Top-Quark Mass and Bottom-Quark Decay
Paul H. Ginsparg and Sheldon L. Glashow
Lyman Laboratory of Physics, Harvard University, Cambridge, Massachusetts 02138

and

Mark B. Wise
Lauritsen Laboratory of High Energy Physics, California Institute of Technology, Pasadena, California 91125

(Received 7 February 1983)

The possibility of a long $B$-meson lifetime is explored, in which case the weak mixing angles $\theta_1$ and $\theta_2$ are quite small. This allows the derivation of a lower bound on the top-quark mass as a function of the $B$-meson lifetime, by comparison of the short-distance prediction for the $CP$-nonconservation parameter $\epsilon$ with its experimental value. The bound is significant for $\tau_B > 4 \times 10^{-15}$ s.

PACS numbers: 14.80.Dq, 12.35.Eq, 12.70.+q, 14.40.Jz

In the standard six-quark model of the strong, weak and electromagnetic interactions the quarks are in left-handed doublets

$$\begin{pmatrix} u \\ d' \end{pmatrix}_L, \begin{pmatrix} c \\ s' \end{pmatrix}_L, \begin{pmatrix} t \\ b' \end{pmatrix}_L$$

and right-handed singlets of the weak SU(2) gauge group. The primed fields are not mass eigenstates, but are related to the mass eigenstate fields $d_L$, $s_L$, and $b_L$ by the unitary transformation\(^1\)

$$\begin{pmatrix} d' \\ s' \\ b' \end{pmatrix}_L = \begin{pmatrix} c_1 & -s_1c_3 & -s_1s_3 \\ s_1c_2 & c_1c_2c_3 - s_2s_3e^{i\delta} & c_1s_2c_3 + s_2s_3e^{i\delta} \\ s_1s_2 & c_1s_2c_3 + c_2s_3e^{i\delta} & c_1s_2s_3 - c_2s_3e^{i\delta} \end{pmatrix} \begin{pmatrix} d \\ s \\ b \end{pmatrix}_L,$$ \hspace{1cm} (2)

where $c_i = \cos \theta_i$ and $s_i = \sin \theta_i$, $i \in \{1, 2, 3, \delta\}$. The phases of the quark fields are chosen so that $\theta_1$, $\theta_2$, and $\theta_3$ lie in the first quadrant. The quadrant of $\delta$ has physical significance and cannot be fixed by convention. Experimental information from beta decay gives $s_\delta^2 \approx 0.05$. The observed validity of Cabibbo universality implies the further constraint\(^6\) $s_\delta < 0.5$.

The $B$ meson can have a lifetime as short as $\sim 10^{-14}$ s. On the other hand, if $s_2$ and $s_3$ are small, the lifetime can be arbitrarily long. With $s_2$ and $s_3$ small, the top quark must be heavy in order to obtain the observed degree of $CP$ nonconservation in kaon decay. In this paper, we compute the lower limit to the top-quark mass as a function of the lifetime of the $B$ particle.

Various constraints of the mixing angles have been derived from comparisons of the measured values of the $CP$-nonconservation parameter $\epsilon$ and the $K_L - K_S$ mass difference with predictions for these quantities based on a short-distance expansion.\(^3\) An upper bound on the top-quark mass has been derived by comparing short-distance predictions for the $K_L - K_S$ mass difference and the $K_L - \mu^+\mu^-$ decay rate with experiment.\(^4\) Unfortunately these predictions are not reliable, since it is difficult to justify a short-distance expansion for the $K_L - K_S$ mass difference. Higher-dimension operators, such as the time-ordered product of two effective Hamiltonians for $\Delta s = 1$ weak nonleptonic decays, are neglected because they lack a factor of $m_t^2$. Matrix-element enhancements, such as those that take place in $K \to \pi(I = 0)$ decay,\(^5\) probably make these higher-dimension operators more important than the short-distance piece. The same criticism does not necessarily apply to the use of a short-distance expansion for the $CP$-nonconservation parameter $\epsilon$. It is likely that the higher-dimension operators do not contribute a significant imaginary part to the $K^0 - \bar{K}^0$ mass matrix in the basis in which the $K^0 - \bar{K}^0$ amplitude is real.

