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THE EFFECT OF QUADRATURE ERRORS IN THE COMPUTATION OF L^2 PIECEWISE POLYNOMIAL APPROXIMATIONS*

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Abstract. In this paper we investigate the L^2 piecewise polynomial approximation problem. L^2 bounds for the derivatives of the error in approximating sufficiently smooth functions by polynomial splines follow immediately from the analogous results for polynomial spline interpolation. We derive L^2 bounds for the errors introduced by the use of two types of quadrature rules for the numerical computation of L^2 piecewise polynomial approximations. These bounds enable us to present some asymptotic results and to examine the consistent convergence of appropriately chosen sequences of such approximations. Some numerical results are also included.

1. Introduction. Interpolation, approximation, and smoothing techniques employing piecewise polynomials have received considerable attention in the literature for many years now. The reader is referred to [21] for an extensive bibliography of the spline literature through 1966 including much of the research into the theory and use of the univariate splines. In particular, the excellent approximation properties of polynomial spline interpolates, coupled with their computability, make their use as approximations quite attractive; cf. [18] and [8]. There are, however, several situations where the use of these interpolation techniques is inappropriate. First of all, these methods should not be used when the data are contaminated by errors since errors cause unwanted inflections in polynomial spline interpolates. Secondly, they become computationally unattractive when there are large quantities of data because of the corresponding increase in the dimensionality of the spline spaces involved. Reinsch [14] and Ritter [15] have considered generalized interpolation techniques for the purpose of data smoothing which, just like interpolation, suffer from the drawback of being computationally unattractive for large data sets.

In [22] and [23], respectively, Schumaker presents an overview of the theory of approximation by spline functions and discusses methods for computing such approximations. An examination of the papers which Schumaker references indicates that the emphasis has been on L^∞ spline approximation. However, Smith [24] as well as de Boor and Rice [6] have documented the implementation of discrete least squares techniques using cubic polynomial splines. Smith approached the problem through the normal equations for the cardinal spline representation introduced by Schoenberg in his early work on splines. The normal matrix was found to be ill-conditioned in all but the simplest cases. Smith recognized that the B-spline representation for the spline spaces might circumvent these numerical difficulties. The approach taken by de Boor and Rice is based on the Gram–Schmidt orthogonalization process and is motivated by their desire to generate an iterative algorithm with which the nonlinear least squares cubic spline approximation problem can be approached (cf. [7]).

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In this paper, we present the results of an investigation of some approximate integral least square polynomial spline approximation techniques. A distinct advantage of these methods over discrete least squares techniques is their stability with respect to the distribution of the data. Bounds for the least square error and its derivatives in approximating sufficiently smooth functions by polynomial splines are derived. We discuss computational aspects of the problem which lead us to the use of quadrature schemes in order to actually compute integral least square polynomial spline approximations. We derive bounds for the errors induced by the use of such schemes and we use these bounds to generate asymptotic results. Moreover, we examine the question of the consistency of the orders of accuracy of the errors induced by the use of such quadrature schemes with the order of accuracy of the least square error itself. We conclude the paper with a presentation of some numerical results.

2. Integral least square polynomial spline approximation. We begin this section with the following definitions. For each nonnegative integer N, let $\mathscr{P}_N[0, 1]$ denote the set of all partitions of the interval $[0, 1]$ of the form $\Delta: 0 = x_0 < x_1 < \cdots < x_N < x_{N+1} = 1$. Moreover, let $\mathscr{P}[0, 1] = \bigcup_{N=0}^{\infty} \mathscr{P}_N[0, 1]$.

For $\Delta \in \mathscr{P}_N[0, 1]$, d a positive integer and z an integer such that $-1 \leqq z \leqq d - 1$, let $S(d, \Delta, z)$ be the set of all real-valued functions $s(x)$ defined on $[0, 1]$ such that $s(x) \in C^z[0, 1]$ and, on each subinterval $[x_i, x_{i+1}]$, $0 \leqq i \leqq N$, determined by Δ, $s(x)$ is a polynomial of degree d. Here, $C^{-1}[0, 1]$ is defined to be the set of all piecewise continuous functions on $[0, 1]$ with each discontinuity a simple jump discontinuity at one of the mesh points x_i, $1 \leqq i \leqq N$; cf. [21].

Note that for $d = 2m - 1, m \geqq 1$, and $m - 1 \leqq z \leqq 2m - 2 = d - 1$, we have defined the spaces of deficient splines of [1]. For generalizations, the reader is invited to study [21]. Following [1], we define the interpolation mapping $\mathscr{I}_m: C^{m-1} \times [0, 1] \to S(2m - 1, \Delta, z)$ by $\mathscr{I}_m f = s$, where

$$D^k s(x_i) \equiv D^k f(x_i) \begin{cases} 0 \leqq k \leqq 2m - 2 - z, & 1 \leqq i \leqq N, \\ 0 \leqq k \leqq m - 1, & i = 0, N + 1. \end{cases} \tag{2.1}$$

This mapping corresponds to Type I interpolation of [21]. We now state the following basic result on this type of polynomial spline interpolation, cf. [1].

THEOREM 1. *The interpolation mapping given by* (2.1) *is well-defined for all* $\Delta \in \mathscr{P}[0, 1]$, $m \geqq 1$, $m - 1 \leqq z \leqq 2m - 2$.

For each positive integer p, let $K^p[0, 1]$ denote the collection of all real-valued functions f defined on $[0, 1]$ such that $f \in C^{p-1}[0, 1]$, $D^{p-1}f$ is absolutely continuous and $D^p f \in L^2[0, 1]$, where $Df \equiv df/dx$ denotes the derivative of f. Also, given any $\Delta \in \mathscr{P}_N[0, 1]$, let $\bar{\Delta} \equiv \max_{0 \leqq i \leqq N} (x_{i+1} - x_i)$ and $\underline{\Delta} \equiv \min_{0 \leqq i \leqq N} (x_{i+1} - x_i)$. The following theorem gives bounds for the error in polynomial spline interpolation and is a composite of special cases of Theorems 3.5 and 4.1 of [18].

THEOREM 2. $\Delta \in \mathscr{P}_N[0, 1]$, $N \geqq 0$. *Let* $m \geqq 1$ *and* $m - 1 \leqq z \leqq 2m - 2$. *Then, for any* $f \in K^{2m}[0, 1]$ *and* $0 \leqq j \leqq 2m$,

$$\|D^j(f - \mathscr{I}_m f)\|_{L^2[0,1]} \leqq K_{m,z,j}(\bar{\Delta})^{2m-j} \|D^{2m} f\|_{L^2[0,1]}, \tag{2.2}$$

where

$$
(2.3)\qquad K_{m,z,j} = \begin{cases}
\dfrac{\{(z+2-m)!\}^2}{\pi^{2m-j}} & \text{if } 0 \leqq j \leqq 2m-2-z, \\[2ex]
\dfrac{\{(z+2-m)!\}^2}{j!\pi^{2m-j}} & \text{if } 2m-2-z < j \leqq m-1, \\[2ex]
\dfrac{(z+2-m)!}{\pi^m} & \text{if } j = m, \\[2ex]
\dfrac{2}{\pi^{2m-j}} + \dfrac{\{(z+2-m)!+2\}}{\pi^m}\left\{\dfrac{(3m)!}{(4m-j)!}\right\}^2 (\bar{\Delta}/\underline{\Delta})^{j-m} & \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } m+1 \leqq j \leqq 2m-2, & \\[2ex]
\dfrac{2}{(2m-1)!\pi} + \dfrac{\{(z+2-m)!+2\}}{\pi^m}\left\{\dfrac{(3m)!}{(2m+1)!}\right\}^2 (\bar{\Delta}/\underline{\Delta})^{m-1} & \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } j = 2m-1, & \\[2ex]
1 & \text{if } j = 2m.
\end{cases}
$$

We remark, as does the author of [18], that $\mathscr{I}_m f$ is not necessarily in $K^j[0,1]$ for $z+1 \leqq j \leqq 2m$ and, in this case, we must make the definition

$$
\|D^j(f - \mathscr{I}_m f)\|_{L^2[0,1]} \equiv \left[\sum_{i=0}^{N} \|D^j(f - \mathscr{I}_m f)\|^2_{L^2[x_i,x_{i+1}]}\right]^{1/2}.
$$

These interpolation results enable us to give bounds for the L^2-norms of the error and its derivatives in approximating elements of $K^{2m}[0,1]$ by polynomial splines using an integral least square technique; i.e., given $f \in K^{2m}[0,1]$, find $\hat{s} \in S(2m-1, \Delta, z)$ such that, for every $s \in S(2m-1, \Delta, z)$, $\|f - \hat{s}\|_{L^2[0,1]} \leqq \|f - s\|_{L^2[0,1]}$. We can immediately conclude that

$$
\|f - \hat{s}\|_{L^2[0,1]} \leqq \|f - \mathscr{I}_m f\|_{L^2[0,1]}
$$

since, by Theorem 1, $\mathscr{I}_m f$ exists and is unique. Combining this with Theorem 2 for $j = 0$, we have the conclusion of the following theorem in the case that $j = 0$.

