Viscous effects in Rayleigh–Taylor instability

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A simple, physical approximation is developed for the effect of viscosity for stable interfacial waves and for the unstable interfacial waves which correspond to Rayleigh–Taylor instability. The approximate picture is rigorously justified for the interface between a heavy fluid (e.g., water) and a light fluid (e.g., air) with negligible dynamic effect. The approximate picture may also be rigorously justified for the case of two fluids for which the differences in density and viscosity are small. The treatment of the interfacial waves may easily be extended to the case where one of the fluids has a small thickness; that is, the case in which one of the fluids is bounded by a free surface or by a rigid wall. The theory is used to give an explanation of the bioconvective patterns which have been observed with cultures of microorganisms which have negative geotaxis. Since such organisms tend to collect at the surface of a culture and since they are heavier than water, the conditions for Rayleigh–Taylor instability are met. It is shown that the observed patterns are quite accurately explained by the theory. Similar observations with a viscous liquid loaded with small glass spheres are described. A behavior similar to the bioconvective patterns with microorganisms is found and the results are also explained quantitatively by Rayleigh–Taylor instability theory for a continuous medium with viscosity.

INTRODUCTION

Viscous effects in Rayleigh–Taylor instability have not been considered in detail beyond the analytical aspects of the problem, and yet there are interesting situations in which the role of viscosity is quite decisive for the behavior of the instabilities. While the analysis of the instability of the interface between immiscible fluids can be worked out in a straightforward way, there are extensive algebraic complications which quite soon become apparent. Since the physical aspects of the problem can be hidden by such formal solutions, it should be of interest to develop the essential results by simple physical arguments. As we shall see, it is easy to obtain an approximate description of the phenomena involved by such arguments.

APPROXIMATE TREATMENT OF INTERFACIAL WAVES

We first consider the simplest possible situation in which a nonviscous fluid of density ρ when undisturbed occupies the semi-infinite region $z < 0$ and is acted on by gravity with acceleration g. Suppose that the region $z > 0$ is occupied by a fluid with negligible kinematic or dynamic effect; i.e., the density of this region is zero, and its viscosity is zero. If the surface is disturbed by a plane wave of small amplitude,

$$\eta(x, t; k) = a_k(t) \sin kx, \tag{1}$$

it is evident that the oscillations are stable and it is also evident that for small amplitudes they must be simple harmonic. Thus,

$$\ddot{a}_k + \omega_0^2 a_k = 0. \tag{2}$$

The angular frequency ω_0 can depend only on g and the wavenumber $k = 2\pi/\lambda$, where λ is the wavelength. Dimensional considerations suggest that

$$\omega_0 = (gk)^{1/2}, \tag{3}$$

which is, of course, the correct and familiar result. One cannot be assured by a dimensional argument that the result of Eq. (2) should not contain some numerical factor, but the precise result, as given, is easily derived.

If now we consider the interface between two fluids, one of density ρ' in the region $z > 0$ and the second of density ρ in the region $z < 0$ and if $\rho' < \rho$, then an interfacial wave of small amplitude is stable. It is easy to see that the effective value of gravity for the wave is

$$g' = \frac{\rho - \rho'}{\rho + \rho'} g, \tag{4}$$

since the downward acceleration is decreased by the factor $(\rho - \rho')/\rho$ and the inertia is increased by the factor $(\rho + \rho')/\rho$ due to the pressure of the upper fluid. The small amplitude oscillation must again be simple harmonic and the angular frequency will be

$$\omega_0 = (g'k)^{1/2}. \tag{5}$$

The effect of surface tension on the surface waves may also be elucidated in the following way. As before, the gravity field is taken to act in the $-z$ direction. We suppose that an element of the fluid of density ρ with cross section $dx\,dy$ is elevated to a height η above $z = 0$ in a fluid of density ρ'. The downward force on the element due to gravity is then $g(\rho - \rho')\eta\, dx\, dy$. The surface tension is given by the product of the surface tension constant T and the curvature. This curvature is approximately $\partial^2\eta/\partial x^2 = -k^2\eta$ since $\eta = a_k \sin kx$ where a_k is a small quantity. Thus, the downward force on the element from surface tension is $Tk^2\eta\, dx\, dy$. The effective inertia of the element is $(\rho + \rho')\eta\, dx\, dy$ and it follows that the net effective acceleration in the $-z$ direction is

$$g'_T = \frac{(\rho - \rho')}{(\rho + \rho')} g + \frac{Tk^2}{(\rho + \rho')}. \tag{6}$$

The simple harmonic oscillation of $a_k(t)$ is now given by

$$\ddot{a}_k + \omega_0^2 a_k = 0, \tag{7}$$

where

$$\omega_0^2 = kg_T' = \frac{(\rho - \rho')}{(\rho + \rho')} gk + \frac{Tk^3}{(\rho + \rho')}. \tag{8}$$

Equation (8) is the well-known dispersion formula for interfacial waves when viscosity is neglected.

