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THE STABILITY OF ONE-STEP SCHEMES FOR FIRST-ORDER TWO-POINT BOUNDARY VALUE PROBLEMS*

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Dedicated to Professor Henrici on the occasion of his 60th birthday

Abstract. The stability of a finite difference scheme is related explicitly to the stability of the continuous problem being solved. At times, this gives materially better estimates for the stability constant than those obtained by the standard process of appealing to the stability of the numerical scheme for the associated initial value problem.

1. Introduction. To paraphrase the first sentence in the preface to Raudkivi [1980], the stability of finite difference schemes for two-point boundary value problems "is well understood, but far from explained." A popular explanation (see, e.g., Keller [1976], Keller and White [1975], and a typical use in Esser and Niederdrenk [1980] or Lynch and Rice [1980]) relates it to the stability of the associated initial value problem. In effect, use is made of the simple fact that, on a finite dimensional linear space (viz. the nullspace of the differential operator) and in any norm, any linear map is bounded. Numerically, the argument is equivalent to solving the problem by shooting. But, much as multiple shooting often is necessary to overcome the large stability constant of the initial value problem, so other means should or must be employed if one is actually after the precise stability constant of the differences scheme employed. Knowledge of this constant is important for judging the condition of the numerical scheme. Also, when solving a problem on an infinite interval by truncation, it is important to know just how the stability constant depends on the interval on which the problem is being solved.

The obvious source for this information is the stability constant of the continuous problem. Usually, the stability constant of the numerical scheme approaches that of the continuous problem as the meshsize goes to zero and hence can be inferred from the latter. This idea is implicit in Kreiss' [1972] treatment of finite difference schemes. The Soviet literature, as exemplified by Kantorovich and Akilov [1964], uses this idea explicitly in the abstract treatment of projection methods for the solution of second kind equations. It can also be found in the literature which follows Stummel (e.g., Grigorieff [1970]).

In this note, we carry out this idea for a first-order system of linear ordinary differential equations and for one-step methods. The slightly more complicated case of multistep methods for a system of mth order equations is treated in the companion paper de Boor and de Hoog [1983]. No mesh restrictions, such as uniformity or quasi-uniformity, are imposed.

We consider the problem of finding the n-vector valued function $y:[0, T]\to\mathbb{R}^n$ which satisfies the differential equation

$$Ly = f \tag{1.1a}$$

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with side condition

$$By = b. \tag{1.1b}$$

Here, L is the first order linear differential operator

$$Ly := y' - Ay$$

with $A: [0, T] \to \mathbb{R}^{n\times n}$ continuous, and

$$By := B_0 y(0) + B_T y(T),$$

with $B_0, B_T \in \mathbb{R}^{n\times n}$. The function $f: [0, T] \to \mathbb{R}^n$ and the n-vector b are given.

It is well known (see, e.g., Keller [1976, p. 1]) that (1.1) has a solution if and only if the matrix BY is nonsingular, with Y any fundamental matrix for L; i.e.,

$$Y: [0, T] \to \mathbb{R}^{n\times n} \quad \text{such that } LY = 0.$$

In particular, assume that Y is the fundamental matrix associated with the initial value problem; i.e.,

$$Y: [0, T] \to \mathbb{R}^{n\times n}, \quad LY = 0, \quad Y(0) = 1.$$

Further assume that BY is invertible. Then, for any $y \in (L_1^{(1)}[0, T])^n$,

$$y = \Phi By + \int_0^T G(\cdot, s)(Ly)(s)\, ds, \tag{1.2}$$

with

$$\Phi(t) := Y(t)(BY)^{-1} \tag{1.3a}$$

and

$$G(t, s) := \begin{cases} \Phi(t)B_0\Phi(0)\Phi(s)^{-1}, & 0 < s < t, \\ -\Phi(t)B_T\Phi(T)\Phi(s)^{-1}, & t < s < T. \end{cases} \tag{1.3b}$$

