RESONANCE FOR SINGULAR PERTURBATION PROBLEMS*

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Abstract. Consider the resonance for a second-order equation \( \epsilon y'' - xp(x, \epsilon)y' + q(x, \epsilon)y = 0 \). Another proof is given for the necessity of the Matkowsky condition and the connection with a regular eigenvalue problem is established. Also, if \( p, q \) are analytic, necessary and sufficient conditions are derived.

1. Introduction. Consider the differential equation

\[
(1.1) \quad \epsilon y'' - xp(x, \epsilon)y' + q(x, \epsilon)y = 0, \quad -c \leq x \leq c.
\]

Here \( \epsilon \) with \( 0 < \epsilon < 1 \) is a small constant, \( p(x, \epsilon) \geq p_0 > 0 \) and \( q(x, \epsilon) \) are smooth functions of \( x, \epsilon \); i.e., \( p \in C^\infty(x, \epsilon) \), \( q(x, \epsilon) \in C^\infty(x, \epsilon) \). Many authors, for example \([2], [6], [11], [12]\) have studied the asymptotic behavior of the solutions of (1.1) when \( \epsilon \to 0 \). In particular Pearson \([10]\) and Ackerberg and O'Malley \([1]\) proved the following basic result. If

\[
(1.2) \quad q(0, 0) \neq 0, 1, 2, \ldots,
\]

then for any \( \delta, c \) with \( 0 < \delta < c \)

\[
(1.3) \quad \lim_{\epsilon \to 0} \| y \|_{-c+\delta, c-\delta} = 0, \quad \| y \|_{a,b} = \max_{a \leq x \leq b} | y(x) |.
\]

Another proof was given in \([5]\) using the maximum principle.

If (1.2) does not hold, then resonance can occur; i.e., for some constants \( c > 0 \) and \( \delta > 0 \) there is a sequence of solutions with

\[
(1.4) \quad \lim_{\epsilon \to 0} \| y \|_{-c+\delta, c-\delta} \neq 0, \quad 0 < \delta < c.
\]

B. Matkowsky \([8]\) has proposed a sequence of conditions which must be satisfied for resonance to occur. N. Kopell \([3]\) has shown that these conditions are necessary but not sufficient. By changing \( q(x, \epsilon) \) by a quantity which is smaller than any power of \( \epsilon \), resonance can be enforced. Similar results are obtained by F. Olver \([9]\).

In this paper we want to extend the methods of \([5]\) to prove the same results. Also, we shall show that resonance occurs if and only if \( \lambda = 0 \) is an eigenvalue of an associated regular eigenvalue problem. Furthermore, if the coefficients of \( p(x, \epsilon), q(x, \epsilon) \) are analytic functions of \( x \) and smooth functions of \( \epsilon \), then resonance occurs if and only if there is a sequence of analytic solutions

\[
(1.5) \quad y(x, \epsilon) = \sum_{\nu=0}^{\infty} a_\nu(\epsilon)x^\nu, \quad |a_\nu| \leq K\zeta^\nu,
\]

\( K, \zeta \) constants independent of \( \epsilon \), which in a neighborhood of \( x = 0 \) converges uniformly to a nontrivial solution of the reduced equation.

2. Necessary conditions for resonance. In this section we want to prove that the Matkowsky conditions are necessary for resonance. We start with a number of lemmata which are slight generalizations of results in \([5]\).
LEMMA 2.1. Consider

\[ ey'' - xp(x, \varepsilon)y' + q(x, \varepsilon)y = F(x, \varepsilon), \quad F(x, \varepsilon) \in C^\infty(x, \varepsilon), \]

in an interval \(0 < a \leq x \leq b \leq c\). There are constants \(K_0, \beta > 0\) which depend only on \(a\) and bounds for \(xp\) and \(q\) such that for all sufficiently small \(\varepsilon > 0\) and all \(\delta\) with \(0 < \delta \leq \frac{1}{2}(b - a)\)

\[ \|y\|_{a,b-\delta} \leq K_0(\|F\|_{a,b} + |y(a)| + e^{-\beta\varepsilon/\varepsilon} |y(b)|). \]

Also, there are constants \(K_i\) which depend only on \(a\) and bounds for \(d^\nu(xp)/dx^\nu\), \(d^\nu q/dx^\nu\), \(\nu = 0, 1, 2, \ldots, j\) such that

\[ \left\| \frac{d^\nu y}{dx^\nu} \right\|_{a,b-\delta} \leq K_j(1 + e^{-i} e^{-\beta\varepsilon/\varepsilon}) \left( \sum_{\nu=0}^{j} \left\| \frac{d^\nu F}{dx^\nu} \right\|_{a,c} + |y(a)| + e^{-i} e^{-\beta\varepsilon/\varepsilon} |y(b)| \right). \]

The same estimates hold if \(-c \leq b \leq x \leq a < 0\).

Proof. The estimates follow from standard results for singular perturbation problems (see, for example, [7], [11]).

We shall now show that we can estimate the derivatives \(dy/dx\) in terms of \(y\) in intervals which contain the point \(x = 0\). These estimates are the main tool of this section.