Thus far ω_0^2 in Eq. (5) or (8) has been taken to be a positive quantity since we have supposed that $\rho > \rho'$. There is no mathematical or physical reason that limits the applicability of the discussion to the case in which $\rho' > \rho$. In place of (7) we would have

$$\ddot{a}_k - \sigma^2 a_k = 0, \tag{9}$$

where σ^2 is a positive quantity,

$$\sigma^2 = -\omega_0^2 = \frac{(\rho' - \rho)}{(\rho' + \rho)} gk - \frac{Tk^3}{(\rho + \rho')}. \tag{10}$$

As is to be expected the interface is now unstable, and the interfacial wave amplitude grows like $e^{\sigma t}$. This growth phenomenon is the familiar Rayleigh–Taylor instability phenomenon. The description of the instability is, of course, valid only so long as the amplitude remains small, but we must expect that the wavelengths for which σ is largest as given by the small amplitude theory will continue to lead in growth beyond the amplitude range for which the small amplitude description is valid.

It is evident from (10) that surface tension can prevent the instability for sufficiently small wavelengths. The limit of instability is given by

$$k_1 = ((\rho' - \rho)g/T)^{1/2}. \tag{11}$$

The stability of small hanging water droplets is easily observed and is a familiar effect. This stability is related to the behavior just indicated.

In an application of Rayleigh–Taylor instability which will be of particular interest here, surface tension is not so important but viscosity will be decisive. To simplify the physical discussion, therefore, we shall drop the term arising from surface tension in the following; its effects can always be included in the way that has just been described. We shall now attempt to develop a simple approach to the damping of stable or unstable interfacial waves. If we first consider the stable case, $\rho' < \rho$, which can be described in terms of a simple harmonic oscillation,

$$\ddot{a}_k + \omega_0^2 a_k = 0,$$

with

$$\omega_0^2 = \frac{(\rho - \rho')}{(\rho + \rho')} gk.$$

Clearly, the effect of viscosity is to give some damping to the oscillations, and the damping may easily be estimated for some particular cases. First, we take the case of a heavy fluid (e.g., glycerol, water) in contact with a fluid with negligible dynamic effect (e.g., air) so that we have only to consider a single fluid. If μ is the dynamic viscosity of this fluid, and $\nu = \mu/\rho$ is its kinematic viscosity, then from dimensional considerations the damping of the oscillations should depend only on νk^2. From the familiar expression for a damped simple harmonic oscillator, the damped surface wave will, in an approximate sense, satisfy an equation of the form

$$\ddot{a}_k + f\nu k^2 \dot{a}_k + \omega_0^2 a_k = 0. \tag{12}$$

The factor f is, of course, unknown, and actually the exact description of damped surface waves cannot be accurately described in such simple terms except in limiting situations. For example, it is well-known (Ref. 1, pp. 623–625) that, for very small damping, surface wave oscillations have the form

$$\exp(i\omega_0 t - 2\nu k^2 t),$$

and this form would be obtained from (12) with $f = 4$:

$$\ddot{a}_k + 4\nu k^2 \dot{a}_k + \omega_0^2 a_k = 0, \qquad \nu k^2 \ll \omega_0, \tag{13}$$

as may be seen by writing

$$a_k(t) = a_k(0)\exp(nt),$$

so that for n we have the equation

$$n^2 + 4\nu k^2 n + \omega_0^2 = 0, \qquad \nu k^2 \ll \omega_0. \tag{13'}$$

Equations (13) and (13′) then describe the long-wavelength limit in which the damping is very small. Our major interest here, however, is in the case in which viscous damping is important. Some guidance in this direction may be obtained from the known behavior in the "creeping motion" limit for which (Ref. 1, pp. 625–628)

$$n \simeq -\omega_0^2/2\nu k^2.$$

This relation suggests that the short wavelength limit, or the limit in which damping is important, may be described by

$$\ddot{a}_k + 2\nu k^2 \dot{a}_k + \omega_0^2 a_k = 0, \qquad \nu k^2 \gg \omega_0, \tag{14}$$

or by

$$n^2 + 2\nu k^2 n + \omega_0^2 = 0, \qquad \nu k^2 \gg \omega_0. \tag{14'}$$

We must expect a corresponding behavior for the unstable case for which we would then have

$$n^2 + 4\nu k^2 n - \sigma^2 = 0, \qquad \nu k^2 \ll \sigma, \tag{15}$$

$$n^2 + 2\nu k^2 n - \sigma^2 = 0, \qquad \nu k^2 \gg \sigma. \tag{16}$$

We may expect Eq. (16) to be of particular interest here since it covers the range in which viscous damping is important. We shall use Eq. (16) over the whole range of k even though it may not be accurately permissible when $\nu k^2 \sim \sigma$. We shall see later that Eq. (16) has acceptable accuracy even when $\nu k^2 \sim \sigma$ for the kind of applications to be made here. If we proceed with this expression for n as a function of k, or λ, we readily see that n has a maximum for