Knowledge of Y, hence of Green's function G, makes it possible to calculate stability constants. Denote by $|\cdot|$ any convenient norm in $\mathbb{R}^n$ as well as the corresponding matrix norm. Also, let

$$\|y\|_p := \left(\int_0^T |y(t)|^p\, dt\right)^{1/p},$$

with $\|y\|_\infty := \sup_{t\in T} |y(t)|$ its limiting value as $p \to \infty$. Then (1.2) implies the *differential stability relation*

$$\|y\|_\infty \leqq K|By| + c_p\|Ly\|_p, \tag{1.4}$$

with *interior stability constant*

$$\begin{aligned} c_p &:= \sup\left\{\left\|\int_0^T G(\cdot, s)z(s)\, ds\right\|_\infty \Big/ \|z\|_p : z \in (L_p[0, T])^n\right\} \\ &= \left\|\left[\int_0^T |G(\cdot, s)|^q\, ds\right]^{1/q}\right\|_\infty, \quad \frac{1}{p} + \frac{1}{q} = 1, \end{aligned} \tag{1.5a}$$

and *side condition stability constant*

$$K := \|\Phi\|_\infty. \tag{1.5b}$$

Of particular interest for us are the special choices $p = 1, \infty$ which give

$$c_1 = \|G\|_\infty \quad \text{and} \quad c_\infty = \left\| \int_0^T |G(\cdot, s)|\, ds \right\|_\infty, \tag{1.6}$$

and correspond to measuring the size of Ly, i.e., of f in (1.1), by $\|f\|_1$ and $\|f\|_\infty$, respectively. It is worthwhile to consider both these choices, as the following example illustrates. Choose

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad B_0 = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \quad B_T = \frac{1}{2}\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}.$$

Then

$$\Phi(t) = \begin{bmatrix} e^{-t} & e^{t-T} \\ -e^{-t} & e^{t-T} \end{bmatrix}, \qquad G(t, s) = \begin{cases} \dfrac{e^{s-t}}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, & s < t, \\ \dfrac{e^{t-s}}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, & t < s. \end{cases}$$

From this, one calculates

$$K = 1 + e^{-T}, \quad c_1 = 1, \quad c_\infty = 2(1 - e^{-T/2}).$$

If T is not large, then $p = 1$ is a desirable choice because it allows $f = Ly$ to have integrable singularities. On the other hand, if T is large (as would be the case when a boundary value problem on a semi-infinite interval is approximated by a problem on a finite interval) then $p = \infty$ may be more appropriate. For example, if $Ly = 1$, then $\|Ly\|_\infty = 1$ regardless of T, while $\|Ly\|_1 = T$, and so use of $p = 1$ would lead to linear growth in T in the estimate (1.4).

It is obvious that c_p depends on the side conditions. In particular, the problem (1.1) may be well conditioned (i.e., have K and c_p of acceptable size) while the associated initial value problem is badly conditioned and vice versa. In any case, using the initial value problem to estimate the stability constants for (1.1) amounts to estimating the size of $|\Phi(t)|$ by $|Y(t)||(BY)^{-1}|$, and this may well be a bad overestimate.

2. Stability of one-step schemes. Let $\Delta := (t_i)_0^N$ be a *mesh* for $[0, T]$; i.e.,

$$0 = t_0 < \cdots < t_N = T.$$

For such a mesh, we use the abbreviations

$$h_j := t_{j+1} - t_j \quad \text{and} \quad h := \max_j h_j.$$

With any function $y : M \to \mathbb{R}^n$ defined on some set M containing Δ, we associate the step function $\underset{\sim}{y}$ and the broken line $\tilde{y}$. Both agree with y on Δ; i.e.,

$$y(t_j) = \underset{\sim}{y}(t_j) = \tilde{y}(t_j), \qquad j = 0, \cdots, N.$$

The function $\underset{\sim}{y}$ is piecewise constant (in each component), with breakpoints $t_1, \cdots, t_N$, and is continuous from the right, as is $D\tilde{y}$ (to be precise about it). The function $\tilde{y}$ is piecewise linear, with breakpoints $t_1, \cdots, t_{N-1}$, and continuous.

