SIAM J. NUMER. ANAL.
Vol. 6, No. 4, December 1969

APPROXIMATION THEORY OF MULTIVARIATE SPLINE FUNCTIONS IN SOBOLEV SPACES*

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1. Introduction. In this paper we study some approximation theory questions which arise from the analysis of the discretization error associated with the use of the Rayleigh-Ritz-Galerkin method for approximating the solutions to various types of boundary value problems, cf. [1], [2], [3], [4], [7], [8], [9], [12], [14], [18], [19], [20] and [22]. In particular, we consider upper and lower bounds for the error in approximation of certain families of functions in Sobolev spaces, cf. [15], by functions in finite-dimensional "polynomial spline types" subspaces, cf. [16].

In doing this, we directly generalize, improve, and extend the corresponding results of [1], [17], [18], [19], [20], and [21]. Throughout this paper, the symbol K will be used repeatedly to denote a positive constant, not necessarily the same at each occurrence and the symbol μ will be used repeatedly to denote a nonnegative, continuous function on $[0, \infty]$, not necessarily the same at each occurrence.

2. One-dimensional spline spaces. In this section we discuss *computable lower* and *upper* bounds for the error in approximating a class of functions of one real variable belonging to a Sobolev space by functions belonging to a one-dimensional spline subspace. These results extend, generalize, and improve the corresponding results of [1] and [17].

Let $-\infty < a < b < \infty$ be fixed and for each nonnegative integer, $H^q(a, b)$ denote the completion of the real-valued functions in $C^\infty(a, b)$ with respect to the Sobolev norm,

$$\|u\|_{H^q(a,b)} \equiv \left(\sum_{j=0}^{q} \int_a^b (D^j u(x))^2 \, dx\right)^{1/2},$$

where $Du \equiv du/dx$ denotes the derivative of u, and $H_n^q(a, b), n \leqq q$, denotes the completion of the real-valued functions in $C_n^\infty(a, b) \equiv \{f \in C^\infty(a, b) | D^\alpha f(a) = D^\alpha f(b) = 0, 0 \leqq \alpha < n - 1\}$ with respect to $\| \cdot \|_{H^q(a,b)}$.

Given two nonnegative integers $p \leqq r$, S a finite-dimensional subspace of $H^p(a, b)$, and S_p a finite-dimensional subspace of $H_p^p(a, b)$, we are interested in upper and lower bounds for the quantities

$$\bar{E}(r, p, S) \equiv \sup \{\inf_{s \in S} (\|D^p(f - s)\|_{H^0(a,b)} / \|f\|_{H^r(a,b)}) | f \in H^r(a, b), f \not\equiv 0\} \tag{2.1}$$

and

$$\bar{E}(r, p, S_p) \equiv \sup \{\inf_{s \in S_p} (\|D^p(f - s)\|_{H^0(a,b)} / \|f\|_{H^r(a,b)}) | f \in H_p^r(a, b), f \not\equiv 0\}. \tag{2.2}$$

* Received by the editors May 22, 1969, and in revised form July 17, 1969.

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This work was supported in part by the National Science Foundation under Grant GP-11236.

For each nonnegative integer, M, let $\mathscr{P}_M(a, b)$ denote the set of partitions, Δ, of $[a, b]$ of the form

$$\Delta : a = x^0 < x^1 < \cdots < x^M < x^{M+1} = b \tag{2.3}$$

and let $\mathscr{P}(a, b) \equiv \bigcup_{M=0}^{\infty} \mathscr{P}_M(a, b)$.

If $\Delta \in \mathscr{P}_M(a, b)$, d is a positive integer, and z is an integer such that $-1 \leqq z \leqq d - 1$, we define the *spline space*, $S(d, \Delta, z)$, to be the set of all real-valued functions $s(x) \in C^z[a, b]$, such that on each subinterval (x^i, x^{i+1}), $0 \leqq i \leqq M$, $s(x)$ is a polynomial of degree d, where by $C^{-1}[a, b]$ we mean those functions which have a simple jump discontinuity at each partition point, x^i, $1 \leqq i \leqq M$. Clearly $S(d, \Delta, z) \subset H^{z+1}(a, b)$ and if $0 \leqq p \leqq z + 1$, $S_p(d, \Delta, z) \equiv S(d, \Delta, z) \cap H_p^p(a, b)$. It is easy to verify that all the results of this paper remain essentially unchanged if one allows the number z to depend on the partition points, x^i, $1 \leqq i \leqq M$, in such a way that $-1 \leqq z(x^i) \leqq d - 1$ for all $1 \leqq i \leqq M$ or if one considers appropriate spaces of "g-splines", cf. [17]. The details are left to the reader.

Finally, following Kolmogorov, cf. [13, p. 146], if k and j are positive integers, let $\lambda_k(j)$ denote the kth eigenvalue of the boundary value problem,

$$(-1)^j D^{2j} y(x) = \lambda y(x), \qquad a < x < b, \tag{2.4}$$

$$D^i y(a) = D^i y(b) = 0, \qquad j \leqq i \leqq 2j - 1, \tag{2.5}$$

where the λ_k are arranged in order of increasing magnitude and repeated according to their multiplicity. We remark that the problem (2.4)–(2.5) has a countably infinite number of eigenvalues, all of which are nonnegative and it may be shown that $\lambda_k(j) = (\pi/(b - a))^{2j} k^{2j}[1 + O(k^{-1})]$, as $j < k \to \infty$.