$$\lambda_m = 4\pi(\nu^2/g')^{1/3}.$$

This result is of great physical significance since the

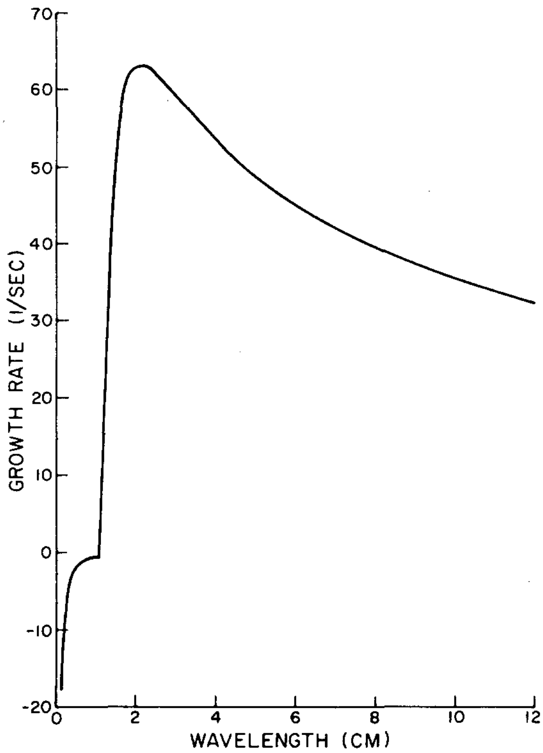

FIG. 1. The ordinate is the instability growth factor n as given by the approximate theory of Eq. (16) with σ given by Eq. (10). Water is accelerated into air with an acceleration of 2 g. For very short wavelengths, the growth rate n is negative and the interface is stable.

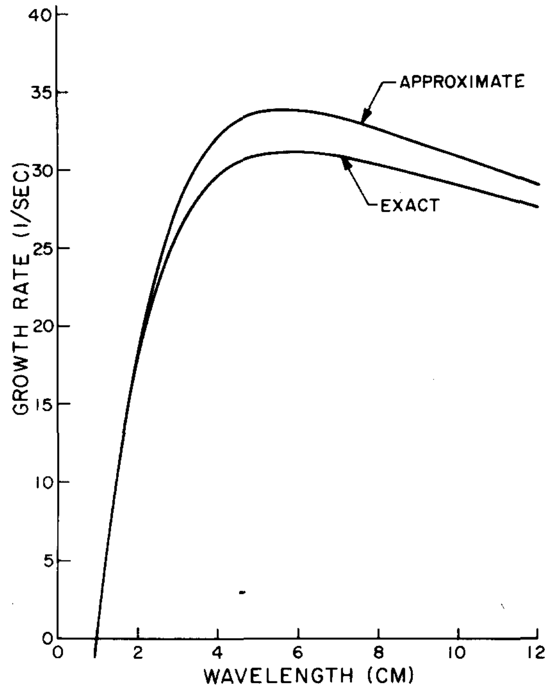

FIG. 2. The exact and the approximate values for the instability growth factor n are shown for glycerin accelerated into air with an acceleration of 2 g.

interfacial disturbance with this wavelength grows more rapidly than any other. It is true that the theory is limited to small disturbances, but the disturbance which grows most rapidly should continue to be the leading one into the range where large amplitudes are reached.

Some objection might be raised to the very simple derivation of λ_m. It may be pointed out that the numerical factor f in Eq. (12) enters only as $f^{2/3}$ so that our expression for λ_m is not sensitive to its value. It is, of course, a straightforward matter to determine the rigorous dispersion formula for $n(k)$, and it is also easy to show that one gets Eq. (16) in the short wavelength limit. Beyond that a comparison of the approximate and the exact result may readily be made. For water accelerated into air with an acceleration of $2g$ the approximate n as given by (16) is shown in Fig. 1; the exact n was also computed and no significant difference between the two sets of value could be detected. Surface tension was included in these calculations. The close agreement between the exact solution and the approximate solution of Eq. (16) for the water–air interface cannot be taken as a justification of the approximate treatment of viscosity since for this combination surface tension, not viscosity, gives the significant modification of the simple gravity effect. Figure 2 shows the approximate relation $n(\lambda)$ and the exact relation for air accelerated into glycerin with $2g$. Surface tension is not important in this case, but viscosity is decisive. The agreement is seen to be quite satisfactory.

The great advantage of the simple model which leads to (16) is that it gives a direct physical insight to an expectation of a maximum in $n(k)$, a maximum which must occur in the unstable physical situation.