THEOREM 3. $\Delta \in \mathscr{P}_N[0,1]$, $N \geqq 0$. *Let* $m \geqq 1$ *and* $m - 1 \leqq z \leqq 2m - 2$. *For* $f \in K^{2m}[0,1]$ *and* $\hat{s}$ *defined as above,*

$$
(2.4)\qquad \|D^j(f - \hat{s})\|_{L^2[0,1]} \leqq C_{m,z,j}(\bar{\Delta})^{2m-j}\|D^{2m}f\|_{L^2[0,1]},
$$

where

$$
(2.5)\qquad C_{m,z,j} = \begin{cases}
K_{m,z,0} & \text{if } j = 0, \\[2ex]
K_{m,z,j} + 2\left[\dfrac{(2m-1)!}{(2m-j-1)!}\right]^2 K_{m,z,0}(\bar{\Delta}/\underline{\Delta})^j & \text{if } 1 \leqq j \leqq 2m-1, \\[2ex]
K_{m,z,2m} & \text{if } j = 2m,
\end{cases}
$$

and the $K_{m,z,j}$ *are given by* (2.3).

Proof. We assume that $1 \leqq j \leqq 2m - 1$ since we have already established the result in the case that $j = 0$ and, in the case that $j = 2m$, it is immediate since $D^{2m}\hat{s} = 0$ a.e. on $[0, 1]$. Then, recalling that $\mathscr{I}_m f$ is a well-defined element of the polynomial spline space $S(2m - 1, \Delta, z)$ to which $\hat{s}$ also belongs,

$$\|D^j(f - \hat{s})\|_{L^2[0,1]} \leqq \|D^j(f - \mathscr{I}_m f)\|_{L^2[0,1]} + \|D^j(\mathscr{I}_m f - \hat{s})\|_{L^2[0,1]}. \tag{2.6}$$

Repeatedly applying the following inequality of E. Schmidt (cf. [4]) relating the L^2-norm of the derivative of a polynomial of degree m to the L^2-norm of the polynomial itself,

$$\|Dp\|_{L^2[a,b]} \leqq \frac{(m+1)^2}{b-a}\|p\|_{L^2[a,b]}, \tag{2.7}$$

to the second term on the right-hand side of (2.6), we obtain

$$\begin{aligned}
\|D^j(\mathscr{I}_m f - \hat{s})\|_{L^2[0,1]} &\leqq \frac{\prod_{i=1}^{j}(2m-i)^2}{(\underline{\Delta})^j}\|\mathscr{I}_m f - \hat{s}\|_{L^2[0,1]} \\
&\leqq \left[\frac{(2m-1)!}{(2m-j-1)!}\right]^2 (\underline{\Delta})^{-j}\{\|\mathscr{I}_m f - f\|_{L^2[0,1]} \\
&\qquad + \|f - \hat{s}\|_{L^2[0,1]}\} \\
&\leqq 2\left[\frac{(2m-1)!}{(2m-j-1)!}\right]^2 (\underline{\Delta})^{-j}\|f - \mathscr{I}_m f\|_{L^2[0,1]}.
\end{aligned} \tag{2.8}$$

Finally, combining (2.6) and (2.8) and applying Theorem 2 to the terms on the right-hand side of the resulting expression, we find that

$$\begin{aligned}
&\|D^j(f - \hat{s})\|_{L^2[0,1]} \\
&\quad \leqq \|D^j(f - \mathscr{I}_m f)\|_{L^2[0,1]} + 2\left[\frac{(2m-1)!}{(2m-j-1)!}\right]^2 (\underline{\Delta})^{-j}\|f - \mathscr{I}_m f\|_{L^2[0,1]} \\
&\quad \leqq K_{m,z,j}(\overline{\Delta})^{2m-j}\|D^{2m}f\|_{L^2[0,1]} \\
&\qquad + 2\left[\frac{(2m-1)}{(2m-j-1)}\right]^2 (\underline{\Delta})^{-j}K_{m,z,0}(\overline{\Delta})^{2m}\|D^{2m}f\|_{L^2[0,1]} \\
&\quad = \left\{K_{m,z,j} + 2\left[\frac{(2m-1)}{(2m-j-1)}\right]^2 (\overline{\Delta}/\underline{\Delta})^j K_{m,z,0}\right\}(\overline{\Delta})^{2m-j}\|D^{2m}f\|_{L^2[0,1]},
\end{aligned}$$

the result of the theorem in the case that $1 \leqq j \leqq 2m - 1$.

We immediately have the following corollary.

COROLLARY. *Given a sequence of partitions $\{\Delta^j\}_{j=1}^{\infty}$ of $[0, 1]$ such that $\lim_{j\to\infty}\overline{\Delta}^j = 0$, let $\hat{s}_j$, for each j, be the least square approximation in $S(2m - 1, \Delta, z)$, $m \geqq 1$ and $m - 1 \leqq z \leqq 2m - 2$, to $f \in K^{2m}[0, 1]$. Then*

$$\lim_{j\to\infty}\|f - \hat{s}_j\|_{L^2[0,1]} = 0.$$

If, in addition, $(\overline{\Delta}^j/\underline{\Delta}^j) \leqq M$ for all j, then

$$\lim_{j\to\infty}\|D^k(f - \hat{s}_j)\|_{L^2[0,1]} = 0, \qquad 1 \leqq k \leqq 2m - 1.$$

Employing an inequality relating the L^∞-norm of an absolutely continuous function to a weighted sum of the L^2-norms of the function and its derivative, we are able to derive results in the L^∞-norm analogous to these except for a loss of 1/2 in the exponent of $\bar{\Delta}$. Numerical results (cf. [13]) did not reflect this degradation in the order of accuracy for the L^∞-norm. Indeed, Schultz [19] has derived the sharper result.

3. Quadrature errors in the numerical solution of the integral least square piecewise polynomial approximation problem. The practical computation of integral least square piecewise polynomial approximations is based on the solution of the normal equations generated with respect to a basis $\{s_i\}_{i=1}^{ns}$ for $S(d, \Delta, z)$ by the integral inner product defined for $g, h \in L^2[0, 1]$ by $(g, h) \equiv \int_0^1 g(x)h(x)\,dx$. The reader is invited to verify that $ns = d + 1 - N(d - z)$ for $\Delta \in \mathscr{P}_N[0, 1]$. Then for $f \in L^2[0, 1]$, the integral least square approximation to f in $S(d, \Delta, z)$ is $\hat{s} = \sum_{i=1}^{ns} \hat{\alpha}_i s_i$, where $\underline{\hat{\alpha}} \equiv (\hat{\alpha}_1, \hat{\alpha}_2, \cdots, \hat{\alpha}_{ns})$ is the unique solution of the normal system of equations

$$A\underline{\hat{\alpha}} = \underline{\hat{k}}, \tag{3.1}$$

where the entries, $a_{i,j}$, of A and the components, k_i, of $\underline{\hat{k}}$ are defined for $1 \leqq i, j \leqq ns$ by

$$a_{i,j} \equiv \int_0^1 s_i(x)s_j(x)\,dx$$

and

$$\hat{k}_i \equiv \int_0^1 f(x)s_i(x)\,dx.$$

Therefore, in order to actually obtain $\hat{s}$, i.e., compute the ns-tuple $\underline{\hat{\alpha}}$, we must have numerical values for the entries of A and the components of $\underline{\hat{k}}$ as well as an effective technique with which we can solve the system of equations (3.1). Since A is positive definite and symmetric, α can be determined by the method of Cholesky (cf. [10, p. 127]). Moreover, if A is a band matrix, which indeed is the case for the B-spline representation of $S(2m - 1, \Delta, z)$, $m \geqq 1$, $m - 1 \leqq z \leqq 2m - 2$ (cf. [17]), band Cholesky or Gaussian elimination can be used to efficiently solve the system (3.1). In any case, this choice is made once the entries of A are determined. In fact, once the basis functions for $S(d, \Delta, z)$ are chosen, the entries of A may be calculated directly as they are just sums of definite integrals of polynomials over the intervals $[x_i, x_{i+1}]$, $0 \leqq i \leqq N$, determined by Δ. Of course, the zero structure of A will then be known and the appropriate technique for the solution of the normal equations can be chosen.