Letting $\bar{\Delta} \equiv \max_{0 \leqq i \leqq M}(x^{i+1} - x^i)$ and $\underline{\Delta} \equiv \min_{0 \leqq i \leqq M}(x^{i-1} - x^i)$, we have the following new results.

THEOREM 2.1. *Let $d \equiv 2m - 1$, where m is a positive integer.*

Part (i)

$$\lambda_t^{-1/2}(m - p) \leqq \bar{E}(m, p, S(2m - 1, \Delta, z)) \leqq K_{m,m,s,p}(\bar{\Delta})^{m-p}, \tag{2.6}$$

and

$$\lambda_t^{-1/2}(m - p) \leqq \bar{E}(m, p, S_p(2m - 1, \Delta, z)) \leqq K_{m,m,s,p}(\bar{\Delta})^{m-p}, \tag{2.7}$$

where $t \equiv (M + 1)(2m - z + 1) + z - p + 2$, $s \equiv \max(z, m - 1)$, *and*

$$K_{m,m,s,p} \equiv \begin{cases} 1, & \text{if } p = m, \\ \left(\dfrac{1}{\pi}\right)^{m-p}, & \text{if } m - 1 = s, \quad 0 \leqq p \leqq m - 1, \\ \dfrac{(s + 2 - m)!}{\pi^{m-p}}, & \text{if } 0 \leqq p \leqq 2m - 2 - s, \\ \dfrac{(s + 2 - m)!}{p!\,\pi^{m-p}}, & \text{if } 2m - 2 - s \leqq p \leqq m - 1, \end{cases} \tag{2.8}$$

for all $1 \leqq m, 0 \leqq M, \Delta \in \mathscr{P}_M(a, b), -1 \leqq z \leqq 2m - 2,$ *and* $0 \leqq p \leqq \min(z + 1, m)$.

Part (ii)

$$\lambda_t^{-1/2}(2m - p) \leqq \bar{E}(2m, p, S(2m - 1, \Delta, z)) \leqq K_{m,2m,s,p}(\bar{\Delta})^{2m-p}, \tag{2.9}$$

$$\lambda_t^{-1/2}(2m - p) \leqq \bar{E}(2m, p, S_p(2m - 1, \Delta, z)) \leqq K_{m,2m,s,p}(\bar{\Delta})^{2m-p}, \tag{2.10}$$

where $t \equiv (M + 1)(2m - z + 1) + z - p + 2, s \equiv \max(z, m - 1),$ *and*

$$K_{m,2m,s,p} \equiv (K_{m,m,s,p})(K_{m,m,s,0}), \tag{2.11}$$

for all $1 \leqq m, 0 \leqq M, \Delta \in \mathscr{P}_M(a,b), -1 \leqq z \leqq 2m - 2,$ *and* $0 \leqq p \leqq \min(z + 1, m)$.

Part (iii)

$$\lambda_t^{-1/2}(r - p) \leqq \bar{E}(r, p, S(2m - 1, \Delta, z)) \leqq K_{m,r,s,p}(\bar{\Delta})^{r-p}, \tag{2.12}$$

and

$$\lambda_t^{-1/2}(r - p) \leqq \bar{E}(r, p, S_p(2m - 1, \Delta, z)) \leqq K_{m,r,s,p}(\bar{\Delta})^{r-p}, \tag{2.13}$$

where $t \equiv (M + 1)(2m - z + 1) + z - p + 2, s \equiv \max(z, 4m - 2r - 1),$ *and*

$$K_{m,r,s,p} \equiv \left\{K_{r,r,2m-1,p} + K_{m,2m,s,p} \cdot 2^{1/2(2m-r)}\left[\frac{r!}{(2r - 2m)!}\right]^2\left(\frac{\bar{\Delta}}{\underline{\Delta}}\right)^{2m-r}\right\}, \tag{2.14}$$

for all $1 \leqq m$, $0 \leqq M$, $\Delta \in \mathscr{P}_M(a,b)$, $-1 \leqq z \leqq 2m - 2$, $m < r < 2m$, *and* $0 \leqq p \leqq \min(z + 1, m)$.

Part (iv)

$$\lambda_t^{-1/2}(r - p) \leqq \bar{E}(r, p, S(2m - 1, \Delta, z)) \leqq K_{m,r,s,p}(\bar{\Delta})^{r-p}, \tag{2.15}$$

where $t \equiv (M + 1)(2m - z + 1) + z - p + 2$ *and* $s \equiv \max(z, 4m - 2r - 1)$ *for all* $1 \leqq m$, $0 \leqq M$, $\Delta \in \mathscr{P}_M(a,b)$, $-1 \leqq z \leqq 2m - 2$, $m < r < 2m$, *and* $m < p < \min(z + 1, r)$.