A second example which will be considered is the case of two fluids in which the density difference is small. Further, it will be supposed that the two fluids have the same kinematic viscosity and that surface tension is not important. In the short wavelength limit, that is, in the limit in which viscosity is important, we again use the damped oscillator equation in the form

$$n^2 + 2\nu k^2 n + \omega_0^2 = 0, \qquad \nu k^2 \gg \omega_0, \tag{17}$$

for the stable case, and

$$n^2 + 2\nu k^2 n - \sigma^2 = 0, \qquad \nu k^2 \gg \sigma, \tag{18}$$

for the unstable case. Our concern here is with the unstable case and as before the maximum value of n occurs for a wavelength

$$\lambda_m = 4\pi(\nu^2/g')^{1/3}.$$

The approximate Eq. (18) may again be justified by examination of the exact solution. A comparison of the approximate solution and the exact solution shows that the approximate formulation is quite accurate (see Fig. 3). We point out that Eqs. (17) and (18) describe a wave on the interface of two fluids for which $\rho = \rho' + \Delta\rho$ and $\mu = \mu' + \Delta\mu$, where $\Delta\rho$ and $\Delta\mu$ are both small compared with ρ and μ, respectively.
Fig. 3.

We shall also be concerned with an experimental situation in which the fluid of density ρ', which overlays a fluid of density ρ, has a thickness h'. The lower fluid will be taken to have a large thickness. The dimensions with which h' is to be compared are, of course, the wavelengths λ of the interfacial waves which are of interest. The upper fluid now lies in the range $0 < z < h'$ and the lower fluid in the entire range $z < 0$. In the first case, we

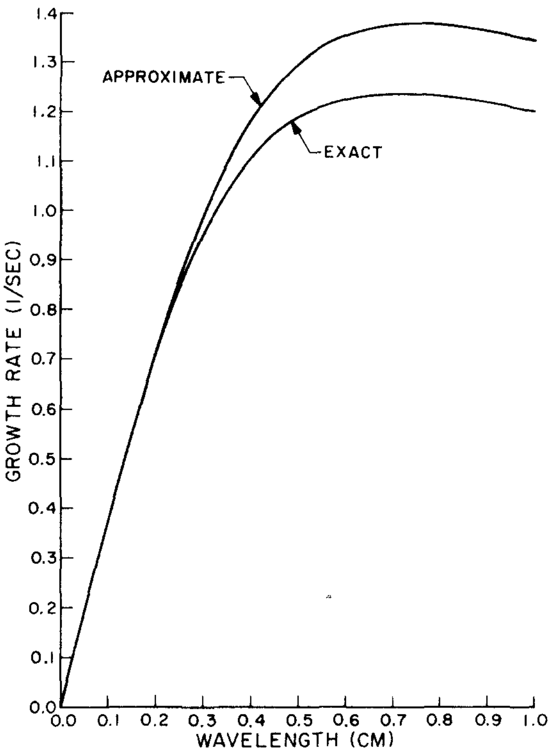

FIG. 3. The exact and the approximate values for the instability growth factor n are shown for a liquid with density 1.000942 g/cm³ over a liquid with unit density in the gravity field 1 g.

suppose that the upper fluid is confined by a rigid plane at $z = h'$. If we consider a wave on the interface with wavenumber k, then it is known for the nonviscous case (Ref. 1, pp. 370–372) that the dispersion formula is

$$\omega_0^2 = \frac{\rho - \rho'}{\rho + \rho' \coth kh'} gk, \qquad \rho > \rho'. \tag{19}$$

We shall be concerned with applications in which the kinematic viscosities, ν and ν', are essentially the same. If we have fluids for which the viscosities are large, or if the density difference is small, viscous damping of the oscillations will be important. As before, if we write

$$a_k(t) = a_k(0)\exp(nt),$$

then

$$n^2 + 2\nu k^2 n + \omega_0^2 = 0, \qquad \nu k^2 \gtrsim \omega_0, \tag{20}$$

for the stable case $\rho > \rho'$. For the unstable case, we have

$$n^2 + 2\nu k^2 n - \sigma^2 = 0, \qquad \nu k^2 \gtrsim \sigma. \tag{21}$$

When the upper boundary at $z = h'$ is a free surface we have (Ref. 1, pp. 370–372) for the stable case

$$\omega_0^2 = \frac{\rho - \rho'}{\rho \coth kh' + \rho'} gk \tag{22}$$

which should be used in Eq. (20). The unstable case has

$$\sigma^2 = \frac{\rho' - \rho}{\rho \coth kh' + \rho'} gk \tag{23}$$

to be used in Eq. (21).