THEOREM 2.1. There are constants \(C_i\) such that for all sufficiently small \(\varepsilon\)

\[ \left\| \frac{d^i y}{dx^i} \right\|_{-c+\delta,c-\delta} \leq C_i(1 + e^{-i} e^{-\beta\varepsilon/\varepsilon}) \left( \sum_{\nu=0}^{i} \left\| \frac{d^\nu F}{dx^\nu} \right\|_{c,c} + \|y\|_{-c,c} \right). \]

Proof. (For more details see [5].) Differentiate (2.1) with respect to \(x\) and let

\[ v_i = \frac{d^i y}{dx^i}, \quad F_i = \frac{d^i F}{dx^i}. \]

A simple calculation shows that

\[ \varepsilon d^i y = xp v_i' + (q - j(p + xp'))v_i = F_i + \sum_{\nu=0}^{i-1} A_{i\nu} v_{\nu}, \]

where

\[ A_{i\nu} = \left( \begin{array}{c} j \vspace{1mm} \\ -1 \end{array} \right)(xp)_{j+1-i} - \left( \begin{array}{c} i \vspace{1mm} \\ \nu \end{array} \right) q_{i-\nu}. \]

By Lemma 2.1 the above estimate is certainly valid for \(|x| \geq \alpha\), where \(\alpha\) is any constant with \(0 < \alpha \leq \frac{1}{2}c\). For sufficiently large \(j, q - j(p + xp') < 0\) in a neighbourhood of \(x = 0\). Here we can estimate \(v_i\) in terms of \(v_{\nu}, \nu = 0, 1, 2, \ldots, j - 1\) using the maximum principle. Therefore we can estimate \(v_i\) also in terms of \(v_0 = y\). This proves the theorem. \(\square\)

We need also

LEMMA 2.2. Let

\[ \frac{q(0, 0)}{p(0, 0)} = l, \quad l = 0, 1, 2, \ldots. \]

The equation

\[ x\phi' - \frac{q(x, 0)}{p(x, 0)} \phi = g(x) = \sum_{\nu=0}^{l} g_{\nu} \frac{x^\nu}{\nu!} + x^{l+1} h(x), \quad h \in C^\infty(x), \]

has a solution belonging to $C^\infty(x)$ if and only if

$$g_l = \frac{d}{dx}g_l \bigg|_{x=0} = 0.$$  

Proof. We can write

$$\frac{q(x, 0)}{p(x, 0)} = l + xq_1(x).$$

Let $\phi = x^l\psi$. Then $\psi$ is the solution of

$$\psi' - xq_1\psi = x^{-(l+1)}g(x).$$

$\psi$ is of the form

$$\psi = \sum_{\nu=-l}^{-1} c_\nu x^\nu + c_0 \log |x| + \psi_1, \quad \psi_1 \in C^\infty(x),$$

and $c_0 = 0$ if and only if $g_l = 0$. This proves the lemma. □

We now can prove that the Matkowsky conditions are necessary. Let (2.5) be satisfied and let $y(x, \epsilon), \|y(x, \epsilon)\|_{-c, c} = 1$ be a sequence of solutions of (1.1) which in the interior of $-c < x < c$ converges to a nontrivial solution $\tilde{y}$ of the reduced equation

$$xp(x, 0)\tilde{y}' + q(x, 0)\tilde{y} = 0.$$  

In particular, let $x_0$ with $x_0 \neq 0$, $|x_0| < c - \delta$ be a point and let $\phi_0$ be the solution of

$$xp(x, 0)\phi'_0 + q(x, 0)\phi_0 = 0, \quad \phi_0(x_0) = y(x_0, \epsilon).$$

Then $\lim \phi_0 = \tilde{y}$, and $y_1 = y - \phi_0$ satisfies

$$\epsilon y_1'' - xp(x, \epsilon)y_1' + q(x, \epsilon)y_1 = \epsilon g_1(x, \epsilon), \quad y_1(x_0, \epsilon) = 0.$$  

$y_1$ is bounded and $g_1(x, \epsilon)$ is a smooth function of $x, \epsilon$. Let

$$\tilde{y}_1 = \frac{y_1}{a}, \quad a = \max (\epsilon, \|y_1\|_{-c+\delta/2, c-\delta/2}).$$

Then $\tilde{y}_1$ is the solution of

$$\epsilon \tilde{y}_1'' - xp(x, \epsilon)\tilde{y}' + q(x, \epsilon)\tilde{y}_1 = \left(\frac{\epsilon}{a}\right)g_1(x, \epsilon), \quad \tilde{y}_1(x_0, \epsilon) = 0.$$  

$\tilde{y}_1$ is smooth for $|x| \leq c - \delta$. We want to show that there is a constant $\tau$ such that

$$0 < \tau \leq \epsilon/\alpha \leq 1.$$  

Assume there is no such $\tau$. Then we can assume that $\lim_{\epsilon \to 0} \epsilon/\alpha = 0$. Using Theorem 2.1 we can also assume that

$$\lim_{\epsilon \to 0} \tilde{y}_1(x, \epsilon) = \psi(x) \quad \text{for } |x| \leq c - \delta.$$

where

$$-xp(x, 0)\psi' + q\psi = 0, \quad \psi(x_0) = 0, \quad \psi(x) = 0.$$  

In particular,

$$\lim_{\epsilon \to 0} \tilde{y}_1(-c + \delta, \epsilon) = \lim_{\epsilon \to 0} \tilde{y}_1(c - \delta, \epsilon) = 0.$$  

By Lemma 2.1, applied to the intervals $-c \leq x \leq -c + \delta$ and $c - \delta \leq x \leq c$ respectively, it follows that $\lim_{\epsilon \to 0} \|y_1\|_{-c+\delta/2, c+\delta/2} = 0$, which is impossible. Thus we can introduce
into (2.6) the new variable \( \tilde{y}_1 = y_1 / \varepsilon \). Then \( \tilde{y}_1 \) is bounded for \( |x| \leq c - \frac{1}{2} \delta \) and satisfies
\[
(2.7) \quad \varepsilon \tilde{y}'_1 - xy \tilde{y}_1 + q \tilde{y}_1 = g(x, \varepsilon).
\]

By Theorem 2.1, \( \tilde{y}_1 \) converges to a smooth solution of
\[
 xp(x, 0) \phi'_1 + q(x, 0) \phi_1 = g(x, 0),
\]
which does not contain any log terms. This is the first of the Matkowsky conditions. We can now repeat the above process and obtain

**Theorem 2.2.** The Matkowsky conditions are necessary for resonance.

3. **An estimate.** Consider a system of differential equations
\[
(3.1) \quad \frac{dy}{dx} = A(x)y = F(x).
\]

Here \( F = (F^{(1)}, \ldots, F^{(n)})' \) is a smooth vector function, and
\[
A = \begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{nn}
\end{pmatrix}
\]
is a smooth complex \( n \times n \) matrix.