The possibility of computing numerical values for the components of $\underline{\hat{k}}$ directly seems remote since we may not have a representation of f which would permit such a calculation. Indeed, in many, if not most, practical applications, f is a tabulated function, i.e., its value is known at only a finite set of discrete points. In such a situation, a quadrature scheme must be employed in order to obtain an ns-tuple $\underline{\tilde{k}}$ as an approximation to $\underline{\hat{k}}$. The system of equations

$$A\underline{\hat{\alpha}} = \underline{\tilde{k}} \tag{3.2}$$

is solved for $\hat{\underline{\alpha}} \equiv (\tilde{\alpha}_1, \tilde{\alpha}_2, \cdots, \tilde{\alpha}_{ns})$ and $\tilde{s} = \sum_{i=1}^{ns} \tilde{\alpha}_i s_i$ is used as an approximation to f. Recalling that $\hat{s}$ denotes the integral least square approximation to f in $S(d, \Delta, z)$, we wish to bound $\|\hat{s} - \tilde{s}\|_{L^2[0,1]}$ in terms of known quantities. Such bounds enable us to consider asymptotic results as well as the consistency of the order of accuracy of this error with the order of accuracy of the least square error itself.

We begin by letting $\hat{L}$ denote the integral over $[0, 1]$ and $\tilde{L}$ the quadrature rule used to determine $\underline{\tilde{k}}$, both regarded as bounded linear functionals on $C[0, 1]$. Then $\hat{k}_i = \int_0^1 f(x)s_i(x)\,dx = \hat{L}[f \cdot s_i]$ and $\tilde{k}_i = \tilde{L}[f \cdot s_i]$, $1 \leqq i \leqq ns$, and, consequently,

$$
\begin{aligned}
\|\hat{s} - \tilde{s}\|^2_{L^2[0,1]} &= \left\| \sum_{i=1}^{ns} (\hat{\alpha}_i - \tilde{\alpha}_i)s_i \right\|^2_{L^2[0,1]} \\
&= \left(\sum_{i=1}^{ns} (\hat{\alpha}_i - \tilde{\alpha}_i)s_i, \sum_{j=1}^{ns} (\hat{\alpha}_j - \tilde{\alpha}_j)s_j \right) \\
&= \sum_{i=1}^{ns} (\hat{\alpha}_i - \tilde{\alpha}_i) \sum_{j=1}^{ns} (\hat{\alpha}_j - \tilde{\alpha}_j)(s_i, s_j) \\
&= (\hat{\underline{\alpha}} - \tilde{\underline{\alpha}})^T A(\hat{\underline{\alpha}} - \tilde{\underline{\alpha}}) = (\hat{\underline{\alpha}} - \tilde{\underline{\alpha}})\ (\underline{k} - \underline{\tilde{k}}) \\
&= \sum_{i=1}^{ns} (\hat{\alpha}_i - \tilde{\alpha}_i)\{\hat{L}[f \cdot s_i] - \tilde{L}[f \cdot s_i]\} \\
&= (\hat{L} - \tilde{L})\left\{ f \cdot \sum_{i=1}^{ns} (\hat{\alpha}_i - \tilde{\alpha}_i)s_i \right\} \\
&= (\hat{L} - \tilde{L})\{f \cdot (\hat{s} - \tilde{s})\}.
\end{aligned} \tag{3.3}
$$

We use this identity to develop our main results much as a similar one leads to similar results in [9].

We first consider composite quadrature rules defined by interpolatory formulas. As in [11, p. 303], given $n + 1$ distinct points $\tau_0 < \tau_1 < \cdots < \tau_n$ in the interval $[a, b]$ and any function $\sigma \in C[a, b]$, we may compute the interpolation polynomial P_n of degree at most n such that $\sigma(\tau_j) = P_n(\tau_j)$ for $j = 0, 1, \cdots, n$. Let $\tilde{L}^*\sigma = \int_a^b P_n(x)\,dx$ be used as an approximation to $\hat{L}^*\sigma = \int_a^b \sigma(x)\,dx$. By using the Lagrange form for the interpolation polynomial, $P_n(x) = \sum_{j=0}^n \varphi_{n,j}(x)\sigma(\tau_j)$ where, for $x \in [a, b]$ and $0 \leqq j \leqq n$, $\varphi_{n,j}(x) = \omega_n(x)/[(x - \tau_j)\omega_n'(\tau_j)]$ and $\omega_n(x) = (x - \tau_0) \times (x - \tau_1) \cdots (x - \tau_n)$, we obtain the representation $\tilde{L}^*\sigma = \sum_{j=0}^n w_{n,j}\sigma(\tau_j)$ with the coefficients $w_{n,j}$ given by $w_{n,j} = \int_a^b \varphi_{n,j}(x)\,dx, 0 \leqq j \leqq n$. Any quadrature formula defined in this manner is called an interpolatory formula. The following theorem provides us with bounds for the errors in interpolatory quadratures and will enable us to bound the error in an arbitrary composite quadrature scheme based on interpolatory formulas. The proof depends on Peano's kernel theorem and can be found in [13]. This result is quite general and sharper bounds can be derived for certain interpolatory schemes such as Gaussian and Newton–Cotes formulas.

THEOREM 4. *Let $\tilde{L}^*$ be defined as above. Then, for any $\sigma \in C^{n+1}[a, b]$, the quadrature error satisfies*

$$
\begin{aligned}
|[\hat{L}^* - \tilde{L}^*]\sigma| &= \left| \int_a^b \sigma(x)\,dx - \sum_{j=1}^n w_{n,j}\sigma(\tau_j) \right| \\
&\leqq Q(b - a)^{n+3/2} \|D^{n+1}\sigma\|_{L^2[a,b]},
\end{aligned} \tag{3.4}
$$

where Q is independent of the length of the interval. Indeed, Q is the constant for the interpolatory quadrature over $[0, 1]$ corresponding to $\tilde{L}^*$ under the change of variables $y = (x - a)/(b - a)$ for $x \in [a, b]$.

Given partitions Δ_i^* of the form $\Delta_i^* : x_i \leqq \tau_{i,0} < \tau_{i,1} < \cdots < \tau_{i,n} \leqq x_{i+1}$ of the subintervals $[x_i, x_{i+1}]$, $0 \leqq i \leqq N$, determined by Δ, we define the composite rule $\tilde{L}$ by

$$\tilde{L}\sigma = \sum_{i=0}^{N} \sum_{j=0}^{n} w_{n,j}^i \sigma(\tau_{i,j}) \tag{3.5}$$

in terms of the weights $w_{n,j}^i$, $0 \leqq j \leqq n$, of the interpolatory quadratures $\tilde{L}_i^*$, $0 \leqq i \leqq N$, defined over the partitions Δ_i^*. This brings us to the following theorem, which can be improved in those cases where sharper bounds exist for the interpolatory schemes employed in the composite rule.

THEOREM 5. *Let $\Delta \in \mathscr{P}_N[0, 1]$, $N \geqq 0$, and let the partitions Δ_i^*, $0 \leqq i \leqq N$, be given as above. For $f \in C^{n+1}[0, 1]$, let $\hat{s}$ be the integral least square approximation to f in $S(d, \Delta, z)$ where $d \leqq n$. Then if $\tilde{s}$ is a discretized integral least square approximation to f in $S(d, \Delta, z)$ based on the composite quadrature scheme $\tilde{L}$ defined by* (3.5),

$$\|\hat{s} - \tilde{s}\|_{L^2[0,1]} \leqq K(\bar{\Delta})^{n-d+1/2}, \tag{3.6}$$

where K is a positive constant depending on the Δ_i^*.

Proof. Using (3.3) and applying (3.4) to the corresponding interpolatory quadrature scheme $\tilde{L}_i^*$ in each subinterval determined by Δ with the appropriate normalized constant denoted Q_i, we obtain

$$\begin{aligned} \|\hat{s} - \tilde{s}\|_{L^2[0,1]}^2 &= |(\hat{L} - \tilde{L})[f(\hat{s} - \tilde{s})]| \\ &\leqq Q \sum_{j=0}^{N} (h_j)^{n+3/2} \|D^{n+1}[f(\hat{s} - \tilde{s})]\|_{L^2[x_j, x_{j+1}]}, \end{aligned} \tag{3.7}$$

where $Q \equiv \max_{0 \leqq i \leqq N} Q_i$ and $h_j = x_{j+1} - x_j$, $0 \leqq j \leqq N$. However, $f \in C^{n+1}[0, 1]$ implies the existence of a positive constant $C_f \equiv \max_{0 \leqq k \leqq n+1} \|D^k f\|_{L^\infty[0,1]}$ and $\hat{s}, \tilde{s} \in S(d, \Delta, z)$ implies that $\hat{s} - \tilde{s}$ is a polynomial of degree $\leqq d$ on each subinterval $[x_j, x_{j+1}]$ and, consequently, $\|D^k(\hat{s} - \tilde{s})\|_{L^2[x_j, x_{j+1}]} = 0$, $0 \leqq j \leqq N$, $d + 1 \leqq k \leqq n + 1$. Therefore,

$$\begin{aligned} \|D^{n+1}[f(\hat{s} - \tilde{s})]\|_{L^2[x_j, x_{j+1}]} &\leqq \sum_{k=0}^{n+1} \binom{n+1}{k} \|D^{n-k+1} f\|_{L^\infty[x_j, x_{j+1}]} \|D^k(\hat{s} - \tilde{s})\|_{L^2[x_j, x_{j+1}]} \\ &\leqq C_f \sum_{k=0}^{d} \binom{n+1}{k} \|D^k(\hat{s} - \tilde{s})\|_{L^2[x_j, x_{j+1}]}. \end{aligned} \tag{3.8}$$