Part (v)

$$\lambda_t^{-1/2}(r - p) \leqq \bar{E}(r, p, S(2m - 1, \Delta, z)) \leqq K_{2m-r,2m-r,s,2m+p-2r}(\bar{\Delta})^{r-p}, \tag{2.16}$$

and

$$\lambda_t^{-1/2}(r - p) \leqq \bar{E}(r, p, S_p(2m - 1, \Delta, z)) \leqq K_{2m-r,2m-r,s,2m+p-2r}(\bar{\Delta})^{r-p}, \tag{2.17}$$

where $t \equiv (M + 1)(2m - z + 1) + z - p + 2$ *and* $s \equiv \max(2m + z - 2r, 2m - 1 - r)$, *for all* $1 \leqq m, 0 \leqq M, \Delta \in \mathscr{P}_M(a, b), -1 \leqq z \leqq 2m - 2, 0 \leqq r \leqq m - 1,$ *and* $0 \leqq p \leqq \min(z + 1, r)$.

Proof. The proof of this theorem depends directly on the interpolation results of [22]. Part (i) follows from Theorem 3.4, (ii) follows from Theorem 3.5, (iii) follows directly from Theorem 3.6, and (iv) and (v) follow from Theorem 4.1.

Since the proofs of the different parts are similar, we explicitly prove only Part (i). To prove the right-hand inequality of (2.6), we need only recognize that

$$\begin{aligned}\bar{E}(m, p, S(2m - 1, \Delta, z)) &\leqq \bar{E}(m, p, S(2m - 1, \Delta, s)) \\ &\leqq \sup\{\|f - \mathscr{I}f\|_{H^p(a,b)}/\|f\|_{H^m(a,b)} | f \in H^p(a, b) \text{ and } f \neq 0\},\end{aligned}$$

where $\mathscr{I}f$ denotes the $S(2m - 1, \Delta, s)$-interpolate of f defined in [22] and then apply the result of Theorem 3.4 of [22].

Finally we prove the left-hand inequality of (2.6). This inequality follows directly from a fundamental result of Kolmogorov, cf. [13, p. 146], which states that

$$\lambda_{t+1}^{-1/2}(m-p) \leqq \bar{E}(m,p,S(2m-1,\Delta,z)), \tag{2.18}$$

where $t \equiv \dim D^p(S(2m-1,\Delta,z))$, for all $1 \leqq m$, $0 \leqq M$, $\Delta \in \mathscr{P}_M(a,b)$, $-1 \leqq z \leqq 2m-2$, and $0 \leqq p \leqq \min(z+1,m)$. But the space $D^p(S(2m-1,\Delta,z))$ has dimension $t = (2m-p)(M+1)-(z+1-p)M = (M+1)(2m-z+1)+z-p+1$, which concludes the proof.

From Theorem 2.1, we have the following analogous result for even order spline subspaces.

THEOREM 2.2. *Let $d \equiv 2m$, where m is a positive integer.*

Part (i)

$$\lambda_t^{-1/2}(m-p) \leqq \bar{E}(m,p,S(2m,\Delta,z)) \leqq K_{m+1,m+1,s,p+1}(\bar{\Delta})^{m-p}, \tag{2.19}$$

and

$$\lambda_t^{-1/2}(m-p) \leqq \bar{E}(m,p,S_p(2m,\Delta,z)) \leqq K_{m+1,m+1,s,p+1}(\bar{\Delta})^{m-p}, \tag{2.20}$$

where $t \equiv (M+1)(2m-z+2)+z-p+2$, $s \equiv \max(z+1,m)$, for all $1 \leqq m$, $0 \leqq M, \Delta \in \mathscr{P}_M(a,b)$, $-1 \leqq z \leqq 2m-1$, and $0 \leqq p \leqq \min(z+1,m)$.

Part (ii)

$$\begin{aligned}\lambda_t^{-1/2}(2m+1-p) &\leqq \bar{E}(2m+1,p,S(2m,\Delta,z)) \\ &\leqq K_{m+1,2m+2,s,p+1}(\bar{\Delta})^{2m+1-p}\end{aligned} \tag{2.21}$$

and

$$\begin{aligned}\lambda_t^{-1/2}(2m+1-p) &\leqq \bar{E}(2m+1,p,S_p(2m,\Delta,z)) \\ &\leqq K_{m+1,2m+2,s,p+1}(\bar{\Delta})^{2m+1-p},\end{aligned} \tag{2.22}$$

where $t \equiv (M+1)(2m-z+2)+z-p+2$, $s \equiv \max(z+1,m)$, for all $1 \leqq m$, $0 \leqq M, \Delta \in \mathscr{P}_M(a,b)$, $-1 \leqq z \leqq 2m-1$, and $0 \leqq p \leqq \min(z+1,m)$.

Part (iii)

$$\lambda_t^{-1/2}(r-p) \leqq \bar{E}(r,p,S(2m,\Delta,z)) \leqq K_{m+1,r+1,s,p+1}(\bar{\Delta})^{r-p} \tag{2.23}$$

and

$$\lambda_t^{-1/2}(r-p) \leqq \bar{E}(r,p,S_p(2m,\Delta,z)) \leqq K_{m+1,r+1,s,p+1}(\bar{\Delta})^{r-p}, \tag{2.24}$$

where $t \equiv (M+1)(2m-z+2)+z-p+2$ and $s \equiv \max(z+1,4m-2r+3)$, for all $1 \leqq m$, $0 \leqq M$, $\Delta \in \mathscr{P}_M(a,b)$, $-1 \leqq z \leqq 2m-1$, $m < r \leqq 2m+1$, and $0 \leqq p \leqq \min(z+1,m)$.