Since the $b$ quark is heavy compared with the scale of the strong interactions, $B$-meson decay can be approximated by the decay of a free $b$
quark. Then
\[ \Gamma_B = \Gamma(b - c) + \Gamma(b - u), \quad (3) \]
where
\[ \Gamma(b - c) = (G_F^2 m_b^5 / 192\pi^3) \left[ (c_s s_3 c_2 s_5 - s_2 c_2 s_5)^2 + s_2^2 c_2^2 s_5^2 \right] \left\{ 2 f(m_c / m_b) + (m_c / m_b)^2 \right\}, \]
\[ + 3 \eta_0 f(m_c / m_b) \left[ (c_s^2 s_3^2 + s_2^2 c_2^2 s_5^2) \left\{ 2 f(m_c / m_b) + (m_c / m_b)^2 \right\} \right], \]
\[ \Gamma(b - u) = (G_F^2 m_b^5 / 192\pi^3) \left[ (s_s^2 s_5^2) \left\{ 2 f(m_r / m_b) + (m_r / m_b)^2 \right\} \right], \]
\[ + 3 \eta_0 f(m_c / m_b) \left[ (c_s^2 s_3^2 + s_2^2 c_2^2 s_5^2) \left\{ 2 f(m_r / m_b) + (m_r / m_b)^2 \right\} \right]. \quad (4a) \]
In Eqs. (4), \( f \) and \( \varphi \) are phase-space suppression factors.\(^6\) For \( m_c = 1.4 \text{ GeV} \) and \( m_b = 4.6 \text{ GeV} \) they have the values \( f(m_c / m_b) = 0.51, \ h(m_c / m_b) = 0.19, \ f(m_r / m_b) = 0.33, \) and \( \varphi(m_c / m_b, m_r / m_b) = 0.09. \) The quantity \( \eta_0 \) takes into account strong-interaction corrections to the effective Hamiltonian for \( |b - u| = 1 \) weak nonleptonic decays, and in the leading logarithmic approximation (neglecting penguin-type contributions)
\[ \eta_0 = \frac{1}{2} \left\{ \frac{\alpha_s(M_W^2)}{\alpha_s(m_t^2)} \right\}^{12/21} \left\{ \frac{\alpha_s(m_b^2)}{\alpha_s(m_t^2)} \right\}^{12/23} + \left\{ \frac{\alpha_s(m_b^2)}{\alpha_s(m_t^2)} \right\}^{24/21} \left\{ \frac{\alpha_s(m_b^2)}{\alpha_s(m_t^2)} \right\}^{24/23}. \quad (5) \]
Using the quark masses mentioned previously, \( M_W = 80 \text{ GeV} \), and \( \Lambda_{QCD} = 0.1 \text{ GeV} \), we find \( \eta_0 = 1.1 \) (note that \( \eta_0 \) is roughly independent of the 

For long \( B \)-meson lifetimes, the angles \( \theta_2 \) and \( \theta_3 \) are small, and to first nontrivial order in these small quantities, Eqs. (4) imply
\[ (s^2 + s_5^2 + 2 s_2 s_5 c_2) = 4.2 \times 10^{-2} R(b - c) \left( \frac{T_B}{10^{-12} \text{ s}} \right)^{-1}, \quad (6a) \]
\[ s_3^2 = 3.9 \times 10^{-2} R(b - u) \left( \frac{T_B}{10^{-12} \text{ s}} \right)^{-1}, \quad (6b) \]
where \( R \) denotes branching ratio. Experimentally,\(^7\) \( R(b - u) < 0.09. \) “Higher-order” contributions to the \( B \)-meson lifetime would give even smaller angles. These effects, however, appear to be very small.\(^8\)