As is customary, we denote by

$$(\mathbb{R}^n)^\Delta$$

the collection of all n-vector valued *mesh functions* $y:\Delta\to\mathbb{R}^n$. We identify each such function y with its step function interpolant. In particular,

$$\|y\|_p := \|\underset{\sim}{y}\|_p = \left(\sum_{i=1}^{N} h_{i-1}|y_{i-1}|^p\right)^{1/p}.$$

Yet we use

$$y_j \text{ instead of } y(t_j)$$

for the value of y at t_j. We write $\underset{\sim}{y}$ instead of y if we want to stress the fact that y is a mesh function. Finally, we associate with any n-vector valued function y on $[0, T]$ the real valued mesh function $|y|_\Delta$ given by

$$|y|_{\Delta,j} := \sup\{|y(t)|: t_j \leqq t \leqq t_{j+1}\}, \qquad j=0,\cdots,N-1.$$

We approximate the solution of (1.1) by the mesh function y which satisfies

$$L_\Delta y = \underset{\sim}{f}, \qquad By = b, \tag{2.1}$$

with

$$(L_\Delta y)_j := \frac{y_{j+1}-y_j}{h_j} - (A_\Delta y)_j, \qquad j=0,\cdots,N-1.$$

Here, A_Δ is a linear map carrying mesh functions to mesh functions. We give examples later on.

In this section, we are not concerned with the details of this approximation to (1.1). We only give suitable conditions on A_Δ which allow us to connect the stability of the continuous problem (1.1) with that of the discrete problem (2.1). We begin with the following.

CONDITION (p). *There exist functions d_1 and d_2 independent of Δ such that*

$$\||A_\Delta y - A\tilde{y}|_\Delta\|_p \leqq d_1(h)\|y\|_\infty + d_2(h)\|L_\Delta y\|_p, \quad \textit{for all } y\in(\mathbb{R}^n)^\Delta. \tag{2.2}$$

PROPOSITION 1. *Let $1\leqq p\leqq\infty$. If Condition (p) holds, then the difference stability relation*

$$\|y\|_\infty \leqq K|By| + c_p(1+d_2(h))\|L_\Delta y\|_p + c_p d_1(h)\|y\|_\infty \tag{2.3}$$

holds for all $y\in(\mathbb{R}^n)^\Delta$. If $d_2(h)$ stays bounded while $d_1(h)\to 0$ as $h\to 0$, this gives a stability constant for the discrete scheme (2.1) *for all sufficiently small h. If also $d_2(h)\to 0$ as $h\to 0$, the resulting stability constant approaches that of the continuous problem.*

Proof. For any mesh function y, we have

$$L\tilde{y} = L_\Delta y + A_\Delta y - A\tilde{y}.$$

Therefore, from the differential stability relation (1.4),

$$\|y\|_\infty = \|\tilde{y}\|_\infty \leqq K|By| + c_p(\|L_\Delta y\|_p + \||A_\Delta y - A\tilde{y}|_\Delta\|_p).$$

This together with (2.2) implies (2.3). $\square$

For any $1\leqq p\leqq\infty$, Condition (p) is implied by the:

LOCAL CONDITION. *There exist a function $\bar{d}_1$ and constants $\bar{d}_2$ and r independent of Δ such that*

$$|A_\Delta y - A\tilde{y}|_{\Delta,j} \leqq \bar{d}_1(h)\|y\|_\infty + \bar{d}_2 \sum_{|i-j|\leqq r} h_i|(L_\Delta y)_i| \quad \textit{for all } y\in(\mathbb{R}^n)^\Delta. \tag{2.4}$$

Explicitly, this Local Condition implies Condition (p) with $d_1 = T^{1/p}\bar{d}_1$ and $d_2(h) = (2r+1)h\bar{d}_2$. To derive this, note that

$$\left(\sum_{|i-j|\leqq r} h_i|(L_\Delta y)_i|\right)^p \leqq ((2r+1)h)^{p-1} \sum_{|i-j|\leqq r} h_i|(L_\Delta y)_i|^p.$$

Therefore

$$\||A_\Delta y - A\tilde{y}|_\Delta\|_p \leqq T^{1/p}\bar{d}_1(h)\|y\|_\infty + \left([(2r+1)h]^{p-1}\sum_j h_j \sum_{|i-j|\leqq r} h_i|(L_\Delta y)_i|^p\right)^{1/p},$$

and the observation

$$\sum_j h_j \sum_{|i-j|\leqq r} h_i|(L_\Delta y)_i|^p \leqq h\sum_j \sum_{|i-j|\leqq r} h_i|(L_\Delta y)_i|^p \leqq (2r+1)h\|L_\Delta y\|_p^p$$

finishes the argument. The Local Condition also implies the following refined stability estimate.