Part (iv)

$$\lambda_t^{-1/2}(r-p) \leqq \bar{E}(r,p,S(2m,\Delta,z)) \leqq K_{m+1,r+1,s,p+1}(\bar{\Delta})^{r-p}, \tag{2.25}$$

where $t \equiv (M+1)(2m-z+2)+z-p+2$ and $s \equiv \max(z+1,4m-2r+3)$, for all $1 \leqq m$, $0 \leqq M$, $\Delta \in \mathscr{P}_M(a,b)$, $-1 \leqq z \leqq 2m-1$, $m < r \leqq 2m+1$, and $m < p \leqq \min(z+1,r)$.

Part (v)

$$\begin{aligned}\lambda_t^{-1/2}(r-p) &\leqq \bar{E}(r,p,S(2m,\Delta,z)) \\ &\leqq K_{2m+2-r,2m+2-r,s,2m+p-2r+1}(\bar{\Delta})^{r-p}\end{aligned} \tag{2.26}$$

and

$$\lambda_t^{-1/2}(r-p) \leqq \bar{E}(r,p,S_p(2m,\Delta,z)) \leqq K_{2m+2-r,2m+2-r,s,2m+p-2r+1}(\bar{\Delta})^{r-p}, \tag{2.27}$$

where $t \equiv (M+1)(2m-z+2)+z-p+2$ *and* $s \equiv \max(2m+z-2r+1, 2m-r)$, *for all* $1 \leqq m$, $0 \leqq M$, $\Delta \in \mathscr{P}_M(a,b)$, $-1 \leqq z \leqq 2m-1$, $0 \leqq r \leqq m-1$, *and* $0 \leqq p \leqq \min(z+1, r)$.

Proof. We prove only Part (i), as the proofs of the other parts are similar. To prove the right-hand inequality of (2.19), we need only recognize that $\bar{E}(m,p,S(2m,\Delta,z)) = \bar{E}(m+1, p+1, S(2m+1, \Delta, z+1))$ and apply Part (i) of Theorem 2.1.

The proof of the left-hand inequality of (2.19) is the same as the proof of the left-hand inequality of (2.6).

3. Multivariate splines defined in rectangular parallelepipeds. In this section we discuss *lower* and *upper* bounds for the error in approximating a class of functions, defined on a rectangular parallelepiped and belonging to a Sobolev space, by functions belonging to a multivariate spline subspace. These results extend, generalize, and improve the corresponding results of [1], [18], [19], and [20].

Let N be a fixed positive integer. For each integer $1 \leqq i \leqq N$, let $-\infty < a_i < b_i < \infty$, $R \equiv \times_{i=1}^{N} [a_i, b_i]$, and $\mathscr{P}(R) \equiv \{\rho \equiv \times_{i=1}^{N} \Delta_i | \Delta_i \in \mathscr{P}(a_i, b_i), 1 \leqq i \leqq N\}$. For each nonnegative integer, q, let $H^q(R)$ denote the completion of the real-valued functions in $C^\infty(R)$ with respect to the Sobolev norm,

$$\|u\|_{H^q(R)} \equiv \left(\sum_{|\alpha| \leqq q} \int_R |D^\alpha u(x)|^2 \, dx \right)^{1/2},$$

where $\alpha \equiv (\alpha_1, \cdots, \alpha_N)$, α_i nonnegative integers, $1 \leqq i \leqq N$, $|\alpha| \equiv \sum_{i=1}^{N} \alpha_i$, $D^\alpha \equiv (\partial^{\alpha_1}/\partial x_1^{\alpha_1}) \cdots (\partial^{\alpha_N}/\partial x_N^{\alpha_N})$, and $H_n^q(R)$, $n \leqq q$, denote the completion of the real-valued functions in $C_n^\infty(R) \equiv \{f \in C^\infty(R) | D^\alpha f(x) = 0 \text{ for all } x \in \partial R, |\alpha| \leqq n-1\}$ with respect to $\|\cdot\|_{H^q(R)}$.

Given two nonnegative integers $p \leqq r$, S a finite-dimensional subspace of $H^p(R)$, and S_p a finite-dimensional subspace of $H_p^p(R)$, we are interested in upper and lower bounds for the quantities.

$$E(r,p,S) \equiv \sup\{\inf_{s \in S}(\|f-s\|_{H^p(R)}/\|f\|_{H^r(R)}) | f \in H^r(R), f \not\equiv 0\} \tag{3.1}$$

and

$$E(r,p,S_p) \equiv \sup\{\inf_{s \in S_p}(\|f-s\|_{H^p(R)}/\|f\|_{H^r(R)}) | f \in H_p^r(R), f \not\equiv 0\}. \tag{3.2}$$

Finally, following Jerome, cf. [10] and [11], if k and j are positive integers, let $\lambda_k(j)$ denote the kth eigenvalue of the boundary value problem,

$$\int_R \sum_{|\alpha| \leqq j} D^\alpha y(x) D^\alpha \varphi(x) \, dx = \lambda \int_R y(x)\varphi(x) \, dx \tag{3.3}$$

for all $\varphi \in H^j(R)$, where the λ_k are arranged in order of increasing magnitude and repeated according to their multiplicity. We remark that it may be shown that $\lambda_k(j) \sim k^{2j/N}$, as $j < k \to \infty$, cf. [11].