The imaginary part of the \( K^0 - \overline{K}^0 \) mass matrix violates \( CP \) conservation and can be reliably calculated by use of a short-distance expansion. Neglecting \( CP \) nonconservation from \( K \to 2 \pi \) decay amplitudes,\(^9\) we find for the \( CP \)-nonconservation parameter \( \epsilon \)
\[ \epsilon = \frac{s_3^2}{16\sqrt{2} \pi^2 (m_s - m_b)} \left [ c_s s_2 s_3 s_5 \left \{ - c_1 c_2^2 c_3 + s_2 c_2^2 s_3 s_5 \right \} + \eta_1 \left \{ - c_1 c_3^2 c_2 - s_2 c_3^2 s_5 \right \} \right ] \right ] e^{i \pi/4}. \quad (7) \]
In Eq. (7), \( \eta_1, \eta_2, \) and \( \eta_3 \) take into account strong-interaction corrections to the effective Hamiltonian for \( |\Delta s| = 2 \ K^0 - \overline{K}^0 \) mixing.\(^10\) They are roughly independent of the value of the top-quark mass, and for \( m_t = 1.4 \text{ GeV} \), \( m_t = 4.6 \text{ GeV} \), \( M_W = 80 \text{ GeV} \), \( \Lambda_{QCD} = 0.1 \text{ GeV} \), and \( \alpha_s(g^2) = 1 \) the following approximate values: \( \eta_1 = 0.7, \eta_2 = 0.6, \) and \( \eta_3 = 0.4. \) \( B \) is the factor that relates the \( K^0 - \overline{K}^0 \) matrix element of the local operator \( \overline{s_s \gamma^0 (1 - \gamma_5) d_d} \times [s_s \gamma_5 (1 - \gamma_5) d_d] \) to \( f m_s^3. \) We use \( f = 0.13 \text{ GeV}. \) In the soft pion and kaon limit the magnitude of \( B \) is determined in terms of the measured \( K \to \pi \pi \) amplitude and the coefficient of the \( I = \frac{3}{2} \) operator in the effective Hamiltonian for \( |\Delta s| = 1 \) weak nonleptonic decays.\(^11\) With the parameters used previously, the magnitude of \( B \) is equal to 0.37. Note that the quantities \( \eta_1, \eta_2, \) and \( \eta_3 \) depend on the subtraction point \( \mu \) as \( \left[ \alpha_s(\mu^2) \right]^{1/2} \) while \( B \) depends on the subtraction point as \( \left[ \alpha_s(\mu^2) \right]^{1/2}. \) leaving the physical quantity \( \epsilon \) independent of our arbitrary choice of subtraction point.

We can derive a lower bound on the top-quark mass, \( m_t, \) by substituting the experimental values\(^2\) \( \epsilon = (2.27 \times 10^{-5}) e^{i \pi/4} \) and \( m_{K_L} - m_{K_S} = 3.5 \times 10^{-12} \text{ GeV} \) in Eq. (7) and considering all values of the angles \( \theta_2, \theta_3, \) and \( \delta \) allowed by Eqs. (6). In Fig. 1, we show the bounds as a function of the \( B \)-meson lifetime, assuming the current experimental limit \( \Gamma(b - u)/\Gamma(b - c) < 0.1 \) in (6). We do not consider values of \( m_t \) greater than 80 GeV.
since the derivation of Eq. (7) requires $m_t$ small compared with $M_W$. Bounds for $c_\delta > 0$ and $c_\delta < 0$ are plotted separately, with $c_\delta > 0$ giving a stronger bound on the top-quark mass. This is because for $c_\delta > 0$ there can be no cancellation in Eq. (6a) and the consequent smallness of $s_\delta$ and $s_o$ keeps the coefficient of the $m_t^2/m_o^2$ term in Eq. (7) small as well. If the top quark is found to have a mass consistent with the $c_\delta < 0$ bound, but not consistent with the $c_\delta > 0$ bound, then the phase $\delta$ is determined to lie in either the second or third quadrant.

We have not assumed any knowledge of the sign of $B$. The measured phase of $\epsilon$ implies that $BS_\delta > 0$ for $c_\delta > 0$. However, $BS_\delta$ can have either sign for $c_\delta < 0$ and the bound in Fig. 1 corresponds to $BS_\delta > 0$. $BS_\delta < 0$ implies a much more stringent constraint on the top-quark mass, corresponding to $m_t > 60$ GeV for all $B$ lifetimes plotted in our figures.

