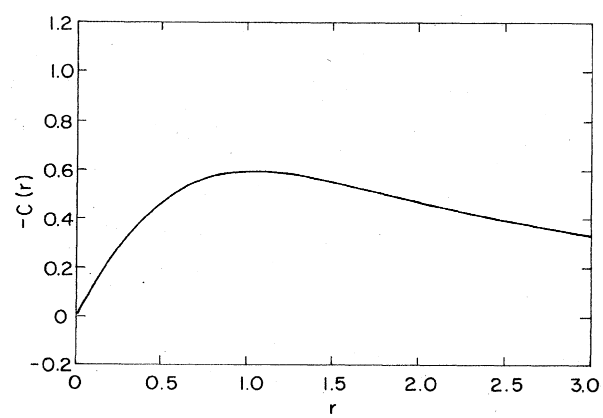

FIG. 3. The electric vector potential $C(r)$ vs r.

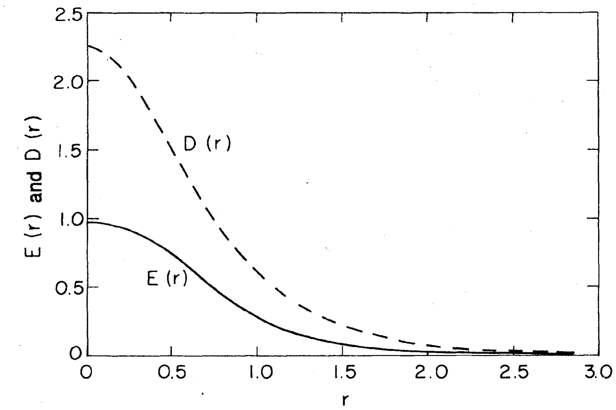

FIG. 4. The color electric field $E(r)$ and the color electric displacement vector $D(r)=-(1/r)(d/dr)[rC(r)]$ vs r.

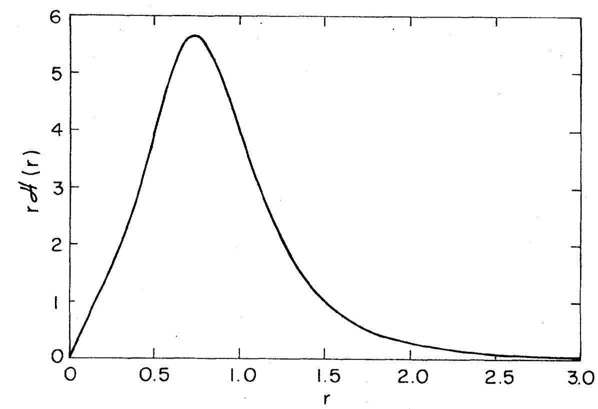

FIG. 6. The energy density $\mathscr{H}(r)$ multiplied by r vs r.

(1) W has a nontrivial minimum; i.e., there exist solutions of Eq. (6.1) with $b\neq 0$.

(2) W has the right curvatures at the minimum so that the deviations of the solutions from their large-distance limiting values decrease exponentially. For the particular case of our sixth-order expression (4.17) for W, this meant that Eq. (6.5) has five positive roots. The parameters λ, λ_+, λ_-, and η in Eq. (6.5) are determined by W according to Eq. (6.6). For the case of a general W (having the correct symmetry properties) the same Eq. (6.5) remains valid, where the parameters λ, λ_+, λ_-, and η are determined in terms of the second derivatives of W at the minimum rather than by Eq. (6.6).

The existence of a flux-tube solution thus depends only upon a few general properties of W. However, the predictive power of the theory depends upon the sensitivity of physical results to the detailed structure of W. From Table I we obtain some information about the sensitivity of the string tension to the parameters determining W. Let us now see how other physical quantities depend upon W.

Since the electric field vanishes at large distances from the flux tube while the magnetic color fields $\mathbf{B}_1$ and $\mathbf{B}_2$ are nonvanishing, the flux tube orients the one and two components of the color magnetic field at large distances, while $\mathbf{B}_3$ and all three components of the color electric field at large distances remain random as in the vacuum. As discussed in Sec. IV, if fluctuations are not large we should expect that the vacuum expectation value of the operator $\mathbf{B}_1^{2\mathrm{OP}}$ is of the order of the square of $\mathbf{B}_1$ at large distances from the flux tube; i.e.,

$$\langle \mathbf{B}_1^{2\mathrm{OP}}\rangle_{\mathrm{vac}} \approx g^2 M^4 b^2 , \tag{7.1}$$

where we have taken account of the rescaling Eq. (3.7).