Combining (3.7) and (3.8) and applying (2.7), Schmidt's inequality, k times to

$\|D^k(\hat{s} - \tilde{s})\|_{L^2[x_j, x_{j+1}]}$, $0 \leqq j \leqq N$, $1 \leqq k \leqq d$, we find that

$$
\begin{aligned}
&\|\hat{s} - \tilde{s}\|^2_{L^2[0,1]} \\
&\quad \leqq QC_f \sum_{j=0}^{N} (h_j)^{n+3/2} \sum_{k=0}^{d} \binom{n+1}{k} \left[\frac{d}{(d-k)}\right]^2 (h_j)^{-k} \|\hat{s} - \tilde{s}\|_{L^2[x_j, x_{j+1}]} \\
&\quad \leqq QC_f \left\{ \sum_{k=0}^{d} \binom{n+1}{k} \left[\frac{d!}{(d-k)!}\right]^2 (\bar{\Delta})^{d-k} \right\} \|\hat{s} - \tilde{s}\|_{L^2[0,1]} \sum_{j=0}^{N} (h_j)^{n-d+3/2} \\
&\quad \leqq QC_f \left\{ \sum_{k=0}^{d} \binom{n+1}{k} \left[\frac{d!}{(d-k)!}\right]^2 \right\} (\bar{\Delta})^{n-d+1/2} \|\hat{s} - \tilde{s}\|_{L^2[0,1]}
\end{aligned}
\tag{3.9}
$$

since $\bar{\Delta} \leqq 1$ and $\sum_{j=0}^{N} h_j = 1$. Cancelling the factor $\|\hat{s} - \tilde{s}\|_{L^2[0,1]}$ from both sides of (3.9), we obtain

$$\|\hat{s} - \tilde{s}\|_{L^2[0,1]} \leqq K(\bar{\Delta})^{n-d+1/2},$$

where

$$K \equiv QC_f \left\{ \sum_{k=0}^{d} \binom{n+1}{k} \left[\frac{d!}{(d-k)!}\right]^2 \right\}$$

is dependent on Δ only through Q. This completes the proof.

Numerical results (cf. [13]) indicated that the exponent of $\bar{\Delta}$ might be increased by 1/2 in some cases. Indeed, a slight modification in the derivation of (3.9) in the proof of the preceding theorem leads to the inequality

$$\|\hat{s} - \tilde{s}\|_{L^2[0,1]} \leqq K(\bar{\Delta}/\underline{\Delta})^{1/2}(\bar{\Delta})^{n-d+1}.$$

This improvement was pointed out to the author by Professor M. H. Schultz.

The following corollary is immediate.

COROLLARY. *Given a sequence of partitions* $\{\Delta^j\}_{j=1}^{\infty}$ *of* $[0, 1]$ *such that* $\lim_{j\to\infty} \bar{\Delta}^j = 0$, *let* $\hat{s}_j$, *for each* j, *be the least square approximation in* $S(d, \Delta^j, z)$ *to* $f \in C^{n+1}[0, 1]$. *Let* $\mathscr{F} \subseteq \mathscr{P}_{n-1}[0, 1]$ *be finite. For each* j, *let* $\tilde{s}_j$ *be a discretized least square approximation in* $S(d, \Delta^j, z)$ *to* f *obtained using a composite quadrature rule* $\tilde{L}_j$ *of the form given in* (3.5), *where all the partitions of the subintervals determined by* Δ^j *over which the interpolatory formulas are defined, when scaled to* $[0, 1]$, *are members of the finite set. Then, if* $d \leqq n$,

$$\lim_{j\to\infty} \|\hat{s}_j - \tilde{s}_j\|_{L^2[0,1]} = 0.$$

This, of course, means that the errors introduced into the approximation by the use of composite schemes of this type tend to zero with $\bar{\Delta}^j$. These errors may or may not be small compared to $\|f - \hat{s}_j\|_{L^2[0,1]}$. Nevertheless, combining the corollary to Theorem 3 with this last corollary, we obtain the following result.

COROLLARY. *If, in addition to the hypothesis of the corollary just given, we also assume that* $d = 2m - 1$ *and* $m - 1 \leqq z \leqq 2m - 2 = d - 1$, *then*

$$\lim_{j\to\infty} \|f - \hat{s}_j\|_{L^2[0,1]} = 0.$$

We proceed to define the concept of the consistency of quadrature schemes for the approximate solution of the least square problem (cf. [9]). Let d be any fixed

positive integer and let $\mathscr{C}$ be a collection of partitions of the interval $[0, 1]$. For each $\Delta \in \mathscr{C}$, let $S(d, \Delta, z)$ be a space of piecewise polynomials and let $\hat{s}_\Delta$, the integral least square approximation to $f \in L^2[0, 1]$ in $S(d, \Delta, z)$, satisfy

$$\|f - \hat{s}_\Delta\|_N \leqq \mathscr{K}(\bar{\Delta})^l, \tag{3.10}$$

where $\|\cdot\|_N$ is some norm on $L^2[0, 1]$ and $\mathscr{K}$ and l are positive constants independent of Δ. In addition, for each $\Delta \in \mathscr{C}$, let $\tilde{s}_\Delta$, that element of $S(d, \Delta, z)$ obtained as an approximation to $\hat{s}_n$ using some bounded linear functional $\tilde{L}_\Delta$, satisfy

$$\|\hat{s}_\Delta - \tilde{s}_\Delta\|_N \leqq \mathscr{K}'(\bar{\Delta})^{l'}, \tag{3.11}$$

where $\mathscr{K}'$ and l' are positive constants independent of Δ. Then the triangle inequality, (3.10) and (3.11) imply that

$$\begin{aligned}\|f - \tilde{s}_\Delta\|_N &\leqq \|f - \hat{s}_\Delta\|_N + \|\hat{s}_\Delta - \tilde{s}_\Delta\|_N \\ &\leqq (\mathscr{K} + \mathscr{K}')(\bar{\Delta})^{\min(l,l')}\end{aligned}$$

for all $\Delta \in \mathscr{C}$ since $\bar{\Delta} \leqq 1$. Consequently, if $l' \geqq l$, the order of accuracy of the polynomial splines $\tilde{s}_\Delta$, $\Delta \in \mathscr{C}$, as approximations to f is no worse than the order of accuracy of the polynomial spline approximations $\hat{s}_\Delta$, $\Delta \in \mathscr{C}$. In this case, we say that the choice of functionals $\tilde{L}_\Delta$, $\Delta \in \mathscr{C}$, is consistent in the norm $\|\cdot\|_N$ with the bounds for the least square error given by (3.10).

The results of Theorems 3 and 5 immediately give us the following result. We note that this result can be improved in those cases for which special bounds exist for the interpolatory formulas which are employed in the composite rule.

THEOREM 6. *Let $\mathscr{C} = \mathscr{P}[0, 1]$, $d = 2m - 1$, $m \geqq 1$, and $m - 1 \leqq z \leqq 2m - 2$. Let $\mathscr{F} \subseteq \mathscr{P}_{n-1}[0, 1]$ be finite. For each $\Delta \in \mathscr{C}$, let the linear functional given in* (3.5) *be defined in terms of interpolation over partitions of subintervals determined by Δ all of which, when scaled to* $[0, 1]$, *are members of $\mathscr{F}$. Then, for $f \in C^{n+1}[0, 1]$ and* $4(2m - 1) \leqq 2n - 1$, *this choice of linear functionals is consistent in the L^2-norm with the bounds for the least square error given by* (2.4).