Letting $\otimes_{i=1}^{N} S(2m-1, \Delta_i, z_i)$ denote the tensor product of the spaces $S(2m-1, \Delta_i, z_i)$, we have the following results.

THEOREM 3.1. *Let $d \equiv 2m-1$, where m is a positive integer.*

Part (i)

$$\lambda_t^{-1/2}(m) \leqq E\left(m, 0, \sum_{i=1}^{N} S(2m-1, \Delta_i, z_i)\right) \leqq \sum_{i=1}^{N} K_{m,m,s_i,0}(\bar{\Delta}_i)^m, \tag{3.4}$$

where $t \equiv 1 + \prod_{i=1}^{N} \{(M_i+1)(2m-z_i+1) + z_i + 1\}$ and $s_i \equiv \max(z_i, m-1)$, for all $1 \leqq m$, $0 \leqq M_i$, $\Delta_i \in \mathscr{P}_{M_i}(a_i, b_i)$, and $-1 \leqq z_i \leqq 2m-2$.

Part (ii)

$$\lambda_t^{-1/2}(2m) \leqq E\left(2m, 0, \bigotimes_{i=1}^{N} S(2m-1, \Delta_i, z_i)\right) \leqq \sum_{i=1}^{N} K_{m,2m,s_i,0}(\bar{\Delta})^{2m}, \tag{3.5}$$

where $t \equiv 1 + \prod_{i=1}^{N} \{(M_i+1)(2m-z_i+1) + z_i + 1\}$ and $s_i \equiv \max(z_i, m-1)$, for all $1 \leqq m$, $0 \leqq M_i$, $\Delta_i \in \mathscr{P}_{M_i}(a_i, b_i)$, and $-1 \leqq z_i \leqq 2m-2$.

Part (iii)

$$\lambda_t^{-1/2}(r) \leqq E\left(r, 0, \bigotimes_{i=1}^{N} S(2m-1, \Delta_i, z_i)\right) \leqq \sum_{i=1}^{N} K_{m,r,s_i,0}(\bar{\Delta})^r, \tag{3.6}$$

where $t \equiv 1 + \prod_{i=1}^{N} \{(M_i+1)(2m-z_i+1) + z_i + 1\}$ and $s_i \equiv \max(z_i, 4m-2r-1)$, for all $1 \leqq m$, $0 \leqq M_i$, $\Delta_i \in \mathscr{P}_{M_i}(a_i, b_i)$, $-1 \leqq z_i \leqq 2m-2$, and $m < r < 2m$.

Part (iv)

$$\lambda_t^{-1/2}(r) \leqq E\left(r, 0, \bigotimes_{i=1}^{N} S(2m-1, \Delta_i, z_i)\right) \leqq \sum_{i=1}^{N} K_{2m-r,s_i,2m-2r}(\bar{\Delta}_i)^r, \tag{3.7}$$

where $t \equiv 1 + \prod_{i=1}^{N} \{(M_i+1)(2m-z_i+1) + z_i + 1\}$ and $s_i \equiv \max(2m+z_i-2r, 2m-1-r)$, for all $1 \leqq m$, $0 \leqq M_i$, $\Delta_i \in \mathscr{P}_{M_i}(a_i, b_i)$, $-1 \leqq z_i \leqq 2m-2$, and $0 \leqq r \leqq m-1$.

Proof. The right-hand inequalities of the above results follow from the corresponding one-dimensional results of Theorem 2.1 and induction on the dimension N. We illustrate the general case by proving Part (i) for $N = 2$.

Letting P_i denote the orthogonal projection of $H^0(a_i, b_i)$ onto $S(2m-1, \Delta_i, z_i)$ for $i = 1$ and 2, we have

$$E\left(m, 0, \bigotimes_{i=1}^{2} S(2m-1, \Delta_i, z_i)\right) \leqq \sup\{\|f - P_1P_2f\|_{H^0(R)}/\|f\|_{H^m(R)} | f \in H^m(R) \text{ and } f \not\equiv 0\}.$$

Moreover, $\|f - P_1P_2f\|_{H^0(R)} \leqq \|f - P_1f\|_{H^0(R)} + \|P_1(f - P_2f)\|_{H^0(R)} \leqq \|f - P_1f\|_{H^0(R)} + \|f - P_2f\|_{H^0(R)}$ and the inequalities follow from Theorem 2.1.

The left-hand inequalities follow directly from [10, Theorem 5.1] and the fact that the dimension of $\otimes_{i=1}^{2} S(2m-1, \Delta_i, z_i) = \prod_{i=1}^{2} \{(M_i+1)(2m-z_i+1) + z_i + 1\}$.

In a similar fashion, we have from Theorem 2.2.

THEOREM 3.2. *Let $d \equiv 2m$, where m is a positive integer.*

Part (i)

$$\lambda_t^{-1/2}(m) \leqq E\left(m, 0, \bigotimes_{i=1}^{N} S(2m, \Delta_i, z_i)\right) \leqq \sum_{i=1}^{N} K_{m+1,m+1,s_i,0}(\overline{\Delta}_i)^m, \tag{3.8}$$

where $t \equiv 1 + \prod_{i=1}^{N} \{(M_i + 1)(2m - z_i + 2) + z_i + 1\}$ and $s_i \equiv \max(z_i + 1, m)$ for all $1 \leqq m, 0 \leqq M_i, \Delta_i \in \mathscr{P}_{M_i}(a_i, b_i)$ and $-1 \leqq z_i \leqq 2m - 1$.