COMMENTS ON THE SOLUTIONS FOR DAMPED INTERNAL WAVES

The complete analysis of small amplitude internal waves with viscous damping for two fluids of infinite thickness has been made by Harrison[2] for the stable case and for the unstable case by Bellman and Pennington.[3] There are some serious errors in Harrison's results and there also appear to be errors in the results presented in the paper by Bellman and Pennington. Correct results have been given by Chandrasekhar.[4] The difficulties in the problem are only algebraic. The dispersion formula for n in the stable case $\rho' < \rho$, is

$$\begin{aligned} &n^2[-(\Delta\rho)^2 + (\rho + \rho')(\rho m' + \rho' m)/k] \\ &\quad + 2n\Delta\mu[\Delta\rho(km' - 2k^2 + km) + (\rho + \rho')(m' - m)k] \\ &\quad + (\omega_0^2/k)[(\rho + \rho')(m'\rho + m\rho') - k(\rho + \rho')^2] \\ &\quad + 4k^2(\Delta\mu)^2(k - m')(m - k) = 0, \end{aligned} \tag{24}$$

where

$$\Delta\mu = \mu - \mu'; \quad \Delta\rho = \rho - \rho';$$
$$m = (k^2 + n/\nu)^{1/2}; \quad m' = (k^2 + n/\nu')^{1/2};$$

and ω_0^2 is given by Eq. (8). For the unstable case $\rho' > \rho$, we use Eq. (24), replacing ω_0^2 by $-\sigma^2$ where σ^2 is positive and is given by Eq. (10).

The general result given in Eq. (24) may readily be specialized to the particular cases mentioned in the previous section. First, if $\rho' \to 0$, $\mu' \to 0$, $m'\mu' \to 0$, then one finds for this one medium case,

$$(n + 2\nu k^2)^2 + \omega_0^2 - 4\nu^2 k^3 m = 0 \tag{25}$$

for the stable case. The unstable case has $-\sigma^2$ in place of ω_0^2. Equation (25) is given in Ref. 1, p. 627. The long wavelength limit where viscous effects are small is found by neglecting the term $4\nu^2 k^3(k - m)$:

$$\begin{aligned} &n^2 + 4\nu k^2 n + \omega_0^2 = 0, \qquad \nu k^2 \ll \omega_0, \text{ stable case;} \\ &n^2 + 4\nu k^2 n - \sigma^2 = 0, \qquad \nu k^2 \ll \sigma, \text{ unstable case .} \end{aligned} \tag{26}$$

To obtain the short wavelength limit, m is approximated:

$$m = (k^2 + n/\nu)^{1/2} \simeq k(1 + n/2\nu k^2).$$

When this approximation is used in (25), one gets, at once,

$$\begin{aligned} &n^2 + 2\nu k^2 n + \omega_0^2 = 0, \qquad \nu k^2 \gg \omega_0, \text{ stable case;} \\ &n^2 + 2\nu k^2 n - \sigma^2 = 0, \qquad \nu k^2 \gg \sigma, \text{ unstable case .} \end{aligned} \tag{27}$$

A second case which has been considered is the one for which $\Delta\,\rho$ and $\Delta\mu$ are small. If all terms in Eq. (23) in $(\Delta\rho)^2$, $(\Delta\mu)^2$, or $\Delta\rho\Delta\mu$ are dropped, one has

$$n^2(\rho m' + \rho' m) + \omega_0^2[(\rho m' + \rho' m) - k(\rho + \rho')] = 0, \tag{28}$$

for the stable case. It is clear that Eq. (28) is valid in the first order in $\Delta\rho$ or $\Delta\mu$. The unstable case gives the corresponding result. We shall now suppose that m'

$= m$; that is, we suppose that the kinematic viscosities of the two media are the same. Equation (28) then gives

$$m(n^2 + \omega_0^2) - k\omega_0^2 = 0. \qquad (29)$$

As before, for the short wavelength region we write $m \simeq k(1 + n/2\nu k^2)$, and from Eq. (29) we find

$$n^2 + 2\nu k^2 n + \omega_0^2 = 0, \qquad \nu k^2 \gg \omega_0,$$

for the stable case. The unstable case is, of course, given by

$$n^2 + 2\nu k^2 n - \sigma^2 = 0, \qquad \nu k^2 \gg \sigma. \qquad (30)$$

Figure 3 shows the behavior of the instability with 1 g in a case for which $\rho' = 1\ \mathrm{g/cm^3} + 9.42 \times 10^{-4}\mathrm{g/cm^3}$, $\rho = 1\ \mathrm{g/cm^3}$ and the kinematic viscosity of both fluids is 10^{-2} Stokes. The exact solution and the approximate solution are found to be in fairly reasonable agreement for the values of n, and the value of λ for which n is a maximum is nearly the same for both solutions. We may also present the long wavelength limit of Eq. (24) for the same situation in which we drop second-order terms in $\Delta\rho$, $\Delta\mu$. We shall require that $\nu' = \nu$; i.e., the kinematic viscosities of the two mediums are exactly the same. We again begin with the form of Eq. (29) which is adapted to the unstable case

$$(k^2 + n/\nu)^{1/2}(n^2 - \sigma^2) + k\sigma^2 = 0. \qquad (31)$$

We now suppose that $\nu k^2/\sigma$ is a small quantity. A straightforward algebraic manipulation of Eq. (31) then gives

$$n = \sigma\left[1 - \tfrac{1}{2}(\nu k^2/\sigma)^{1/2} - \tfrac{1}{4}\nu k^2/\sigma\right]. \qquad (32)$$

We may now find that value of k, or λ, for which n is a maximum, and we readily obtain

$$\lambda_m = 4.4\pi(\nu^2/g')^{1/3} \qquad (33)$$

as the wavelength for maximum n for the long-wavelength limit expression. The long-wavelength limit form gives a larger error than the short wavelength limit form when compared with the exact result. Further, the value of n itself is also poorer than the short wavelength limit value.