Approximate methods based on composite interpolatory quadrature schemes require only point evaluations of the basis functions $\{s_i\}_{i=1}^{ns}$, when, in fact, we have explicit piecewise polynomial representations for them. We now discuss the use of quadrature rules based on interpolating the function f by a piecewise polynomial $\tilde{f}$ and using the representations of the basis functions directly in calculating the approximations in question. Specifically, we define $\underline{\tilde{k}}$, and approximation to $\underline{\hat{k}}$, by

$$\tilde{k}_i \equiv \tilde{L}[f \cdot s_i] = \int_0^1 \tilde{f}(x) s_i(x)\, dx, \qquad 1 \leqq i \leqq ns. \tag{3.12}$$

Quadrature schemes for integrals of product integrands in which only one of the factors requires approximation are said to be of the Filon type (cf. [5, p. 62]). Since $\tilde{f}$ and all the basis functions are piecewise polynomials, each component of $\underline{\tilde{k}}$ is just the sum of definite integrals of polynomials and can be calculated directly. Here, again, we denote $\tilde{s} = \sum_{i=1}^{ns} \tilde{\alpha}_i s_i$, where $\tilde{\alpha} \equiv (\tilde{\alpha}_1, \cdots, \tilde{\alpha}_{ns})$ is the unique solution of the linear system (3.2) when $\underline{\tilde{k}}$ is defined by (3.12). We state and then prove

the following theorem which gives bounds for the L^2-norm of the error in approximating $\hat{s}$, the integral least square approximation to f, by $\tilde{s}$ in terms of the L^q-norm, $2 \leqq q \leqq \infty$, of the error in approximating f by the interpolate $\tilde{f}$. We remark that the proof of a multivariate analogue of this theorem is as straightforward as the proof of the univariate result.

THEOREM 7. *Let $\Delta \in \mathscr{P}_N[0, 1]$, $N \geqq 0$. For $f \in C[0, 1]$, let $\hat{s}$ be the integral least square approximation to f in the space of piecewise polynomials $S(d, \Delta, z)$. If $\tilde{s}$ is a discretized integral least square approximation to f in $S(d, \Delta, z)$ based on $\tilde{k}$, an approximation of $\hat{k}$, defined by* (3.12), *then*

$$\|\hat{s} - \tilde{s}\|_{L^2[0,1]} \leqq \|f - \tilde{f}\|_{L^q[0,1]}, \qquad 2 \leqq q \leqq \infty. \tag{3.13}$$

Proof. Beginning with (3.3) and using the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned}\|\hat{s} - \tilde{s}\|^2_{L^2[0,1]} &= (\hat{L} - \tilde{L})[f(\hat{s} - \tilde{s})] \\ &= \int_0^1 [f(x) - \tilde{f}(x)][\hat{s}(x) - \tilde{s}(x)]\, dx \\ &\leqq \|f - \tilde{f}\|_{L^2[0,1]}\|\hat{s} - \tilde{s}\|_{L^2[0,1]} \\ &\leqq \|f - \tilde{f}\|_{L^q[0,1]}\|\hat{s} - \tilde{s}\|_{L^2[0,1]}\end{aligned}$$

for $2 \leqq q \leqq \infty$. Cancelling $\|\hat{s} - \tilde{s}\|_{L^2[0,1]}$ from each side of this inequality yields (3.13), the result of the theorem.

In order to complete the analysis of quadrature errors in the approximate solution of the integral least square problem based on Filon-type schemes, we must specify the type of interpolation to be employed and error bounds for the chosen technique. We choose piecewise Lagrange interpolation and derive global error bounds for this technique using the following rather general error bounds for Lagrange interpolation over the partition $\Delta^*: a \leqq \tau_0 < \tau_1 < \cdots < \tau_s \leqq b$ of the interval $[a, b]$ (cf. [16, p. 105]). We first generalize the spaces $K^p[0, 1]$ defined in § 2. For any positive integer p and any extended real number r such that $1 \leqq r \leqq \infty$, let $K^{p,r}[a, b]$ be the collection of all real-valued functions f defined on $[a, b]$ such that $f \in C^{p-1}[a, b]$, $D^{p-1}f$ is absolutely continuous, and $D^p f \in L^r[a, b]$.

THEOREM 8. *Let $f \in K^{p,r}[a, b]$, $1 \leqq p \leqq s + 1$, and $1 \leqq q, r \leqq \infty$. Let f^* be the Lagrange interpolate to f over the partition Δ^* as defined above. Then*

$$\|f - f^*\|_{L^q[a,b]} \leqq (b - a)^{p-1/r+1/q} C_{\Delta^*,p,r,q} \|D^p f\|_{L^r[a,b]}. \tag{3.14}$$

Here, the constant $C_{\Delta^*,p,r,q}$ is interpreted to mean the appropriate constant for Lagrange interpolation over a partition of $[0, 1]$ corresponding to Lagrange interpolation over the partition Δ^* of $[a, b]$.

Given a partition $\Delta_\tau: a = x_0 < x_1 < \cdots < x_{N+1} = b$ of the interval $[a, b]$ and partitions $\Delta_i^*: x_i \leqq \tau_{i,0} < \tau_{i,1} < \cdots < \tau_{i,s} \leqq x_{i+1}$ of the subintervals $[x_i, x_{i+1}]$, $0 \leqq i \leqq N$, we define a piecewise Lagrange interpolate to $f \in C[a, b]$ by $\tilde{f}(x) \equiv f_i^*(x)$, $x \in [x_i, x_{i+1}]$, $0 \leqq i \leqq N$, where f_i^* is the Lagrange interpolate to f defined over Δ_i^*. Note that $\tilde{f}$ need not be continuous at the points x_i, $1 \leqq i \leqq N$, although continuity at x_i is guaranteed by $\tau_{i-1,s} = x_i = \tau_{i,0}$. In the following theorem, we give error bounds for piecewise Lagrange interpolation. The proof parallels the derivation of error bounds for piecewise Hermite interpolation given in [3].

THEOREM 9. *Given any $f \in K^{p,r}[a, b]$, $1 \leqq p \leqq s + 1$, $1 \leqq r \leqq \infty$, let $\tilde{f}$ be the piecewise Lagrange interpolate to f as defined above. Then for $q \geqq r$,*

$$\|f - \tilde{f}\|_{L^q[a,b]} \leqq (\bar{\Delta}_\tau)^{p-1/r+1/q} \cdot \max_{0 \leqq i \leqq N} C_{\Delta_i^*,p,r,q} \cdot \|D^p f\|_{L^r[a,b]}, \tag{3.15}$$

and, for $1 \leqq q \leqq r$,

$$\|f - \tilde{f}\|_{L^q[a,b]} \leqq (\bar{\Delta}_\tau)^p \cdot (b - a)^{(r-q)/rq} \cdot \max_{0 \leqq i \leqq N} C_{\Delta_i^*,p,r,r} \cdot \|D^p f\|_{L^r[a,b]}. \tag{3.16}$$

Proof. With the definition of $K^{p,r}[a, b]$ and the hypothesis of the theorem, it is clear that $D^p f \in L^r[a, b]$ and $f - \tilde{f} \in L^q[a, b]$ for $1 \leqq q \leqq \infty$. For $0 \leqq i \leqq N$, let

$$v_i \equiv \left\{ \int_{x_i}^{x_{i+1}} |f(t) - \tilde{f}(t)|^q \, dt \right\}^{1/q} \quad \text{and} \quad \omega_i \equiv \left\{ \int_{x_i}^{x_{i+1}} |D^p f|^r \, dt \right\}^{1/r}.$$

Then, employing the result of Theorem 8, we have, for $0 \leqq i \leqq N$,

$$v_i \leqq (x_{i+1} - x_i)^{p-1/r+1/q} \cdot C_{\Delta_i^*,p,r,q} \cdot \omega_i.$$

Here, the constants $C_{\Delta_i^*,p,r,q}$ are interpreted to be the normalized constants defined over the interval $[0, 1]$ in terms of the appropriately scaled partitions. But then

$$\begin{aligned} \|f - \tilde{f}\|_{L^q[a,b]} &= \left\{ \sum_{i=0}^{N} \int_{x_i}^{x_{i+1}} |f(t) - \tilde{f}(t)|^q \, dt \right\}^{1/q} = \left\{ \sum_{i=0}^{N} (v_i)^q \right\}^{1/q} \\ &\leqq \left\{ \sum_{i=0}^{N} [(x_{i+1} - x_i)^{p-1/r+1/q} \cdot C_{\Delta_i^*,p,r,q} \cdot \omega_i]^q \right\}^{1/q} \\ &\leqq (\bar{\Delta}_\tau)^{p-1/r+1/q} \cdot \max_{0 \leqq i \leqq N} C_{\Delta_i^*,p,r,q} \cdot \left\{ \sum_{i=0}^{N} (\omega_i)^q \right\}^{1/q}. \end{aligned} \tag{3.17}$$

However, for $q \geqq r$, Jensen's inequality [2, p. 18] gives

$$\begin{aligned} \left\{ \sum_{i=0}^{N} (\omega_i)^q \right\}^{1/q} \leqq \left\{ \sum_{i=0}^{N} (\omega_i)^r \right\}^{1/r} &= \left\{ \sum_{i=0}^{N} \int_{x_i}^{x_{i+1}} |D^p f(t)|^r \, dt \right\}^{1/r} \\ &= \left\{ \int_a^b |D^p f(t)|^r \, dt \right\}^{1/r} = \|D^p f\|_{L^r[a,b]}. \end{aligned} \tag{3.18}$$

Combining (3.17) and (3.18) yields (3.15), the first result of the theorem. Now, for $1 \leqq q \leqq r$, the integral Hölder's inequality gives

$$\|f - \tilde{f}\|_{L^q[a,b]} \leqq (b - a)^{(r-q)/rq} \|f - \tilde{f}\|_{L^r[a,b]},$$

which, when combined with (3.15) in the case $q = r$, gives (3.16), the second result of the theorem.