If the experimental limit on $\Gamma(b \to u)/\Gamma(b \to c)$ is improved, then our bound on $m_t$ is also improved. Figure 2 shows the lower bounds on $m_t$ for $c_\delta < 0$ and $c_\delta > 0$ when we require $\Gamma(b \to u)/\Gamma(b \to c) < 0.05$.

The bounds shown in Fig. 1 become useful for $\tau_B > 4 \times 10^{-13}$ s. For example, if $\tau_B$ exceeds $10^{-12}$ s, the top quark must be heavier than 30 GeV and toponium is inaccessible to TRISTAM. If the experimental upper limit on $\Gamma(b \to u)/\Gamma(b \to c)$ is improved by a factor of 2, the bound becomes 40 GeV. The current experimental upper limit on $\tau_B$ is $\tau_B < 1.4 \times 10^{-12}$ s. If the values of $m_t$ and $\tau_B$ turn out to lie in the excluded region, new physics (like the existence of a fourth generation) is mandatory. If they lie between our bounds, we have determined that $\cos \delta < 0$, thus resolving a quadrant ambiguity of the Kobayashi-Maskawa model.

This work was supported by the National Science Foundation under Grants No. PHY77-22864 and No. PHY82-15249 and by the U. S. Department of Energy under Contract No. DE-AC03-81ER40050.

Note added.—For $\tau_B = 10^{-12}$ s, the lower bound ($c_\delta < 0$) on the top-quark mass in Fig. 1 occurs at $s_\delta \approx 0.1$, $s_o = 0.06$, and $s_t = 0.06$. For these values, the middle term (proportional to $m_t^2$) in Eq. (7) contributes about 45% of $\epsilon$. We note that the lower bound always comes from extreme values of the range of angles consistent with Eq. (6). Generic values naturally give a larger top-quark mass.

A major theoretical uncertainty in our bound is the value of $B$. If $B$ were increased 25% to 0.46 by, for example, higher-momentum dependence
in the amplitudes for $K^{-}\pi(I=2)$ and for $K^{0}\bar{K}^{0}$ mixing, then the lower bound on $m_{t}$ in Fig. 1 would decrease to 24 GeV for $\tau_{B}=10^{-12}$ s. Smaller values of $B$ would give a correspondingly more stringent bound.

We thank F. Gilman for useful discussions.

2M. Roos et al. (Particle Data Group), Phys. Lett. 111B, 1 (1982).


If the $|\Delta s| = 1$ weak nonleptonic Hamiltonian is dominated by a single operator, the contribution to the $K_{L}$–$K_{S}$ mass difference from higher-dimensional operators ought to be nearly real in a basis where the $K^{-}\pi(I=0)$ amplitude is real. Then the contribution to $\epsilon$ from $CP$ nonconservation in kaon decay amplitudes has magnitude $4\epsilon^{*}||ReM_{12}^{box}(m_{KL}-m_{KS})||$, where $ReM_{12}^{box}$ is the short-distance contribution to the $K^{0}\bar{K}^{0}$ mass matrix element [F. J. Gilman and M. B. Wise, Phys. Lett. 83B, 83 (1979)]. For long $B$-meson lifetimes $|ReM_{12}^{box}/(m_{KL}-m_{KS})| \leq \frac{1}{16}$, except when $m_{t}$ is enormous. In any case, the measurement of a small value for $|\epsilon^{*}/\epsilon|$ would support the approximation.

20The JADE upper limit $\tau_{B} < 1.4 \times 10^{-12}$ s appears in W. Bartel et al., Phys. Lett. 114B, 71 (1982). Roy Weinstein in Proceedings of the American Physical Society Division of Particles and Fields Meeting, College Park, October 1982 (to be published) reports $\tau_{B} = (1.7 \pm 1.0) \times 10^{-12}s$. 

1418