**Definition 3.1.** We say that the matrix \( A \) is negative dominant if there is a constant \( z > 0 \) such that for all \( i, j = 1, 2, \ldots, n \) and all \( x \)
\[
\text{Real } a_{ij} < 0, \quad \sum_{j=1}^{n} |a_{ij}| \leq (1 - \tau) |\text{Real } a_{ij}|.
\]

In [4, Lemma 2.3] we have proved

**Lemma 3.1.** Assume that \( A \) is negative dominant and let
\[
\Lambda(x) = \begin{pmatrix}
a_{11}(x) & 0 & \cdots & 0 \\
0 & a_{22}(x) & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & a_{nn}(x)
\end{pmatrix}.
\]

Then the solutions of (3.1) satisfy the estimate
\[
(3.2) \quad |y(x)| \leq \tau^{-1} \max_{0 \leq \eta \leq x} |\Lambda^{-1}(\eta) F(\eta)| + s(x) |y(0)|, \quad x \geq 0,
\]
where
\[
s(x) = \exp \left[ \tau \int_{0}^{x} a(\xi) \, d\xi \right], \quad a(x) = \max \text{Real } a_{ii}(x).
\]

We want to use Lemma 3.1 to estimate the solutions of (2.1). We assume that (2.5) holds.

Then
\[
(3.3) \quad q(0, \varepsilon) \geq -\rho \varepsilon, \quad \rho = \text{const.} > 0.
\]

We write (2.1) in the form
\[
(3.4) \quad \varepsilon y'' = (xpy)' + q_1 y = F, \quad q_1 = q + (xp)'.
\]
Let
\[ v' = q_1 y - F. \]

Then we can integrate (3.4) to obtain
\[ \varepsilon y' - xpy + v = 0, \]
which gives us
\[ (3.5) \]
\[ \begin{pmatrix} y' \\ v \end{pmatrix} = \begin{pmatrix} \frac{xp}{\varepsilon} - \frac{1}{\varepsilon} \\ q_1 \end{pmatrix} \begin{pmatrix} y \\ v \end{pmatrix} - \begin{pmatrix} 0 \\ F \end{pmatrix}. \]

We want to transform (3.5) into negative dominant form. Introduce new dependent variables by
\[ \begin{pmatrix} y \\ v \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} s(x) u, \quad u = \begin{pmatrix} u^{(1)} \\ u^{(2)} \end{pmatrix}, \quad s(x) = (xp)^{-1}. \]

Then \( u \) satisfies
\[ u' = \begin{pmatrix} 1 & -s(x) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} xp/\varepsilon \\ q_1 \end{pmatrix} u + \begin{pmatrix} s(x)' - q_1 \end{pmatrix} \]
\[ = \begin{pmatrix} \frac{xp}{\varepsilon} - q_1/xp \\ q_1/xp \end{pmatrix} \begin{pmatrix} \frac{(xp)' - q_1}{xp} \end{pmatrix} u + \left( \frac{F}{xp} \right). \]

Let \( \alpha > 0, \beta > 0 \) be constants which we choose later. Then
\[ w = e^{-\beta x} \begin{pmatrix} \alpha x \\ 0 \\ 0 \end{pmatrix} u \]
is the solution of
\[ (3.6) \]
\[ w' = \begin{pmatrix} \frac{xp}{\varepsilon} - \beta - \frac{1}{x} \left( \frac{q_1}{p} + 1 \right) \frac{\alpha}{xp} (xp)' - q_1 \end{pmatrix} w + \begin{pmatrix} F \\ \frac{F}{\alpha x} \end{pmatrix}. \]

We consider now an interval \(-c \leq x \leq -\gamma \sqrt{\varepsilon}, \gamma > 0 \) constant independent of \( \varepsilon \), and shall show that we can choose \( \alpha, \beta, \gamma \) such that the system (3.6) is negative dominant.

1) Choose \( \alpha \) such that \( |q_1/\alpha x| = \frac{1}{2}|q_1/xp| \).

2) By (4.3), \( q_1(0, \varepsilon) = 1 + O(\varepsilon) \), and we can choose \( \beta \) such that \( q_1/xp - \beta \leq -1 \) for all \( x \) with \(-c \leq x < 0 \).

3) Choose \( \gamma \) so large that for \(-c \leq x \leq -\gamma \sqrt{\varepsilon} \)
\[ \frac{xp}{\varepsilon} - \beta - \frac{1}{x} \left( \frac{q_1}{p} + 1 \right) \leq -\frac{1}{2} \frac{p_0 |x|}{\varepsilon}, \quad \frac{\alpha}{xp^{2/2}} (xp)' - q_1 < \frac{1}{4} \frac{p_0 |x|}{\varepsilon}. \]

With these choices of \( \alpha, \beta, \gamma \) the system (3.6) is negative dominant with \( \tau = \frac{1}{4} \).