As a corollary to Theorems 7 and 9, we have the following result. We do not employ these theorems in their greatest generality. Indeed, we assume that $q = 2$ in Theorem 7 and $q = r = 2$ in Theorem 9.

COROLLARY. *Let $\Delta \in \mathscr{P}[0, 1]$ and let $\hat{s}$ be the integral least square approximation to $f \in K^p[0, 1]$ in $S(d, \Delta, z)$. Given a finite subset $\mathscr{F} \subseteq \mathscr{P}_{s-1}[0, 1]$ and a sequence of partitions $\{\Delta_\tau^j\}_{j=1}^{\infty}$ of $[0, 1]$ such that $\lim_{j \to \infty} \bar{\Delta}_\tau^j = 0$, let $\tilde{s}_j$, for each j, be a discretized least square approximation in $S(d, \Delta, z)$ to f obtained using a Filon-type quadrature scheme based on piecewise Lagrange interpolation where the partitions of the sub-*

intervals of Δ_τ^j, *scaled to the interval* $[0, 1]$, *are all elements of the finite set* $\mathscr{F}$. *Then, if* $p \leqq s + 1$,

$$\lim_{j \to \infty} \|\hat{s} - \tilde{s}_j\|_{L^2[0,1]} = 0.$$

This result tells us that the L^2-errors introduced into the approximation by the use of these Filon-type schemes tend to zero with $\bar{\Delta}_\tau^j$. These errors may or may not be small compared to $\|f - \hat{s}\|_{L^2[0,1]}$. By combining the corollary to Theorem 3 with this last result, we immediately obtain the following result.

COROLLARY. *Let* $\{\Delta^j\}_{j=1}^\infty$ *be a sequence of partitions of* $[0, 1]$ *such that* $\lim_{j\to\infty}\bar{\Delta}^j = 0$ *and, for each* j, *let* $\hat{s}_j$ *be the least square approximation to* $f \in K^{2m}[0, 1]$ *in* $S(2m - 1, \Delta, z)$, $m \geqq 1$, $m - 1 \leqq z \leqq 2m - 2$. *Given a finite subset* $\mathscr{F} \subseteq \mathscr{P}_{s-1}[0, 1]$ *and a sequence of partitions* $\{\Delta_\tau^j\}_{j=1}^\infty$ *of* $[0, 1]$ *such that* $\lim_{j\to\infty}\bar{\Delta}_\tau^j = 0$, *let* $\tilde{s}_j$, *for each* j, *be a discretized least square approximation in* $S(2m - 1, \Delta^j, z)$ *to* f *obtained using a Filon-type quadrature scheme based on piecewise Lagrange interpolation where the partitions of the subintervals of* Δ_τ^j, *scaled to the interval* $[0, 1]$, *are all elements of the finite set* $\mathscr{F}$. *Then, if* $2m \leqq s + 1$,

$$\lim_{j \to \infty} \|f - \tilde{s}_j\|_{L^2[0,1]} = 0.$$

Our final result of this section deals with the concept of the consistency of collections of Filon-type schemes and follows from Theorems 2, 7 and 9.

THEOREM 10. *Let* $\mathscr{C} = \mathscr{P}[0, 1]$, $\mathscr{F} \subseteq \mathscr{P}_{2m-2}[0, 1]$, $\mathscr{F}$ *finite*, $m - 1 \leqq z \leqq 2m - 2$ *and, for* $\Delta \in \mathscr{C}$, *consider approximating the integral least square spline in* $S(2m - 1, \Delta, z)$ *to* $f \in K^{2m}[0, 1]$ *using a linear functional of the form* (3.12) *based on piecewise Lagrange interpolation with* $\bar{\Delta}_\tau \leqq \bar{\Delta}$ *and the partitions of the subintervals of* Δ_τ, *scaled to the interval* $[0, 1]$, *all in* $\mathscr{F}$. *Then this choice of linear functionals is consistent in the* L^2-*norm with the bounds for the least square error given by* (2.4).

4. Numerical results. In this section we present some numerical results based on FORTRAN codes of the techniques which we have considered in this paper. We begin with a documentation of experiments designed to illustrate the theoretical results on the consistency of collections of quadrature schemes for the numerical solution of the least square polynomial spline approximation problem. We conclude with plots of least square cubic spline approximations to data sets which are considered difficult to approximate by polynomials. Note that all numerical results were computed on a UNIVAC 1108.

Let Δ_N be the uniform partition of $[0, 1]$ with mesh length $h_N = 1/(N - 1)$. Fix $m = 1$ or 2 and let $m - 1 \leqq z \leqq 2m - 1$. We shall examine the errors in approximating the exponential function, $\exp(x) \equiv e^x$, over $[0, 1]$ by elements of the spline space $S(2m - 1, \Delta_N, z)$ using four different least square techniques. We define the polynomial spline functions $\hat{s}_N$, $\tilde{s}_N^1$, $\tilde{s}_N^2$, and $\tilde{s}_N^3 \in S(2m - 1, \Delta_N, z)$ as follows:

$\hat{s}_N \equiv$ integral least square approximation to exp as defined in § 2;
$\tilde{s}_N^1 \equiv$ discretized least square approximation to exp based on a composite interpolatory quadrature scheme as defined in § 3;
$\tilde{s}_N^2 \equiv$ discretized least square approximation to exp based on a Filon-type quadrature scheme using piecewise Lagrange interpolation as defined in § 3;

and

$\tilde{s}_N^3 \equiv$ standard discrete least squares approximation to exp.

We note that $\hat{s}_N$ can be computed since, for the exponential function, we can obtain numerical values for the components of the vector $\underline{\hat{k}}$ of the system (3.1). The discretized least square approximations, $\tilde{s}_N^1$ and $\tilde{s}_N^2$, are obtained by solving the system (3.2) where $\underline{\tilde{k}}$, an approximation to $\underline{\hat{k}}$, in each case is determined through the use of the appropriate quadrature scheme. The standard discrete least squares technique, which is used to determine $\tilde{s}_N^3$, is based on the solution of the normal equations for a (discrete) semi-inner product. In this case the normal matrix cannot be guaranteed to be positive definite. Of course, the potential instability in solving the corresponding system must be considered when employing this purely discrete technique.

Theorems 3, 5, 7, and 9 are employed to obtain the following appraisals where K, K_1, and K_2 are positive constants independent of h_N:

$$\|\exp - \hat{s}_N\|_{L^2[0,1]} \leqq K(h_N)^{2m}, \tag{4.1}$$

$$\|\hat{s}_N - \tilde{s}_N^1\|_{L^2[0,1]} \leqq K_1(h_N)^{n-2m+3/2}, \tag{4.2}$$

where n is the order of interpolatory quadrature in terms of which $\tilde{s}_N^1$ is defined, and

$$\|\hat{s}_N - \tilde{s}_N^2\|_{L^2[0,1]} \leqq K_2(h_\tau)^{s+1}, \tag{4.3}$$

where s is the degree of piecewise Lagrange interpolation employed in the Filon quadrature in terms of which $\tilde{s}_N^2$ is defined and h_τ is the mesh width for the distribution of data for this interpolation technique. Combining (4.1) and (4.2) yields

$$\|\exp - \tilde{s}_N^1\|_{L^2[0,1]} \leqq \{K + K_1\}(h_N)^{\min(2m, n-2m+3/2)}, \tag{4.4}$$

and (4.1) with (4.3) yields

$$\|\exp - \tilde{s}_N^2\|_{L^2[0,1]} \leqq \{K + K_2\} \max(h_N, h_\tau)^{\min(2m, s+1)}. \tag{4.5}$$

We have no bounds for the error in the fourth approximation. However, for certain weighted discrete techniques, the result of Theorem 5 is valid. We observe that the interpolatory schemes of order n employed in § 3 are exact for polynomials of degree $\leqq n$ on subintervals of the partition Δ. Consequently, if $n \geqq 4m - 1$, composite interpolatory schemes of order n are exact for products of splines in $S(2m - 1, \Delta, z)$ and, in particular, for the entries of the least square matrix defined in (3.1). Then, for the discrete techniques with weights from such composite interpolatory schemes, Theorem 5 holds and we have appraisals in these special cases.