SOME COMPARISONS WITH OBSERVATIONS

It has been known for over a hundred years that microorganisms which are negatively geotaxic, i.e., which swim upward against gravity, develop bioconvective patterns when they have collected into a sufficiently dense layer at the top of a culture. Characteristic of these bioconvective patterns are fingers falling from the top layer into the lower liquid. Further, these fingers are separated in a rather regular pattern (see Figs. 4 and 5). Thermal instabilities have been excluded as a mechanism for these patterns. It seems quite evident that these patterns are the consequence of Rayleigh–Taylor instability. One makes the approximation that the upper layer which contains a dense swarm of the microorganisms is a homogeneous liquid which differs from the liquid below only by being slightly heavier. The greater density of the upper layer follows from the fact that the organisms are denser than water, or the culture medium. The assumption that the layer containing the microorganisms is homogeneous is justified since, in the experimental situation, the organisms are uniformly distributed in the upper layer and they are very numerous. Their separations are small compared with the layer thickness and with other distances in the problem such as the wavelengths λ. Extensive observations have been made[5] with the microorganisms *Tetrahymena pyriformis*. A typical measurement[5] shows a density increment in the upper layer over the lower liquid of $\Delta\rho = 1.21 \times 10^{-4}\mathrm{g/cm^3}$. The upper layer in this experiment has a thickness of 0.15 cm, and the observed distance between the falling fingers is approximately 1.0 cm.

FIG. 4. A strobe-illuminated photograph of bioconvective sedimentation fingers descending from precipitation nodes; view is from the side. This *Tetrahymena* culture is enclosed in a 0.15 cm thick perfusion chamber of diameter 2.1 cm. The multiple flash mode utilized here gives one an impression of the bioconvective motion.

FIG. 5. Precipitation nodes and cross connections in a *Tetrahymena* culture as viewed from above. All light areas are masses of cells reflecting light. White lines are connections, and junctions are precipitation nodes. The circle at top center is about 0.7 cm in diameter.

In order to apply the Rayleigh–Taylor instability theory it is necessary to take into account the small thickness of the upper layer. A precise calculation for this case has not previously been available. While the exact problem is straightforward, there are appreciable algebraic complications. The unbounded two medium problem leads to a four by four determinantal equation for the dispersion relation. When one of the mediums has small thickness, the determinant which gives the dispersion relation is six by six and several of the terms in the determinant are quite lengthy. The algebraic details are complicated and not instructive, but exact solutions have been carried out and the details will be presented elsewhere. The approximate formulation discussed in the previous section leads, of course, to much simpler calculations. The growth factor n is taken as the solution of the simple quadratic equation

$$n^2 + 2\nu k^2 n - \sigma^2 = 0,$$

where for the free surface condition for the upper layer we use the value of σ^2 given by Eq. (23). The effect of thickness of the upper layer is shown in Fig. 6. As might be expected the approximate theory becomes quite inaccurate as $h' \to 0$ since the free surface boundary condition and the viscous interfacial boundary conditions are inadequately considered. Figure 7 shows n as a function of λ for $h' = 0.15$ cm which is an observed value[6]; the approximate theory gives the maximum value of n at $\lambda_m = 1.05$ cm and the exact theory gives the maximum at $\lambda_m = 0.80$ cm. The measured value is 1.0 cm which is in good agreement with the theoretical prediction.

An experiment of a somewhat different kind has been performed in which a very viscous liquid (Dow–Corning DC-200) was loaded with solid glass spherical particles with radii of approximately 0.01 cm. When such a mixture is placed in a chamber with flat top and bottom, the glass particles will settle on the bottom surface and a fairly uniform layer can be obtained. With a liquid of such high viscosity, the container can be inverted without the production of unwanted circulatory flows. The effective density and thickness of the upper layer can be determined before the container is inverted. A typical value for the density of the loaded liquid is $\rho' = 1.4$ g/cm^3 and the density of the unloaded DC-200 is $\rho = 0.943$ g/cm^3. The observed instability pattern is shown in Fig. 8 in which the thickness of the unstable layer was $h' = 0.20$ cm. Again the unstable layer is treated as a homogeneous fluid and the instability pattern can be determined from the Rayleigh–Taylor instability theory. Figure 9 shows the wavelength predicted by the theory where now the upper boundary of the unstable layer is taken to be a rigid boundary. The approximate theory is also shown in this figure. The agreement is not good for small values of h' for two reasons. First, the density difference, $\Delta\rho$, is not small for this example, and second, the no-slip condition at the upper boundary of the unstable layer is not properly accounted for in the approximate picture. Figure 10 shows the instability growth factor, n, as a function of λ computed from the exact theory for an unstable layer thickness $h' = 0.2$ cm. The maximum is n occurs at $\lambda = 0.7$ cm which is in very good agreement with the observations.