PROPOSITION 2. *If the Local Condition holds, then the difference stability relation*

$$\|y\|_\infty \leqq K|By| + c_1(1+(2r+1)\bar{d}_2 h)\|L_\Delta y\|_1 + c_\infty \bar{d}_1(h)\|y\|_\infty \tag{2.5}$$

holds for all $y \in (\mathbb{R}^n)^\Delta$.

Proof. Now, from (1.2),

$$\|y\|_\infty \leqq K|By| + c_1\|L_\Delta y\|_1 + \left\|\int |G(\cdot,s)|\,|A_\Delta y - A\tilde{y}|_\Delta(s)\,ds\right\|_\infty.$$

Here, with (2.4), the last term is bounded by

$$c_\infty \bar{d}_1(h)\|y\|_\infty + \bar{d}_2\left\|\sum_j \int_{t_j}^{t_{j+1}} |G(\cdot,s)|\,ds \sum_{|i-j|\leqq r} h_i|(L_\Delta y)_i|\right\|_\infty$$

and, in this, the last term is bounded by

$$\bar{d}_2 \max_j h_j c_1(2r+1)\|L_\Delta y\|_1. \qquad \square$$

Remark. From (1.6) we see that $c_\infty \leqq Tc_1$. Thus (2.5) improves the final term of (2.3) and makes the h-dependence of the function d_2 in (2.3) explicit.

The argument for Proposition 1 is easily reversed.

PROPOSITION 3. *Let* $1 \leqq p \leqq \infty$. *If there exist functions* e_1 *and* e_2 *independent of* Δ *such that*

$$\||A_\Delta \underset{\sim}{y} - Ay|_\Delta\|_p \leqq e_1(h)\|y\|_\infty + e_2(h)\|Ly\|_p \quad \textit{for all } y \in (L_p^{(1)}[0,T])^n \tag{2.6}$$

and functions K *and* c_p *so that, for all* Δ *with* $h < h_0$,

$$\|y\|_\infty \leqq K(h_0)|By| + c_p(h_0)\|L_\Delta y\|_p \quad \textit{for all } y \in (\mathbb{R}^n)^\Delta,$$

then

$$\|y\|_\infty \leqq K(h_0)|By| + c_p(h_0)(1+e_2(h))\|Ly\|_p + c_p(h_0)e_1(h)\|y\|_\infty \tag{2.7}$$

for all $y \in (L_p^{(1)})^n$ *and all* Δ *with* $h < h_0$. *This provides a stability estimate for the continuous problem provided* $e_2(h)$ *remains bounded and* $e_1(h) \to 0$ *as* $h \to 0$.

Proof. Let Δ be any mesh with $h < h_0$ and let $y \in (L_p^{(1)})^n$. By assumption,

$$\|\underset{\sim}{y}\|_\infty \leqq K(h_0)|By| + c_p(h_0)\|L_\Delta \underset{\sim}{y}\|_p,$$

while

$$L_\Delta \underset{\sim}{y} = L\tilde{y} - (A_\Delta \underset{\sim}{y} - A\tilde{y}).$$

Therefore

$$\|y\|_\infty \leqq K(h_0)|By| + c_p(h_0)\|L\tilde{y}\|_p + c_p(h_0)\| |A_\Delta \underset{\sim}{y} - A\tilde{y}|_\Delta\|_p,$$

and the bounds (2.6) now allow the conclusion that (2.7) holds, with y there replaced by $\tilde{y}$, for any sufficiently fine Δ. But then it must hold for y, too. □