Part (ii)

$$\begin{aligned} \lambda_t^{-1/2}(2m) &\leqq E\left(2m + 1, 0, \bigotimes_{i=1}^{N} S(2m, \Delta_i, z_i)\right) \\ &\leqq \sum_{i=1}^{N} K_{m+1,2m+2,s_i,1}(\overline{\Delta}_i)^{2m+1}, \end{aligned} \tag{3.9}$$

where $t \equiv 1 + \prod_{i=1}^{N} \{(M_i + 1)(2m - z_i + 2) + z_i + 1\}$ and $s_i \equiv \max(z_i + 1, m)$ for all $1 \geqq m, 0 \leqq M_i, \Delta_i \in \mathscr{P}_{M_i}(a_i, b_i)$ and $-1 \leqq z_i \leqq 2m - 1$.

Part (iii)

$$\lambda_t^{-1/2}(r) \leqq E\left(r, 0, \bigotimes_{i=1}^{N} S(2m, \Delta_i, z_i)\right) \leqq \sum_{i=1}^{N} K_{m+1,r+1,s_i,1}(\overline{\Delta}_i)^r, \tag{3.10}$$

where $t \equiv 1 + \prod_{i=1}^{N}\{(M + 1)(2m - z_i + 2) + z_i + 1\}$ and $s_i \equiv \max(z_i + 1, 4m - 2r + 3)$ for all $1 \leqq m, 0 \leqq M_i, \Delta_i \in \mathscr{P}_{M_i}(a_i, b_i)$, $-1 \leqq z_i \leqq 2m - 1$ and $m < r \leqq 2m + 1$.

Part (iv)

$$\begin{aligned} \lambda_t^{-1/2} &\leqq E\left(r, 0, \bigotimes_{i=1}^{N} S(2m, \Delta_i, z_i)\right) \\ &\leqq \sum_{i=1}^{N} K_{2m+2-r,2m+2-r,s_i,2m-2r+1}(\overline{\Delta}_i)^r, \end{aligned} \tag{3.11}$$

where $t \equiv 1 + \prod_{i=1}^{N} \{(M_i + 1)(2m - z_i + 2) + z_i + 1\}$ and $s_i \equiv (2m + z_i - 2r + 1, 2m - r)$ for all $1 \leqq m, 0 \leqq M_i, \Delta_i \in \mathscr{P}_{M_i}(a_i, b_i)$, $-1 \leqq z_i \leqq 2m - 1$ and $0 \leqq r \leqq m - 1$.

We now discuss bounds for the error of approximation in the "higher-order" Sobolev spaces. To simplify the presentation the constants in the bounds are not explicitly computed. Their computation, which is similar to computations in Theorems 3.1 and 3.2, is left to the reader. Letting $\bar{\rho} \equiv \max_{1 \leqq i \leqq N} \overline{\Delta}_i$ and $\bar{\rho} \equiv \min_{1 \leqq i \leqq N} \Delta_i$ for all $\rho \in \mathscr{P}(R)$, we have the following theorem.

THEOREM 3.3. *If $d \equiv 2m - 1$, where m is a positive integer, there exists a nonnegative, continuous function, μ, on $[0, \infty)$ such that*

$$\lambda_t^{-1/2}(r - p) \leqq E\left(r, p, \bigotimes_{i=1}^{N} S(2m - 1, \Delta_i, z_i)\right) \leqq \mu(\rho/\rho)(\bar{\rho})^{r-p}, \tag{3.12}$$

where $t \equiv 1 + \prod_{i=1}^{N} \{(M_i + 1)(2m - z_i + 1) + z_i + 1\}$, for all $1 \leqq m$, $0 \leqq M_i$, $\Delta_i \in \mathscr{P}_{M_i}(a_i, b_i)$, $-1 \leqq z_i \leqq 2m - 2, 0 \leqq r \leqq 2m, 0 \leqq p \leqq \min_{1 \leqq i \leqq N} \min(z_i + 1, r)$ and

$$\lambda_t^{-1/2}(r - p) \leqq E\left(r, p, \bigotimes_{i=1}^{N} S_p(2m - 1), \Delta_i, z_i)\right) \leqq \mu(\rho/\underline{\rho})(\bar{\rho})^{r-p}, \tag{3.13}$$

where $t \equiv 1 + \prod_{i=1}^{N} \{(M_i + 1)(2m - z_i + 1) + z_i + 1\}$, *for all* $1 \leqq m$, $0 \leqq M_i$, $\Delta_i \in \mathscr{P}_{M_i}(a_i, b_i)$, $-1 \leqq z_i \leqq 2m - 2$, $0 \leqq r \leqq 2m$ *and* $0 \leqq p \leqq \min_{1 \leqq i \leqq N} \min(z_i + 1, r, m)$.