On the other hand, by Lorentz invariance we have

$$\langle \mathbf{E}_\alpha^{2\mathrm{OP}}\rangle_{\mathrm{vac}} = -\langle \mathbf{B}_\alpha^{2\mathrm{OP}}\rangle_{\mathrm{vac}},\quad \alpha=1,2,3 , \tag{7.2}$$

and since the vacuum is a color singlet

$$\langle \mathbf{B}_1^{2\mathrm{OP}}\rangle_{\mathrm{vac}} = \langle \mathbf{B}_2^{2\mathrm{OP}}\rangle_{\mathrm{vac}} = \langle \mathbf{B}_3^{2\mathrm{OP}}\rangle_{\mathrm{vac}} . \tag{7.3}$$

Recalling that $\mathbf{E}$ and $\mathbf{B}$ are defined in terms of $\tilde{F}^{\mu\nu}$, Eq. (3.16), we have

$$-\langle \tilde{F}^{\mu\nu}_{\mathrm{OP}} \tilde{F}^{\mathrm{OP}}_{\mu\nu}\rangle_{\mathrm{vac}} = 2\sum_{\alpha=1}^{3} (\langle \mathbf{B}_\alpha^{2\mathrm{OP}}\rangle_{\mathrm{vac}} - \langle \mathbf{E}_\alpha^{2\mathrm{OP}}\rangle_{\mathrm{vac}})$$

$$\approx 12 g^2 M^4 b^2 . \tag{7.4}$$

We can use Eq. (7.4) to determine b provided we can make the identification

$$\langle F^{\mu\nu}_{\mathrm{OP}} F^{\mathrm{OP}}_{\mu\nu}\rangle_{\mathrm{vac}} = -\langle \tilde{F}^{\mu\nu}_{\mathrm{OP}} \tilde{F}^{\mathrm{OP}}_{\mu\nu}\rangle_{\mathrm{vac}} , \tag{7.5}$$

for we then have from Eqs. (7.4) and (7.5)

$$G_2 \equiv \frac{\alpha_s}{\pi} \langle F^{\mu\nu}_{\mathrm{OP}} F^{\mathrm{OP}}_{\mu\nu}\rangle_{\mathrm{vac}} \approx 48 b^2 M^4 , \tag{7.6}$$

where $\alpha_s = e^2/4\pi$.

The assumption on which (7.6) is based is that the asymptotic magnetic fields $\mathbf{B}_1$ and $\mathbf{B}_2$, in the presence of an electric flux tube, have the same magnitude as do the fluctuating fields $\mathbf{B}_1^{\mathrm{OP}}$ and $\mathbf{B}_2^{\mathrm{OP}}$ in the physical vacuum. The flux tube aligns the fields but does not change their magnitude.

We could, however, instead make the assumption that the flux tube leaves unchanged the Lorentz-invariant combination $F^{\mathrm{OP}}_{\mu\nu} F^{\mu\nu}_{\mathrm{OP}}$. If we do this, then (7.6) would be replaced by

$$G_2 \simeq -\frac{\alpha_s}{\pi} \tilde{F}_{\mu\nu} \tilde{F}^{\mu\nu} = 16 b^2 M^2 , \tag{7.7}$$

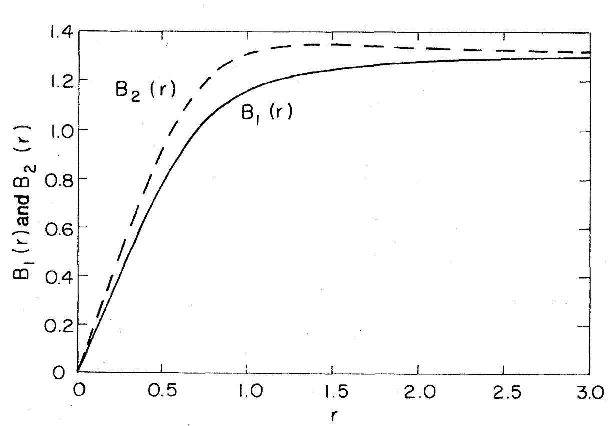

FIG. 5. The color magnetic fields $B_1(r)$ and $B_2(r)$ vs r.

TABLE I. Values of the string tension κ and the magnetic condensate b for four choices of the parameters determining W.

Solution	z	x	w	κ/M^2	b
(a)	0.0107	-0.155	0.0854	12.2	2.2
(b)	0.0213	-0.310	0.1707	10.1	2.2
(c)	0.25	-1.5	1.125	8.5	$\sqrt{2}$
(d)	0.506	-2.61	4.63	15.5	1.3

a result three times smaller.

We do not know which of these two assumptions is more plausible, and the difference between (7.6) and (7.7) therefore indicates the degree of uncertainty in our estimate of G_2. For specific numerical calculations, we shall make use of (7.7) in what follows.

The operator $F_{\mu\nu}$ on the left-hand side of Eq. (7.5) is the usual Yang-Mills field tensor defined in terms of the potentials A_μ. As mentioned earlier, the quantities $\widetilde{F}_{\mu\nu}$ and $\frac{1}{2}\epsilon_{\mu\nu\lambda\rho}F^{\lambda\rho}$ are related by an unknown color matrix. Furthermore as pointed out by Mandelstam[12] one cannot even make the identification, Eq. (7.5), because of singularities in the product of operators at the same point. However if we neglect such problems and provisionally accept the identification (7.5), then we obtain Eq. (7.6) or (7.7) determining the product b^2M^4 in terms of G_2. Although this relation involves assumptions we cannot as yet justify, it makes explicit how a vacuum correlation function can be related to the asymptotic limit of a nonvacuum problem in the classical approximation.