3. Examples. A very transparent example is provided by the *centered Euler* scheme. In this scheme,

$$(L_\Delta y)_j = (L\tilde{y})(t_{j+1/2}), \qquad \text{all } j.$$

Hence

$$(A_\Delta y)_j = (A\tilde{y})(t_{j+1/2}) = A(t_{j+1/2})\frac{y_j + y_{j+1}}{2}.$$

Therefore, on the interval (t_j, t_{j+1}),

$$A_\Delta y - A\tilde{y} = (A\tilde{y})(t_{j+1/2}) - A\tilde{y} = (A(t_{j+1/2}) - A)\tilde{y} + A(t_{j+1/2})(\tilde{y}(t_{j+1/2}) - \tilde{y}).$$

Further,

$$\tilde{y}(t_{j+1/2}) - \tilde{y}(t) = (t - t_{j+1/2})\tilde{y}'_{j+}$$

and

$$\tilde{y}'_{j+} = (L_\Delta y)_j + (A\tilde{y})(t_{j+1/2}).$$

Thus,

$$|A_\Delta y - A\tilde{y}|_{\Delta,j} \leqq (\|A'\|_\infty \|y\|_\infty + \|A\|_\infty |(L_\Delta y)_j| + \|A\|_\infty^2 \|y\|_\infty) h_j/2.$$

This shows that the Local Condition holds with $r = 0$, $\bar{d}_1(h) = h(\|A'\|_\infty + \|A\|_\infty^2)/2$, and $\bar{d}_2 = \|A\|_\infty/2$. Correspondingly, the stability constant for the centered Euler scheme is within $O(h)$ of the stability constant for the continuous problem being solved.

It is also possible to bound $|A_\Delta \underset{\sim}{y} - Ay|_{\Delta,j}$ in terms of Ly. We have

$$A_\Delta \underset{\sim}{y} - Ay = (A(t_{j+1/2}) - A)\tilde{y}(t_{j+1/2}) + A(\tilde{y}(t_{j+1/2}) - y)$$

on (t_j, t_{j+1}). Further,

$$\tilde{y}(t_{j+1/2}) - y(t) = \frac{1}{2}\left(\int_t^{t_j} + \int_t^{t_{j+1}}\right) y'$$

and

$$y' = Ly + Ay.$$

Therefore,

$$|A_\Delta \underset{\sim}{y} - Ay|_{\Delta,j} \leqq \|A'\|_\infty h_j/2 \|y\|_\infty + \|A\|_\infty \frac{1}{2}\left(\int_{t_j}^{t_{j+1}} |(Ly)(s)|\, ds + \|A\|_\infty h_j \|y\|_\infty\right).$$

This shows that (2.6) holds with $e_1(h) = h(\|A'\|_\infty + \|A\|_\infty^2)/2$ and $e_2(h) = h\|A\|_\infty/2$.

A slightly more involved example is given by the choice

$$(A_\Delta y)_j := \sum_{|k| \leqq s} \alpha_{jk} A(t_{j+k}) y_{j+k},$$

with

$$\sum_k \alpha_{jk} = 1, \qquad \alpha_{jk} = 0 \quad \text{for } k \notin [0, N], \qquad \max_{j,k} |\alpha_{jk}| \leqq a.$$

For this scheme and on (t_j, t_{j+1}),

$$A_\Delta y - A\tilde{y} = \sum_k \alpha_{jk} A(t_{j+k})\tilde{y}_{j+k} - A\tilde{y} = \sum_k \alpha_{jk}((A\tilde{y})(t_{j+k}) - A\tilde{y}),$$

and

$$(A\tilde{y})(t_{j+k}) - A\tilde{y} = (A(t_{j+k}) - A)\tilde{y} - A(t_{j+k})(\tilde{y}(t_{j+k}) - \tilde{y}).$$