Proof. The right-hand inequalities of (3.12) and (3.13) follow from the corresponding one-dimensional results of Theorem 2.1 and induction on the dimension N. We illustrate the general case by proving (3.12) for the special case of $r = m$ and $N = 2$. Given $f \in H^m(R)$, there exists a sequence $\{g_n\}_{n=1}^{\infty} \subset C^M(R)$ such that

$$\|f - g_n\|_{H^m(R)} \to 0 \qquad \text{as } n \to \infty \tag{3.14}$$

and

$$\|g_n\|_{H^m(R)} \to \|f\|_{H^m(R)} \qquad \text{as } n \to \infty. \tag{3.15}$$

Hence, if $f \equiv 0$ and $s \in \otimes_{i=1}^{2} s(2m - 1, \Delta_i, z_i)$,

$$\|f - s\|_{H^m(R)}/\|f\|_{H^m(R)} \leqq \|f - g_n\|_{H^m(R)}/\|f\|_{H^m(R)} + \|g_n - s\|_{H^m(R)}/\|f\|_{H^m(R)}.$$

Using (3.14) and (3.15) and taking the limit as $n \to \infty$, we have that to prove the right-hand inequality of (3.12) it suffices to define E by taking the infinum over all $f \in C^m(R)$.

Letting P_i denote the orthogonal projection of $H^0(a_i, b_i)$ onto $S(2m - 1, \Delta_i, z_i)$ for $i = 1$ and 2 and α_1 and α_2 be such that $\alpha_1 + \alpha_2 \leqq p$, then

$$\begin{aligned} \|D_1^{\alpha_1} D_2^{\alpha_2}(f - P_1 P_2 f)\|_{H^0(R)} &\leqq \|D_2^{\alpha_2}(D_1^{\alpha_1} f - P_2 D_1^{\alpha_1} f)\|_{H^0(R)} \\ &\quad + \|D_2^{\alpha_2} P_2(D_1^{\alpha_1} f - D_1^{\alpha_1} P_1 f)\|_{H^0(R)}, \end{aligned} \tag{3.16}$$

where we have used the fact that D_1 commutes with D_2, P_1 commutes with P_2, and D_1 commutes with P_2. Moreover, using the inequality of E. Schmidt, cf. [19], and Theorem 2.1, we have that if $f \in H^r(a_i, b_i)$, $0 \leqq r \leqq 2m$, there exists an $s \in S(2m - 1, \Delta_i, z_i)$ and positive constants K_1, and K_2 and a nonnegative, continuous function μ on $[0, \infty)$ such that

$$\begin{aligned} \|D_i^{\alpha_i}(f - P_i f)\|_{H^0(a_i,b_i)} &\leqq \|D_i^{\alpha_i}(f - s)\|_{H^0(a_i,b_i)} + \|D_i^{\alpha_i}(s - P_i f)\|_{H^0(a_i,b_i)} \\ &\leqq K_1(\bar{\Delta}_i)^{r-\alpha_i} + \frac{K_2}{(\underline{\Delta}_i)^{\alpha_i}} K(\bar{\Delta}_i)^{\alpha_i} \mu(\bar{\Delta}_i/\underline{\Delta}_i)(\bar{\Delta}_i)^{r-\alpha_i} \end{aligned} \tag{3.17}$$

for $0 \leqq \alpha_i \leqq r$, $\quad i = 1$ and 2.

Combining (3.16), (3.17), and the inequality of E. Schmidt, we have

$$\begin{aligned} \|D_1^{\alpha_1} D_2^{\alpha_2}(f - P_1 P_2 f)\|_{H^0(R)} &\leqq \mu(\bar{\Delta}_2/\underline{\Delta}_2)(\bar{\Delta}_2)^{(r-a_1)-\alpha_2} \|D_2^{r-\alpha_1} D_1^{\alpha_1} f\|_{H^0(R)} \\ &\quad + \frac{K}{(\underline{\Delta}_2)^{\alpha_2}} \lambda(\bar{\Delta}_1/\underline{\Delta}_1)(\bar{\Delta}_1)^{r-\alpha_1} \|D_1^r f\|_{H^0(R)}, \end{aligned} \tag{3.18}$$

which yields the required result.

The left-hand inequalities of (3.2) and (3.13) follow directly from [10, Theorem 5.1] as in Theorem 3.1.

A set, $C \subset \mathscr{P}(R)$, of partitions of R is said to be *quasi-uniform* if and only if there exists a positive constant, K, such that $(\bar{\rho}/\underline{\rho}) \leqq K$ for all $\rho \in C$.

COROLLARY. *Let $d \equiv 2m - 1$, where m is a positive integer and C be a quasi-uniform set of partitions of R. There exists a positive constant, K, such that*

$$E\left(r, p \bigotimes_{i=1}^{N} S(2m-1, \Delta_i, z_i)\right) \leqq K(\bar{\rho})^{r-p}, \tag{3.19}$$

for all $1 \leqq m$, $\rho \in C$, $-1 \leqq z_i \leqq 2m-2$, $0 \leqq r \leqq 2m$, $0 \leqq p \leqq \min_{1 \leqq i \leqq N} \min(z_i + 1, r)$, *and*

$$E\left(r, p, \bigotimes_{i=1}^{N} S_p(2m-1, \Delta_i, z_i)\right) \leqq K(\bar{\rho})^{r-p}, \tag{3.20}$$

for all $1 \leqq m$, $\rho \in C$, $-1 \leqq z_i \leqq 2m-2$, $0 \leqq r \leqq 2m$ *and* $0 \leqq p \leqq \min_{1 \leqq i \leqq N} \min(z_i + 1, r, m)$.