Corresponding to the spline spaces $S(1, \Delta_N, 0)$, $S(3, \Delta_N, 2)$ and $S(3, \Delta_N, 1)$, in Tables 1, 2 and 3 we present approximate numerical values for the quantities $\|\exp - \hat{s}_N\|_{L^2[0,1]}$ and $\|\exp - \tilde{s}_N^i\|_{L^2[0,1]}$, $i = 1, 2, 3$, where the quadrature schemes used to determine the discretized spline approximations are chosen to be consistent with the L^2-bounds for the least square error as given by (4.1). Specifically, the composite interpolatory formula employed to determine $\tilde{s}_N^1$ is based on $(4m - 1)$-point open ended Newton–Cotes formulas, and the Filon-type scheme used to determine $\tilde{s}_N^2$ is based on piecewise Lagrange interpolation of degree $2m - 1$.

TABLE 1

Consistent quadrature schemes for linear spline spaces

h_N	$\|\exp - \hat{s}_N\|$	α	$\|\exp - \tilde{s}_N^1\|$	α	$\|\exp - \tilde{s}_N^2\|$	α	$\|\exp - \tilde{s}_N^3\|$	α
1/2	$1.68\ 10^{-2}$	—	$1.68\ 10^{-2}$	—	$1.72\ 10^{-2}$	—	$1.69\ 10^{-2}$	—
1/3	$7.44\ 10^{-3}$	2.01	$7.48\ 10^{-3}$	2.01	$7.62\ 10^{-3}$	2.00	$7.51\ 10^{-3}$	2.00
1/4	$4.18\ 10^{-3}$	2.00	$4.18\ 10^{-3}$	2.00	$4.29\ 10^{-3}$	2.00	$4.23\ 10^{-3}$	2.00
1/5	$2.68\ 10^{-3}$	2.00	$2.68\ 10^{-3}$	2.00	$2.75\ 10^{-3}$	2.00	$2.70\ 10^{-3}$	2.00

TABLE 2

Consistent quadrature schemes for cubic spline spaces

h_N	$\|\exp - s_N\|$	α	$\|\exp - \tilde{s}_N^1\|$	α	$\|\exp - \tilde{s}_N^2\|$	α	$\|\exp - \tilde{s}_N^3\|$	α
1/2	$4.53\ 10^{-5}$	—	$4.54\ 10^{-5}$	—	$4.53\ 10^{-5}$	—	$4.64\ 10^{-5}$	—
1/3	$1.63\ 10^{-5}$	2.52	$1.63\ 10^{-5}$	2.52	$1.63\ 10^{-5}$	2.52	$1.63\ 10^{-5}$	2.58
1/4	$5.30\ 10^{-6}$	3.90	$5.30\ 10^{-6}$	3.90	$5.30\ 10^{-6}$	3.90	$5.31\ 10^{-6}$	3.90
1/5	$2.30\ 10^{-6}$	3.73	$2.30\ 10^{-6}$	3.73	$2.30\ 10^{-6}$	3.73	$2.30\ 10^{-6}$	3.74
1/6	$1.13\ 10^{-6}$	3.91	$1.13\ 10^{-6}$	3.91	$1.13\ 10^{-6}$	3.91	$1.13\ 10^{-6}$	3.91
1/7	$6.21\ 10^{-7}$	3.87	$6.22\ 10^{-7}$	3.87	$6.22\ 10^{-7}$	3.87	$6.22\ 10^{-7}$	3.87
1/8	$3.68\ 10^{-7}$	3.92	$3.68\ 10^{-7}$	3.92	$3.68\ 10^{-7}$	3.92	$3.68\ 10^{-7}$	3.92

TABLE 3

Consistent quadrature schemes for cubic Hermite spline spaces

h_N	$\|\exp - s_N\|$	α	$\|\exp - \tilde{s}_N^1\|$	α	$\|\exp - \tilde{s}_N^2\|$	α	$\|\exp - \tilde{s}_N^3\|$	α
1/2	$4.25\ 10^{-5}$	—	$4.26\ 10^{-5}$	—	$4.26\ 10^{-5}$	—	$4.36\ 10^{-5}$	—
1/3	$1.16\ 10^{-5}$	3.20	$1.16\ 10^{-5}$	3.20	$1.16\ 10^{-5}$	3.20	$1.18\ 10^{-5}$	3.22
1/4	$4.32\ 10^{-6}$	3.44	$4.32\ 10^{-6}$	3.45	$4.32\ 10^{-6}$	3.44	$4.36\ 10^{-6}$	3.46
1/5	$1.94\ 10^{-6}$	3.60	$1.94\ 10^{-6}$	3.60	$1.94\ 10^{-6}$	3.60	$1.95\ 10^{-6}$	3.61
1/6	$9.87\ 10^{-7}$	3.69	$9.87\ 10^{-7}$	3.69	$9.87\ 10^{-7}$	3.69	$9.93\ 10^{-7}$	3.70
1/7	$5.53\ 10^{-7}$	3.75	$5.53\ 10^{-7}$	3.75	$5.53\ 10^{-7}$	3.75	$5.56\ 10^{-7}$	3.76
1/8	$3.33\ 10^{-7}$	3.79	$3.33\ 10^{-7}$	3.79	$3.33\ 10^{-7}$	3.79	$3.35\ 10^{-7}$	3.80

For any fixed value of N, the data points used to determine the approximations are the same for each (discrete) technique. Following [9], for each pair of consecutive entries, we have included the quantity

$$\alpha \equiv \log(\|\exp - s_{n_1}\|/\|\exp - s_{n_2}\|)/\log(h_{n_1}/h_{n_2})$$

defined in terms of successive values of the mesh spacing, $h_{n_1} > h_{n_2}$. The motivation for this definition is that as $h_n \to 0$ we have

$$\|\exp - s_n\| \sim \mathscr{K}(h_n)^\alpha$$

for some constants α and $\mathscr{K}$ depending on the norm $\|\cdot\|$, but not on h_n. Then for two successive values of h, $h_{n_1} > h_{n_2}$,

$$\|\exp - s_{n_1}\|/\|\exp - s_{n_2}\| \sim (h_{n_1}/h_{n_2})^\alpha,$$

from which the definition of α follows. Note that both discretized least square approximations, $\tilde{s}_N^1$ and $\tilde{s}_N^2$, exhibit the consistent behavior predicted by Theorems 6 and 10. We also note that the standard discrete least square technique leads to approximations, $\tilde{s}_N^3$, which also exhibit this consistent behavior.

In Figs. 1, 2 and 3, we present plots of cubic polynomial spline approximations determined by two different least square techniques to each of three data sets (cf. [12], [6] and [20], respectively) considered difficult to approximate using

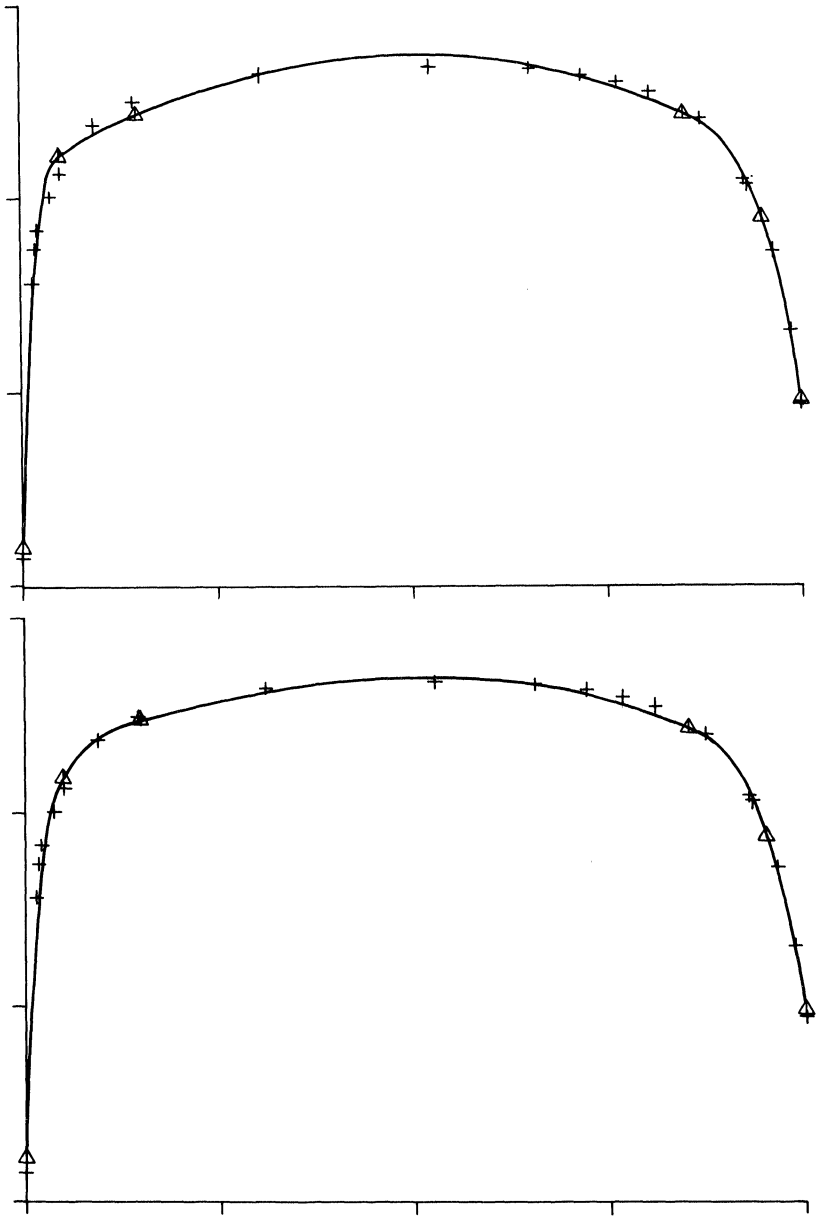

FIG. 1.