We write the solution of (3.6) in the form
\[ w = w_1 + \begin{pmatrix} 0 \\ u_{s} \end{pmatrix} \]
where \( u_s \) is the solution of

\[
(3.7) \quad u_s' = \left( \frac{q_1(x, 0)}{xp(x, 0)} - \beta \right) u_s - F, \quad u_s(-1) = 0,
\]

and \( w_1 \) satisfies

\[
(3.8) \quad w_1' = A w_1 + G, \quad G = (G^{(1)}, G^{(2)}),
\]

where \( A \) is the same matrix as in (3.6) and

\[
G^{(1)} = \frac{F}{ap} - \frac{1}{ap^2} (xp)' - q_1)u_s, \quad G^{(2)} = \frac{e}{x} \left( \frac{q_1(x, \epsilon)}{p(x, \epsilon)} - \frac{q_1(x, 0)}{p(x, 0)} \right) u_s.
\]

The solution of (3.7) can be estimated by

\[
|u_s(x)| \leq K_1 \max_{-c < \xi < x} |F(\xi)| \quad \text{if } q_1(0, 0)/p(0, 0) = 1,
\]

\[
|u_s(x)| \leq K_2 \max_{-c < \xi < x} \left( \frac{e}{\xi} |F(\xi)| + \frac{e}{\xi^2} u_s(\xi) \right) + |w_1(-c)|
\]

\[
\leq K_2 \max_{-c < \xi < x} \left( \frac{e}{\gamma} |F(\xi)| + \frac{1}{\gamma^2} u_s(\xi) \right) + |w_1(-c)|.
\]

The system (3.8) is negative dominant. Therefore, by Lemma 3.1,

\[
|w_1(x)| \leq K_2 \max_{-c < \xi < x} \left( \frac{e}{\xi} |F(\xi)| + \frac{e}{\xi^2} u_s(\xi) \right) + |w_1(-c)|
\]

We now return to the original variables and obtain

\[
|y(x)| + |v(x)| \leq \frac{e^\theta}{\alpha |x|} (|u_s(x)| + |w_1(x)|)
\]

\[
\leq K_3 |x|^{-1} (|y(-c)| + e |y'(-c)|) + K_3 \max_{-c < \xi < x} |F(\xi)|
\]

\[
\left\{ \begin{array}{l}
\log |x| \quad \text{if } q_1(0, 0)/p(0, 0) = 1, \\
1 \quad \text{otherwise.}
\end{array} \right.
\]

Here \( K_i \) are constants which do not depend on \( \epsilon \).

The corresponding estimate holds in the interval \( \gamma \sqrt{\epsilon} \leq x \leq c \).

4. The analytic case. In this section we consider a sequence of solutions \( y(x, \epsilon) \), \( \epsilon \to 0 \), of (2.1) which are uniformly bounded in an interval \( |x| \leq a \), i.e.,

\[
(4.1) \quad \|y(x, \epsilon)\|_{-a, a} \leq K_0.
\]

We assume that \( p(x, \epsilon), q(x, \epsilon), F(x, \epsilon) \) are smooth functions of \( \epsilon \) and analytic functions of \( x \) for \( x \in \Omega \). Here \( \Omega \) is an open domain in the complex plane which contains the interval \( |x| \leq a \). We also assume that \( p, q, F \) are uniformly bounded for all \( \epsilon \) and all \( x \in \Omega \). Therefore, by Cauchy's integral formula, there are constants \( \zeta_1, K \) such that

\[
(4.2) \quad \|F_v\|_{-a, a} + \|q_v\|_{-a, a} + \|(xp)_v\|_{-a, a} \leq K_1 \nu^\zeta \zeta_1^\nu,
\]

\[
F_v = d^v F/dx^v.
\]

It is well known that for every fixed \( \epsilon \) the solutions of (2.1) are analytic functions of \( x \)
for $x \in \Omega$, i.e. we can continue the above solutions analytically into $\Omega$. Let $x = -b$ be a point with $-a < -b < 0$. We want to show that the $y(x, \epsilon)$ are uniformly bounded in a complex neighborhood of $x = -b$.

By (4.1) and Theorem 2.1 there is a constant $K$ such that

$$|y(-a, \epsilon)| + \epsilon |y'(-a, \epsilon)| \leq K.$$  \hspace{1cm} (4.3)

Now rewrite the differential equation (2.1) as the first-order system (3.5) and consider it on the half-lines

$$x = r e^{i\phi} - a, \quad r \geq 0, \quad -\phi_0 \leq \phi \leq \phi_0.$$

There it has the form

$$\frac{d}{dr} \begin{pmatrix} y \\ v \end{pmatrix} = e^{i\phi} \begin{pmatrix} \frac{xp}{e} & -1 \\ q & 0 \end{pmatrix} \begin{pmatrix} y \\ v \end{pmatrix} + e^{i\phi} \begin{pmatrix} 0 \\ F \end{pmatrix}.$$

For all sufficiently small $\phi$ the interval $0 \leq r \leq a - \frac{1}{2}b$ belongs to $\Omega$. Also, the estimates of \S 3 are still valid. Therefore, by (4.3) and (3.9) the solutions $y(x, \epsilon)$ are uniformly bounded in a complex neighbourhood of $x = -b$. The same is true for $x = b$. Using Cauchy's integral formula we obtain

$$|v_r(-b, \epsilon)| + |v_r(b, \epsilon)| \leq K_2 \nu! \xi_2^\nu, \quad v_r = \frac{d^\nu v}{dx^\nu}.$$  \hspace{1cm} (4.4)

Without restriction we can assume that $K_2 = K_1, \xi_2 = \xi_1$.

Now choose for $b$ the largest number with the properties

$$b \leq \frac{1}{2}a, \quad \min_{|x| = a} (p + xp') \geq \frac{1}{2}p_0, \quad p_0 = \min_{|x| = a} p(x).$$