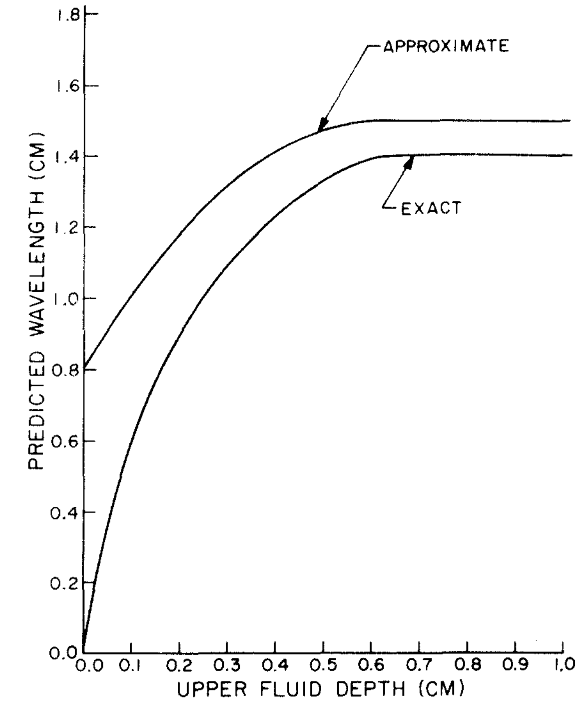

FIG. 6. The wavelength λ_m at which the instability growth factor n is a maximum is shown as a function of the thickness h' of the upper layer. The upper layer density is $\rho' = \rho + 1.21 \times 10^{-4}$ g/cm^3 where the lower layer density is $\rho = 1$ g/cm^3. The fluids are in a 1 g gravity field. The approximate result shown is for the short wavelength approximation.

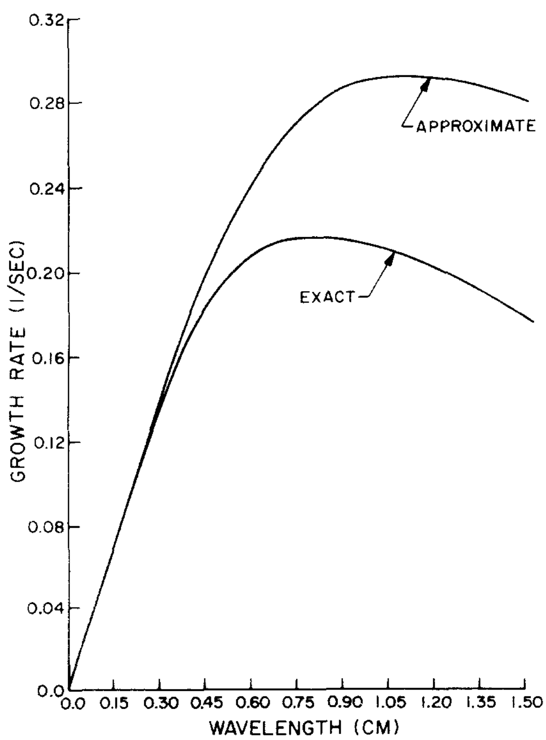

FIG. 7. The instability growth factor n is shown as a function of λ for $\Delta\rho = 1.21 \times 10^{-4}$ g/cm^3, $h' = 0.15$ cm. The approximate result is for the short wavelength approximation.

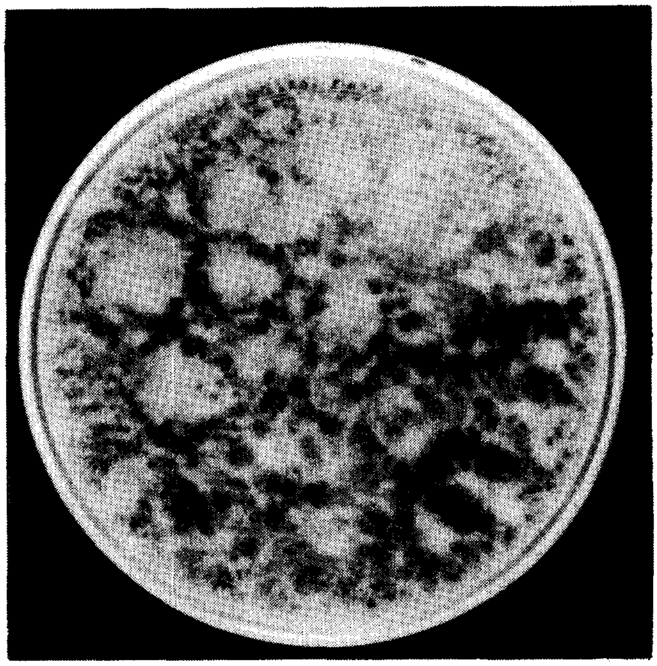

FIG. 8. The photograph shows the instability pattern in a very viscous fluid (Dow–Corning DC-200) which has a kinematic viscosity of 10 stokes. The unstable layer has a thickness $h' = 0.20$ cm and has been loaded with small glass spheres to a net density $\rho' = 1.4$ g/cm^3. The lower layer has a density $\rho = 0.943$ g/cm^3.