Now consider $\tilde{y}(t_{j+}k) - \tilde{y}$. If $k > 0$, then

$$\begin{aligned}\tilde{y}(t_{j+k}) - \tilde{y}(t) &= \tilde{y}(t_{j+k}) - \tilde{y}(t_{j+k-1}) + \cdots + \tilde{y}(t_{j+1}) - \tilde{y}(t)\\ &= \sum_{m=1}^{k-1} h_{j+m}\tilde{y}'_{j+m} + (t_{j+1} - t)\tilde{y}'_j\\ &= \sum_{m=1}^{k-1} H_{j+m}\left[(L_\Delta y)_{j+m} + \sum_{|l|\leqq s} \alpha_{j+m,l} A(t_{j+m+l}) Y_{j+m+l}\right],\end{aligned}$$

with

$$H_{j+m} := \begin{cases} h_{j+m}, & m > 0\\ t_{j+1} - t, & m = 0.\end{cases}$$

A similar formula holds for $k < 1$. From this, we conclude that the Local Condition is satisfied with $r = s$, $d_1(h) = hr^2 a(\|A'\|_\infty + \|A\|_\infty^2 ra)$ and $d_2 = ra\|A\|_\infty$.

4. Related considerations. Certain discrete schemes are so closely related to the continuous problem that it is natural and advantageous to exploit this interplay directly. We consider two specific instances, multiple shooting and algorithms based on approximating the differential equation.

In multiple shooting applied to (1.1), we are led to the system

$$y_{j+1} - Y(t_{j+1})Y(t_j)^{-1} y_j = g_j, \qquad j = 0, \cdots, N-1,$$

$$B_0 y_0 + B_T y_N = b,$$

with $g_j := \int_{t_j}^{t_{j+1}} Y(t_{j+1})Y(s)^{-1} f(s)\, ds$, all j. A simple analysis of this system can be based on the fact that $y_j = \bar{y}(t_j)$, with $\bar{y}$ the solution to the problem

$$L\bar{y} = \bar{g}, \qquad B\bar{y} = b,$$

and

$$\bar{g} := YY(t_{j+1})^{-1} g_j / h_j \quad \text{on} \quad (t_j, t_{j+1}).$$

Therefore

$$\bar{y} = \Phi b + \sum_{j=0}^{N-1} G(\cdot, t_{j+1}) d_j,$$

and so

$$y_i = \Phi(t_i) b + \sum_{j=0}^{N-1} G(t_i, t_{j+1}) d_j.$$

Under suitable assumptions, this leads to bounds for $\|y\|_\infty$ as, e.g., in Mattheij [9].

As an example of the second kind of method, consider the approximating problem

$$L_\Delta y = \underset{\sim}{f}, \qquad By = b \tag{4.1}$$

with $L_\Delta y = y' - \underset{\sim}{A}y$ and $\underset{\sim}{A}, \underset{\sim}{f}$ piecewise constant approximations to A and f, respectively. Since $L_\Delta y = Ly + (L_\Delta - L)y$, we obtain

$$\|y\|_\infty \leqq c_\infty \|Ly\|_\infty + K|By| + c_\infty \|A - \underset{\sim}{A}\|_\infty \|y\|_\infty. \tag{4.2}$$

The approximating problem is therefore stable if

$$c_\infty \|A - \underset{\sim}{A}\|_\infty < 1.$$

Often, c_∞ may be quite small. For example, in many singular perturbation problems,

$$A = C/\varepsilon \quad \text{and} \quad c_\infty = d\varepsilon;$$

therefore, $1 - c_\infty \|A - \underset{\sim}{A}\|_\infty = 1 - d\|C - \underset{\sim}{C}\|_\infty$. In such a case, we conclude from (4.2) that

$$\|y\|_\infty \leqq \frac{K|By| + d\varepsilon \|Ly\|}{1 - d\|C - \underset{\sim}{C}\|}.$$

This implies that the convergence of the solution $\underset{\sim}{y}$ of (4.1) to the solution y of (1.1) is *uniform*, since $L(y - \underset{\sim}{y}) = -(A - \underset{\sim}{A})\underset{\sim}{y}$ and $B(y - \underset{\sim}{y}) = 0$; therefore

$$\|y - \underset{\sim}{y}\|_\infty \leqq d\|C - \underset{\sim}{C}\|_\infty \|\underset{\sim}{y}\|_\infty / (1 - d\|C - \underset{\sim}{C}\|).$$

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