Let $j_0$ be the smallest positive integer such that $q_0 - \frac{j_0}{2}p_0 < 0$, $q_0 = \max_{|x| = a} q$. Using the maximum principle we obtain from (2.4) for $j \geq j_0$

$$\|v_j\|_{-b,b} \leq \frac{1}{(1/2)p_0 - q_0} \left( \|F\|_{-b,b} + \sum_{\nu=0}^{j-1} A_{j\nu} v_\nu \right) + |v_j(-b, \epsilon)| + |v_j(b, \epsilon)|$$

$$\leq K_3 j! \xi_1^j + \frac{K_1}{(1/2)p_0 - q_0} \sum_{\nu=0}^{j-1} \left( \begin{array}{c} j \\ \nu - 1 \end{array} \right) (j + 1 - \nu)! \xi_1^{j+1-\nu} + \left( \begin{array}{c} j \\ \nu \end{array} \right) (j - \nu)! \xi_1^{j-\nu} \|v_\nu\|_{-b,b}$$

$$\leq K_3 j! \xi_1^j + \frac{K_1 j! \xi_1^j}{(1/2)p_0 - q_0} \sum_{\nu=0}^{j-1} (\xi_1 \nu + 1) \|v_\nu\|_{-b,b} \xi_1^{\nu+1}$$

$$\leq j! \xi_1^j \left( K_3 + K_4 \sum_{\nu=0}^{j-1} \|v_\nu\|_{-b,b} \xi_1^{\nu+1} \right).$$

Here

$$K_3 = K_1 \left( 1 + \frac{1}{(1/2)p_0 - q_0} \right), \quad K_4 = K_1 \max_{j \geq j_0} \frac{(j-1)\xi_1 + 1}{(1/2)p_0 - q_0}.$$

Let $\alpha$ be the lower bound of all $\alpha$ satisfying

$$\alpha^j \geq K_3 + K_4 \sum_{\nu=0}^{j-1} \alpha^\nu \quad \text{for} \ j \geq j_0, \quad \alpha^j \geq \frac{\|v_j\|_{-b,b}}{\xi_1^{j+1} j!} \quad \text{for} \ j < j_0.$$

Then the above inequality for $\|v_j\|_{-b,b}$ gives us

$$\|v_j\|_{-b,b} \leq \tilde{\alpha}^j \xi_1^j j!.$$
Therefore the solutions $y(x, \varepsilon)$ can be expanded into the power series (1.5) with $\zeta = \lambda \zeta_1$, and the result stated in the introduction is proved.

5. The associated eigenvalue problem. We assume in this section that the Matkowsky conditions are satisfied and that $c = 1$. We want to construct the general solution of (1.1) in the interval $-1 \leq x \leq -\gamma \sqrt{\varepsilon}$.

Consider the reduced equation

$$-xp(x, 0)u_0' + q(x, 0)u_0 = 0, \quad \frac{d'u_0}{dx} \bigg|_{x=0} = 1.$$ 

By assumption it has a smooth solution of the form

$$u_0(x) = x^l \phi_0(x), \quad \phi_0(x) \geq C_0 > 0, \quad C_0 = \text{const.}$$

Let $y(x)$ be a solution of the full equation (1.1). Then $y_1 = y - u_0$ satisfies

$$\varepsilon y_1'' - xp(x, \varepsilon)y_1' + q(x, \varepsilon)y_1 = \varepsilon (u_0''(x) + F_1(x, \varepsilon)),$$

where

$$u_0''(x) = x^l \phi_0'' + 2lx^{l-1} \phi_0' + l(l-1)x^{l-2} \phi_0,$$

$$\varepsilon F_1 = x(p(x, \varepsilon) - p(x, 0))u_0' - (q(x, \varepsilon) - q(x, 0))u_0 = \varepsilon x^l \psi_1(x)$$

are smooth functions of $x, \varepsilon$. By assumption the equation

$$-xp(x, 0)u_1' + q(x, 0)u_1 = u_0''(x) + F_1(x, 0), \quad \frac{d'u_1}{dx} \bigg|_{x=0} = 0$$

also has a smooth solution which is of the form

$$u_1(x) = x^{l-2}(a_{10}(l-1) + a_{11}lx + a_{12}x^3 + \cdots) = x^{l-2} \phi_1(x).$$

Therefore $y_2 = y_1 - u_1 = y - u_0 - \varepsilon u_1$ satisfies

$$\varepsilon y_2'' - xp(x, \varepsilon)y_2' + q(x, \varepsilon)y_2 = \varepsilon^2 (u_0'' + F_2(x, \varepsilon)),$$

which is of the same form as (5.2). Thus we can repeat the process and obtain after $n$ steps

$$y = \sum_{\nu=0}^{n-1} \varepsilon^n u_\nu + y_n,$$

where

$$u_\nu = x^\tau \phi_\nu(x), \quad \phi_\nu(x) \text{ smooth, } \tau_\nu = \max (l - 2\nu, 0),$$

and $y_n = y_n(x, \varepsilon)$ is the solution of

$$\varepsilon y_n'' - xp(x, \varepsilon)y_n' + q(x, \varepsilon)y_n = \varepsilon^n F_n(x, \varepsilon),$$

with initial values

$$y_n(-1, \varepsilon) = y(-1, \varepsilon) - \sum_{\nu=0}^{n-1} \varepsilon^n u_\nu(-1),$$

$$y_n'(-1, \varepsilon) = y'(-1, \varepsilon) - \sum_{\nu=0}^{n-1} \varepsilon^n u_\nu'(-1).$$

The above expansion is valid for all solutions of (1.1). In particular we can choose the
initial values such that

\[(5.6) \quad y_n(-1, \varepsilon) = y'_n(-1, \varepsilon) = 0.\]

By (3.9)

\[|y_n(x, \varepsilon)| \leq \text{const.} \, \varepsilon^n |\log |x||, \quad 0 < \varepsilon \leq \varepsilon_0, \quad -1 \leq x \leq -\gamma \sqrt{\varepsilon}.\]