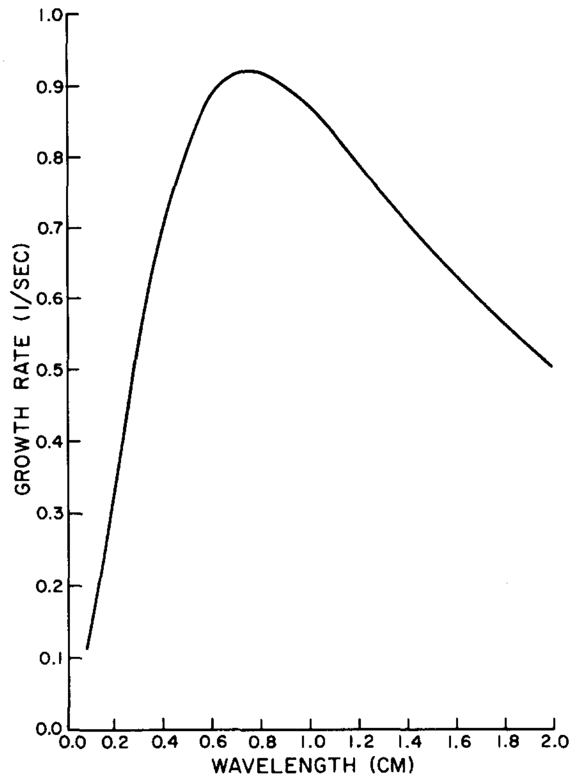

FIG. 10. The dependence of the instability growth factor n is shown as a function of wavelength for the conditions as given for Figs. 9 and 10. The curve is calculated from the exact theory for a fixed upper boundary.

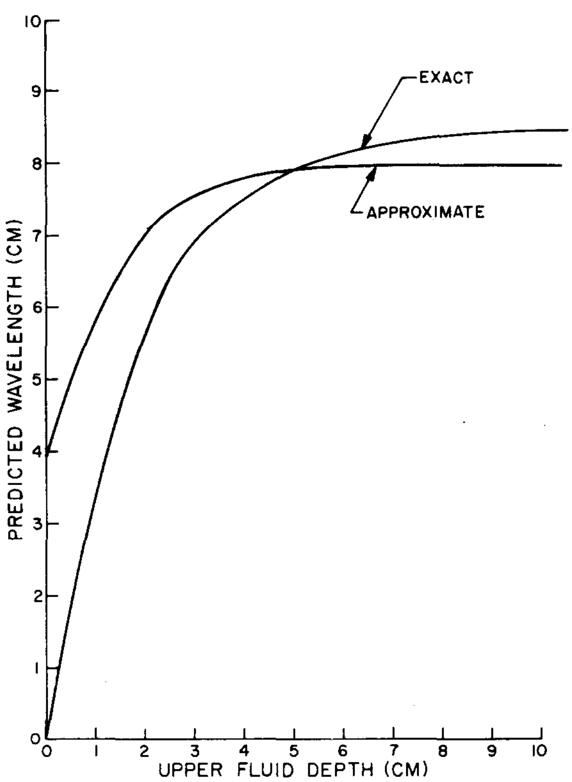

FIG. 9. The wavelength λ_m for which the instability growth factor, n, is a maximum is shown as a function of the thickness of the unstable layer, h'. The upper fluid has density $\rho' = 1.4$ g/cm^3, the lower fluid has density $\rho = 0.943$ g/cm^3. The viscosity of both fluids is 9.43 poise.

A final remark may be made to emphasize the importance of the "cooperative" aspects of the fluid motions as contrasted with individual particle motions for the two cases described here. For the case of the *Tetrahymena pyriformis*, a microorganism which has a radius of approximately 2×10^{-3} cm, the terminal Stokes velocity of fall in water of a spherical particle with this radius is 6.6×10^{-3} cm/sec; the swim speed of organism is 4.5 $\times 10^{-2}$ cm/sec; and the fall velocity of the Rayleigh–Taylor instability jets is approximately 10^{-1} cm/sec. In the experiments with DC-200 liquid with a layer loaded with spherical glass beads, the particles have a radius of 10^{-2} cm so that the terminal Stokes velocity of fall is 3.3 times 10^{-2} cm/sec. The observed fall velocity of the instability jets exceeds this Stokes particle velocity by a factor greater than 20.

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