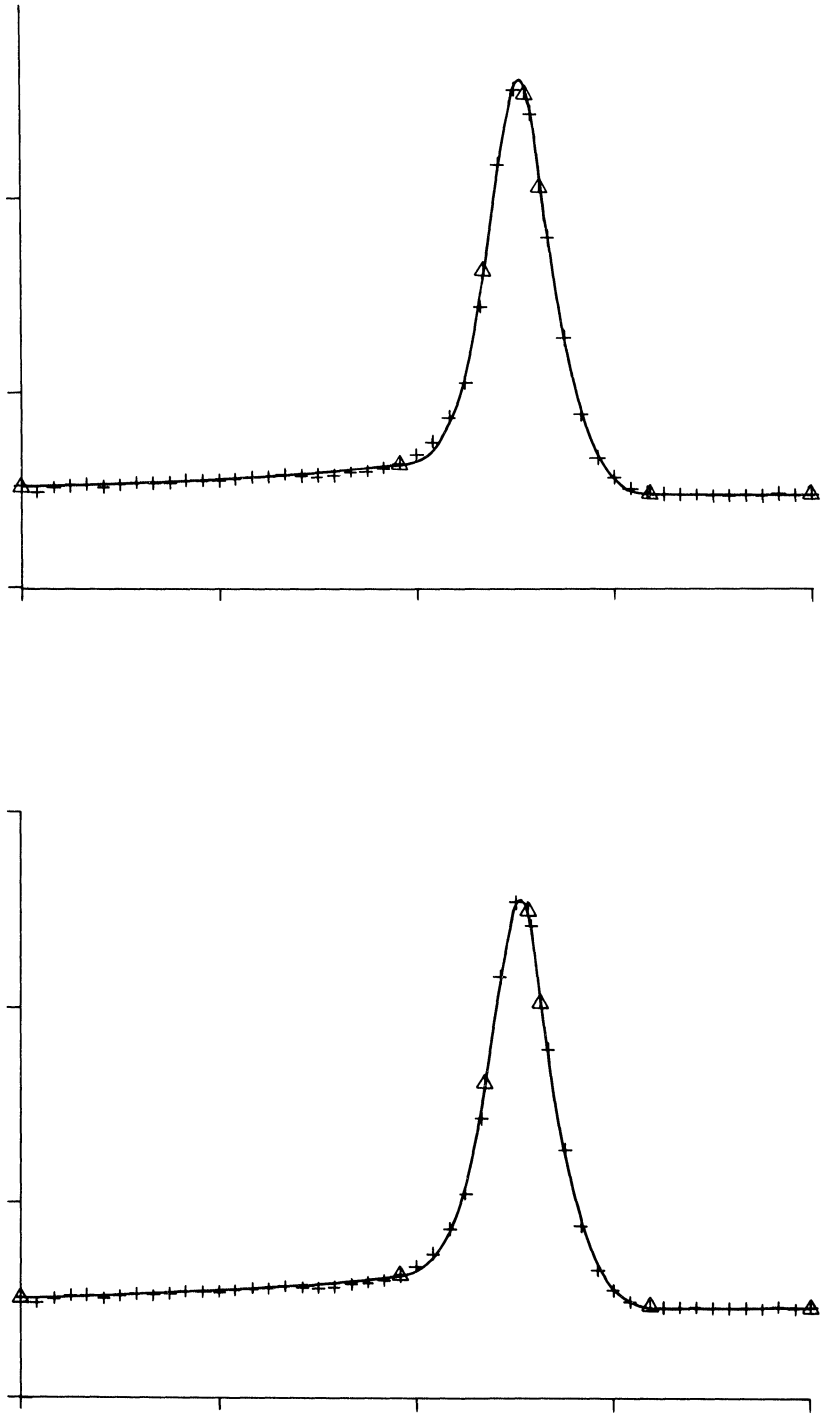

FIG. 2.

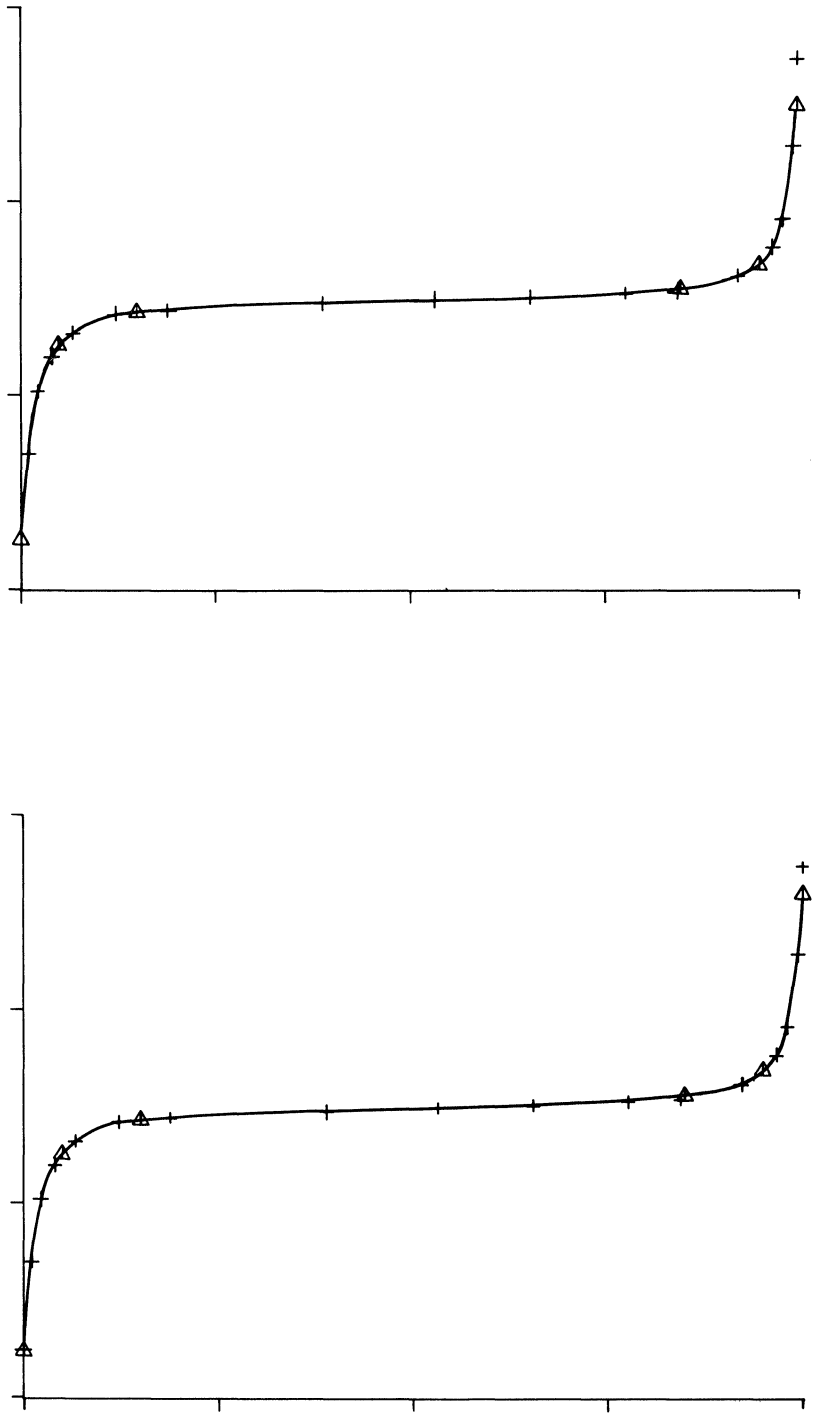

FIG. 3.

polynomials. The partitions were chosen on the basis of trial and error. On these plots, each knot is denoted by Δ and each data point by $+$. In each figure, the upper plot represents a discrete least square cubic spline approximation and the lower plot corresponds to a discretized least square cubic spline approximation determined by Filon quadrature based on piecewise linear interpolation.

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