Also, if we write (5.5) in the form

\[
\begin{align*}
\varepsilon y_n'(x, \varepsilon) &= \int_{-1}^{x} (\xi p(\xi, \varepsilon) y_n(\xi, \varepsilon))' - q(\xi, \varepsilon) y_n(\xi, \varepsilon) + \varepsilon^n F_n(\xi, \varepsilon) \, d\xi \\
&= x p(x, \varepsilon) y_n(x, \varepsilon) - \int_{-1}^{x} ((\xi p(\xi, \varepsilon))' + q(\xi, \varepsilon)) y_n(\xi, \varepsilon) - \varepsilon^n F_n(\xi, \varepsilon) \, d\xi,
\end{align*}
\]

it follows that

\[|\frac{\partial y_n(x, \varepsilon)}{\partial x}| \leq \text{const.} \, \varepsilon^{n-1} |\log |x||,
\]

and therefore we obtain from (5.5)

\[(5.7) \quad |\frac{\partial^{n+1} y_n(x, \varepsilon)}{\partial x^{n+1} \partial \varepsilon}| \leq \text{const.} \, \varepsilon^{n-2} |\log |x||.
\]

Differentiating (5.5) and the boundary conditions (5.6) with respect to $\varepsilon$ gives us, for $v = \frac{\partial y_n}{\partial \varepsilon},$

\[
\varepsilon v'' - x p v' + q v = \varepsilon^{n-2} \ddot{\Phi},
\]

\[v(-1, \varepsilon) = v(-1, \varepsilon) = 0,
\]

which is an equation of the same type as (5.5). Therefore

\[|v| = \left|\frac{\partial y_n}{\partial \varepsilon}\right| \leq \text{const.} \, \varepsilon^{n-2} |\log |x||,
\]

and by (5.7)

\[|\frac{\partial^{n+1} y_n}{\partial x^{n+1} \partial \varepsilon}| \leq \text{const.} \, \varepsilon^{n-2} |\log |x||.
\]

In general we have

\[(5.8) \quad \left|\frac{\partial^{n+1} y_n}{\partial x^{n+1} \partial \varepsilon^{n+1}}\right| \leq \text{const.} \, \varepsilon^{n-2 - \nu_1 - \nu_2} |\log |x||.
\]

The above process can be applied for any $n, \nu_1, \nu_2.$ Therefore, by choosing $n$ sufficiently large, we can construct a solution

\[w_n(x, \varepsilon) = \sum_{\nu=0}^{n-1} \varepsilon^{\nu} u_\nu(x) + y_n(x, \varepsilon),
\]

which for $-1 \leq x \leq -\gamma \sqrt{\varepsilon}$ has any (but fixed) number of derivatives bounded independently of $\varepsilon$.

**Remark.** The important point of (5.8) is that we can estimate the derivatives up to $x$-values of order $O(-\sqrt{\varepsilon}).$ In any other interval $-1 < b \leq x \leq a < 0$, $a, b$ independent of $\varepsilon$, these estimates follow already from (2.3) for all bounded solutions of (1.1), provided $y(-1, \varepsilon)$ and $y'(-1, \varepsilon)$ are smooth functions of $\varepsilon.$
By (5.4) and (5.1) we have, for $-1 \leq x \leq -\gamma \sqrt{\epsilon}$,
\[
|e^{x}u_\nu| \leq \text{const.} \left\{ \begin{array}{ll}
|\epsilon|^{1/2} & \text{for } \nu \leq l/2 \\
|\epsilon|^{\nu} & \text{for } \nu > l/2.
\end{array} \right.
\]

Therefore, by (5.1), there are constants $c_\nu > 0$ such that for $2n \geq l$, sufficiently large $\gamma$ and sufficiently small $\epsilon$
\begin{equation}
(5.9) \quad c_\nu |x|^{l} \leq |w_n(x, \epsilon)| \leq c_\nu |x|^{l}, \quad -1 \leq x \leq -\gamma \sqrt{\epsilon}.
\end{equation}

We determine now the general solution of (1.1) in the usual way. Let $w_n(x, \epsilon)$ be the above solution. Then all other solutions satisfy
\[
v'w_n - vw_n' = \left( \epsilon^{-1} \exp \epsilon^{-1} \int_{-1}^{x} \xi p(\xi, \epsilon) d\xi \right).
\]

An easy calculation shows that
\[
v_n(x, \epsilon) = -\epsilon^{-1}w_n(x, \epsilon) \int_{-1}^{x} (w_n(\eta, \epsilon))^{-1} \exp \left( \epsilon^{-1} \int_{-1}^{\eta} \xi p(\xi, \epsilon) d\xi \right) d\eta
\]
is another linearly independent solution. By (5.9) it has the properties
\begin{equation}
(5.10) \quad v_{n}(-1, \epsilon) = \frac{-1}{p(-1, 0)} + O(\epsilon),
\end{equation}
\[
|v_n(x, \epsilon)| \leq \text{const.} \exp \left( -\frac{1}{\epsilon} \int_{-1}^{x} \xi p(\xi, \epsilon) d\xi \right).
\]
$v_n(x, \epsilon)$ and all its derivatives are exponentially small outside the boundary layer at $x = -1$.

All bounded solutions of (1.1) which are not exponentially small outside the boundary layer can be written as
\begin{equation}
(5.11) \quad y(x, \epsilon) = \rho_1(w_n(x, \epsilon) + \rho_2v_n(x, \epsilon)), \quad \rho_1, \rho_2 \text{ bounded.}
\end{equation}

Therefore at $x_0 = -\gamma \sqrt{\epsilon}$
\begin{equation}
(5.12) \quad \sqrt{\epsilon} \frac{y'(x_0, \epsilon)}{y(x_0, \epsilon)} = \sqrt{\epsilon} \frac{w_n'(x_0, \epsilon)}{w_n(x_0, \epsilon)} + O(\epsilon^{-\alpha/\epsilon})
\end{equation}
\[
= a_0(\epsilon) + O(\epsilon^{-\alpha/\epsilon}), \quad \alpha \sim \left| \int_{-1}^{0} xp(x, \epsilon) dx \right|.
\]