The quantity G_2 has been calculated in SU(2) lattice gauge theory[17] with the result

$$G_2 \approx 0.42\kappa^2 .$$

Taking the string tension to be its experimental value of about 0.2 GeV^2 then gives

$$G_2 \approx 0.017\ \mathrm{GeV}^4 ,$$

which is roughly the value determined from QCD sum rules.[18]

A further constraint on W can be obtained by using the SU(2) relation[17]

$$\epsilon_{\mathrm{vac}} = -\tfrac{11}{48}G_2 , \tag{7.8}$$

where ϵ_{vac} is the difference between the energy of the nonperturbative vacuum in the energy of the perturbative vacuum. Equation (7.8) is a consequence of the stress-energy-tensor trace anomaly.[18] Combining Eq. (7.8) with $G_2 \approx 0.42\kappa^2$ gives the SU(2) lattice value for ϵ_{vac}:

$$\epsilon_{\mathrm{vac}} \approx -0.096\kappa^2 . \tag{7.9}$$

We can compare this with ϵ_{vac} calculated from the large-distance limit of the Hamiltonian density (3.24). Since $\mathbf{E}\to 0$ as $r\to\infty$, the only term which remains is W. We thus have

$$\epsilon_{\mathrm{vac}} = M^2M_f{}^2\{W[\widetilde{F}_{\mu\nu}=\widetilde{F}_{\mu\nu}(r=\infty)] - W(\widetilde{F}_{\mu\nu}=0)\} . \tag{7.10}$$

Using the expression (4.17) for W we obtain

$$\epsilon_{\mathrm{vac}} = \left[\frac{x}{2}b^2 - zb^6\right]M^2M_f{}^2 = -\tfrac{1}{2}zb^6M^2M_f{}^2 . \tag{7.11}$$

The scale M is determined by the string tension, and the scale M_f can then be found from ϵ_{vac}.

In Table II we list M^2, G_2, and $M_f{}^2$ for the four solutions listed in Table I, in order to indicate how these physical quantities depend upon the choice of W. We must remember that these formulas for G_2 and ϵ_{vac} involve assumptions and should be regarded only as estimates.

These should be compared with the result for SU(2) lattice gauge theory, $G_2 \approx 0.42\kappa^2$. (The parameters in the four solutions were not specifically chosen to fit this quantity because of the uncertainty in how to estimate to G_2.) We see that although in most cases the quantities have the right order of magnitude there is significant sensitivity to the choice of W. Using the experimental value of about 0.2 GeV^2 for κ the value of M ranges from 113 to 153 MeV, while the value of M_f varies from 400 to 700 MeV. We recall that the size of the flux tube is determined by the mass αM_f [see Eq. (6.4)]. This ranges from 255 MeV to 1 GeV for the different solutions. This mass is related to the lowest glueball mass.

TABLE II. Values of M^2, G_2, and $M_f{}^2$ for the four solutions listed in Table I.

Solution	z	b	M^2	$G_2 \equiv 16b^2M^4$	$M_f{}^2$
(a)	0.0107	2.2	$\frac{\kappa}{12.2}$	$0.53\kappa^2$	1.95κ
(b)	0.0213	2.2	$\frac{\kappa}{10.1}$	$0.77\kappa^2$	0.80κ
(c)	0.25	$\sqrt{2}$	$\frac{\kappa}{8.5}$	$0.44\kappa^2$	0.81κ
(d)	0.506	1.3	$\frac{\kappa}{15.5}$	$0.12\kappa^2$	1.17κ

VIII. SUMMARY AND DISCUSSION

We have shown that the action (3.12) describing long-distance Yang-Mills theory in terms of electric vector potentials C_μ and dual tensors $\tilde{F}_{\mu\nu}$ yields a confining theory. We have studied to some extent the sensitivity of the parameters of the flux-tube solution to the choice of the potential W. This question must be studied more carefully to see whether the action (3.12) can become a quantitative tool for studying long-distance gluon physics. There is also the possibility that one can find physical principles which could specify W in more detail. Such principles must of course go beyond gauge invariance and our solution of the Dyson equation which to a large extent leave W undetermined. In the meantime we are looking for solutions of Eqs. (3.19)–(3.23) which could correspond to glueballs to check the applicability of the action to other problems.

Aside from the relevance of the specific action (3.12), some of the general ideas presented here may prove useful. The first is that electric vector potentials are natural variables for studying long-distance hadron physics and they can be used concretely to obtain physical results. The second is that it is natural to use tensor fields to describe spontaneous symmetry breaking. The third and related idea is that vacuum properties can be studied by investigating nonvacuum configurations in the classical approximation where tensor fields can produce the spontaneous symmetry breakdown. Finally we emphasize that the spontaneous symmetry-breaking mechanism is essentially non-Abelian. An Abelian configuration of fields reduces the Lagrangian Eq. (3.8) to a quadratic form describing a linear dielectric medium with dielectric constant $-\nabla^2$. In particular there is no contribution to the potential $W(F)$ from Abelian field configurations.

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