For $a_0(\epsilon)$ we obtain the asymptotic expansion
\begin{equation}
(5.13) \quad \sqrt{\epsilon} \frac{w_n'(x_0, \epsilon)}{w_n(x_0, \epsilon)} = a_0(\epsilon)
\end{equation}
\[
= \sqrt{\epsilon} \sum_{\nu=0}^{n-1} \epsilon^{\nu}u_{\nu}(x_0) / \sum_{\nu=0}^{n-1} \epsilon^{\nu}u_{\nu}(x_0) + O(\epsilon^{n-(l+1)/2} \log \epsilon).}
\end{equation}

For $\gamma \sqrt{\epsilon} \leq x \leq 1$ we can proceed in the same way. Thus all bounded solutions which away from the boundary layer are not exponentially small are of the form
\[
y(x, \epsilon) = \rho_1(\bar{w}_n(x, \epsilon) + \rho_2\bar{v}_n(x, \epsilon)), \quad \rho_1, \rho_2 \text{ bounded},
\]
where
\[ \tilde{w}_n(x, \varepsilon) = \sum_{\nu=0}^{n-1} \varepsilon^\nu u_\nu(x) + \tilde{y}_n(x, \varepsilon), \quad |\tilde{y}_n| \leq \text{const.} \varepsilon^n |\log \varepsilon|. \]

(Observe that the \( u_\nu(x) \) are defined for all \( |x| \leq 1 \).) At \( x_1 = \gamma \sqrt{\varepsilon} \) these solutions satisfy the relation
\[ (5.14) \]
\[ \sqrt{\varepsilon} \tilde{y}'(x_1, \varepsilon)/\tilde{y}(x_1, \varepsilon) = \sqrt{\varepsilon} \tilde{w}_n(x_1, \varepsilon)/\tilde{w}_n(x_1, \varepsilon) + O(\varepsilon^{-\beta/\varepsilon}) \]
\[ = a_1(\varepsilon) + O(\varepsilon^{-\beta/\varepsilon}), \quad \beta \sim \left| \int_0^1 xp(x, \varepsilon) \, dx \right| \]

where
\[ a_1(\varepsilon) = \frac{\sqrt{\varepsilon} \sum_{\nu=0}^{n-1} \varepsilon^\nu u_\nu(x_1)}{\sum_{\nu=0}^{n-1} \varepsilon^\nu u_\nu(x_1)} + O(\varepsilon^{-(l+1)/2} |\log \varepsilon|). \]

The associated eigenvalue problem is given by
\[ (5.15) \]
\[ e\phi'' - xp(x, \varepsilon)\phi' + q(x, \varepsilon)\phi = \lambda \phi, \quad -\gamma \sqrt{\varepsilon} \leq x \leq \gamma \sqrt{\varepsilon}, \]
\[ \sqrt{\varepsilon} \phi'(x_0) - a_0 \phi(x_0) = 0, \quad j = 0, 1, \]
which can be written as the regular eigenvalue problem
\[ (5.16) \]
\[ \ddot{\psi} - zp(\sqrt{\varepsilon} z, \varepsilon) \dot{\psi} + q(\sqrt{\varepsilon} z, \varepsilon) \psi = \lambda \psi, \quad -\gamma \leq z \leq \gamma, \]
\[ \dot{\psi}(-\gamma) - a_0 \psi(-\gamma) = 0, \quad \dot{\psi}(\gamma) - a_1 \psi(\gamma) = 0, \quad \psi = \frac{d\psi}{dz}, \]

by introducing \( x = \sqrt{\varepsilon} \cdot z \) as a new variable.

**Theorem 5.1.** Resonance can only occur if \( (5.16) \) has an eigenvalue \( \lambda \) of order \( O(\varepsilon^{-\gamma/\varepsilon}) \), \( \gamma = \text{const.} > 0 \). If \( \lambda = 0 \) is an eigenvalue of \( (5.16) \) for a sequence \( \varepsilon \to 0 \), then resonance occurs. If the Matkowsky conditions are satisfied \( (5.16) \) has an eigenvalue \( h \) with
\[ (5.17) \]
\[ |\lambda| \leq \text{const.} |\varepsilon^m|, \quad \text{for any positive integer } m. \]

Therefore we can find a function \( \tilde{q}(z, \varepsilon) \in C^\infty \) with \( \tilde{q}(z, \varepsilon) = 0 \) for \( z \leq 1/2 \gamma \) and \( \tilde{q}(z, \varepsilon) > 0 \) otherwise such that for the modified problem
\[ ey'' - xp(x, \varepsilon)y' + (q(x, \varepsilon) + \varepsilon^m \tilde{q}(x/\sqrt{\varepsilon}, \varepsilon))y = 0 \]
resonance occurs.

**Proof.** If resonance occurs then there is a sequence of solutions of \( (1.1) \),
\[ y(x, \varepsilon) = \rho_1(w_n(x, \varepsilon) + \rho_2 v_n(x, \varepsilon)), \quad \rho_1 > \delta > 0, \quad \rho_2 \text{ bounded}. \]

These solutions satisfy
\[ ey'' - xp(x, \varepsilon)y' + q(x, \varepsilon)y = 0, \quad x_0 = -\gamma \sqrt{\varepsilon} \leq x \leq \gamma \sqrt{\varepsilon} = x_1, \]
\[ \sqrt{\varepsilon} y'(x_j) - a_j y(x_j) = O(\varepsilon^{-\gamma/\varepsilon}). \]

Therefore, by Lemmata A1 and A3 of the Appendix, \( (5.15) \) and \( (5.16) \) must have an exponentially small eigenvalue.
Assume now that $\lambda = 0$ is an eigenvalue of (5.16) and (5.15) for a sequence $\epsilon \to 0$. Let $\phi(x, \epsilon)$ be the corresponding eigenfunction. Then
\[
y(x, \epsilon) = \begin{cases} 
    w_n(x, \epsilon) & \text{for } x \leq -\gamma \sqrt{\epsilon}, \\
    \frac{w_n(x_0, \epsilon)}{\phi(x_0, \epsilon)} \phi(x, \epsilon) & \text{for } |x| \leq \gamma \sqrt{\epsilon}, \\
    \frac{\phi(x_1, \epsilon)}{\phi(x_0, \epsilon)} w_n(x_0, \epsilon) \tilde{w}_n(x, \epsilon) & \text{for } x \geq \gamma \sqrt{\epsilon},
\end{cases}
\]
denotes a sequence of solutions of (1.1) which do not converge to zero.

If the Matkowsky conditions are satisfied then by (5.3), (5.13) and (5.14) the function
\[
s(x) = \epsilon^{-1/2} \sum_{\nu=0}^{n-1} u_\nu(x)
\]
satisfies
\[
\epsilon s'' - xp(x, \epsilon)s' + q(x, \epsilon)s = O(\epsilon^{n-(l/2)}),  \\
\sqrt{\epsilon} s'(x_i) - a_\beta s(x_i) = O(\epsilon^{n-(l+1)/2} |\log \epsilon|).
\]
Here $n$ is arbitrary and therefore (5.17) follows from Lemmata A1 and A3. The resonance of the modified problem follows from Lemmata A2 and A3. (Observe that $q(x/\sqrt{\epsilon}, \epsilon)$ does not affect $w_n, \tilde{w}_n$ because $\tilde{q} = 0$ for $|x| \geq \gamma \sqrt{\epsilon}$.) \qed

**Appendix.** In this section we collect some facts about eigenvalue problems. We consider selfadjoint equations
\[
L[y] = y'' + f(x)y = \lambda y, \quad 0 \leq x \leq 1, \quad f(x) \in C^\infty, \\
y'(0) + ay(0) = 0, \quad y'(1) + \beta y(1) = 0,
\]
and denote by
\[
(u, v) = \int_0^1 \tilde{u}v \, dx, \quad (u, u) = \|u\|^2
\]
the usual $L_2$-scalar product and norm. The following lemma is well known.

**Lemma A1.** Let $u(x)$ with $\|u\| = 1$ be a function satisfying
\[
u'' + f(x)u = G(x), \\
u'(0) + \alpha u(0) = g_0, \quad u'(1) + \beta u(1) = g_1.
\]
Then (A1) has an eigenvalue $\lambda$ and a corresponding eigenfunction $\varphi$ with
\[
|\lambda| + \|\varphi - u\| \leq \text{const.} (\|G\| + |g_0| + |g_1|).
\]

We need also

**Lemma A2.** Assume that (A1) has an eigenvalue $\lambda_0$ with $|\lambda_0| \ll 1$ and let $\varphi_0$ denote the corresponding eigenfunction. Let $g(x)$ be a smooth function with
\[
(\varphi_0(x), g(x)\varphi_0(x)) \neq 0 \quad (\text{e.g., } g(x) \geq 0, g(x) \neq 0).
\]
If $|\lambda_0|$ is sufficiently small then we can find a $\sigma$ with
\[
|\sigma| \leq \text{const.} |\lambda_0|.
\]
such that \( \lambda = 0 \) is an eigenvalue of the perturbed eigenvalue problem

\[
L[w] = w'' + (f(x) + \sigma g(x))w = \lambda w,
\]

(A2)

\[w'(0) + \alpha w(0) = 0, \quad w'(1) + \beta w(1) = 0.
\]

**Proof.** Let \( \lambda_j \) with \( |\lambda_0| < |\lambda_1| < |\lambda_2| \cdots \) and \( \varphi_j \) denote the eigenvalues and eigenfunctions of (A1) respectively. We want to construct the desired solution in the form

\[w = \varphi_0 + \sum_{j=1}^{\infty} \tilde{w}_j \varphi_j\]

i.e.

\[
0 = L[w] = \lambda_0 \varphi_0 + \sigma g \varphi_0 + \sum_{j=1}^{\infty} \lambda_j \tilde{w}_j \varphi_j + \sigma q \sum_{j=1}^{\infty} \tilde{w}_j \varphi_j.
\]

(A3)

Multiplying (A3) by \( \varphi_j \) and forming the scalar product gives us the equivalent system

\[
\lambda_0 + \sigma (\varphi_0, g \varphi_0) + \sigma \sum_{j=1}^{\infty} \tilde{w}_j (\varphi_0, g \varphi_j) = 0,
\]

(A4)

\[
\lambda_j \tilde{w}_j + \sigma \sum_{j=1}^{\infty} \tilde{w}_j (\varphi_j, g \varphi_j) = -\sigma (\varphi_j, g \varphi_0).
\]

Let \( \tilde{w}_j = \lambda_j \tilde{w}_j \); then (A4) becomes

\[
\lambda_0 + \sigma (\varphi_0, g \varphi_0) + \sigma \sum_{j=1}^{\infty} \tilde{w}_j (\varphi_0, \varphi_j) = 0,
\]

(A5)

\[
\tilde{w}_j + \sigma \sum_{j=1}^{\infty} \tilde{w}_j (\varphi_j, \varphi_j) = -\sigma (\varphi_j, \varphi_0).
\]

(A6)

Observing that \( |\lambda_j^{-1}| \leq \text{const.} / j^2 \) for \( j \geq 1 \) it follows that for sufficiently small \( |\sigma| \) the system (A6) has a solution \( \tilde{w}_j = \sigma (\varphi_j, g \varphi_0) \). Therefore, if \( |\lambda_0| \) is sufficiently small, (A5) can be solved for \( \sigma = -\lambda_0 / (\varphi_0, g \varphi_0) \). This proves the lemma.

More general eigenvalue problems

\[
y'' + h(x)y' + f(x)y = \lambda y,
\]

(A7)

\[y'(0) + \alpha y(0) = 0, \quad y'(1) + \beta y(1) = 0
\]

can be transformed into self-adjoint form and we have

**Lemma A3.** *The results of Lemmata A1 and A2 are also valid for the problems (A7).*

**REFERENCES**