THEOREM 3.4. *Let $d \equiv 2m$, where m is a positive integer. There exists a nonnegative, continuous function, μ, on $[0, \infty)$ such that*

$$\lambda_t^{-1/2}(r-p) \leqq E\left(r, p, \bigotimes_{i=1}^{N} S(2m, \Delta_i, z_i)\right) \leqq \mu(\bar{\rho}/\underline{\rho})(\bar{\rho})^{r-p}, \tag{3.21}$$

where $t \equiv 1 + \prod_{i=1}^{N} \{(M_i + 1)(2m - z_i + 2) + z_i + 1\}$ *for all* $1 \leqq m$, $0 \leqq M_i$, $\Delta_i \in \mathscr{P}_{M_i}(a_i, b_i)$, $-1 \leqq z_i \leqq 2m-1$, $0 \leqq r \leqq 2m+1$, $0 \leqq p \leqq \min_{1 \leqq i \leqq N}(z_i + 1, r)$ *and*

$$\lambda_t^{-1/2}(r-p) \leqq E\left(r, p, \bigotimes_{i=1}^{N} S_p(2m, \Delta_i, z_i)\right) \leqq \mu(\bar{\rho}/\rho)(\bar{\rho})^{r-p}, \tag{3.22}$$

where $t \equiv 1 + \prod_{i=1}^{N} \{(M_i + 1)(2m - z_i + 2) + z_i + 1\}$ *for all* $1 \leqq m$, $0 \leqq M_i$, $\Delta_i \in \mathscr{P}_{M_i}(a_i, b_i)$, $-1 \leqq z_i \leqq 2m-1$, $0 \leqq r \leqq 2m+1$ *and* $0 \leqq p \leqq \min_{1 \leqq i \leqq N} \min(z_i + 1, r, m)$.

COROLLARY. *Let $d \equiv 2m$, where m is a positive integer and C be a quasi-uniform set of partitions of R. There exists a positive constant, K, such that*

$$E\left(r, p, \bigotimes_{i=1}^{N} S(2m, \Delta_i, z_i)\right) \leqq K(\bar{\rho})^{r-p} \tag{3.23}$$

for all $1 \leqq m$, $\rho \in C$, $-1 \leqq z_i \leqq 2m-1$, $0 \leqq r \leqq 2m+1$, $0 \leqq p \leqq \min_{1 \leqq i \leqq N} \min(z_i + 1, r)$, *and*

$$E\left(r, p, \bigotimes_{i=1}^{N} S_p(2m, \Delta_i z_i)\right) \leqq K(\bar{\rho})^{r-p} \tag{3.24}$$

for all $1 \leqq m$, $\rho \in C$, $-1 \leqq z_i \leqq 2m-1$, $0 \leqq r \leqq 2m+1$ *and* $0 \leqq p \leqq \min_{1 \leqq i \leqq N} \min(z_i + 1, r, m)$.

We remark that the results of this section show that spline subspaces are optimal in the sense that no other subspaces of the same dimension give asymptotically smaller approximation errors under the stated hypotheses.

4. General domains. In this section we first discuss *lower* and *upper* bounds for the error in approximating a class of functions, defined on a domain, Ω, of the type first considered by Harrick, cf. [8] and [9], and belonging to a Sobolev space, $H_p^p(\Omega)$, by functions belonging to a "finite-dimensional weighted multivariate spline subspace." Second, we discuss *lower* and *upper* bounds for the error in approximating a class of functions, defined on a Lipschitz domain, Ω, cf. [15], and belonging to a Sobolev space $H^p(\Omega)$, by restrictions of functions belonging to a finite-dimensional spline subspace. The results extend, generalize, and improve the corresponding results of [21].

If Ω is a bounded, measurable subset of R^N and p is a positive integer, let $H^p(\Omega)$ denote the closure of the real-valued functions in $C^\infty(\Omega)$ with respect to the norm

$$\|u\|_{H^p(\Omega)} \equiv \left(\int_\Omega \sum_{0 \leqq |\alpha| \leqq p} |D^\alpha u(x)|^2 \, dx \right)^{1/2}$$

and $H^p_n(\Omega)$, $n \leqq p$, denote the closure of the real-valued functions in

$$C^\infty_n(\Omega) \equiv \{ f \in C^\infty(\Omega) | D^\alpha f(x) = 0 \text{ for all } 0 \leqq |\alpha| \leqq n-1, x \in \partial\Omega \}$$

with respect to $\| \cdot \|_{H^p(\Omega)}$.

Given two nonnegative integers $p \leqq r$, S a finite-dimensional subspace of $H^p(\Omega)$, and S_p a finite-dimensional subspace of $H^p_p(\Omega)$, we are interested in upper and lower bounds for the quantities

$$E(r,p,S) \equiv \sup \{ \inf_{s \in S} (\|f - s\|_{H^p(\Omega)} / \|f\|_{H^r(\Omega)}) | f \in H^r(\Omega), f \not\equiv 0 \} \tag{4.1}$$

and

$$E(r,p,S_p) \equiv \sup \{ \inf_{s \in S_p} (\|f - s\|_{H^p(\Omega)} / \|f\|_{H^r(\Omega)}) | f \in H^r_p(\Omega), f \not\equiv 0 \}. \tag{4.2}$$

Finally, following [10] and [11], if k and j are positive integers, let $\lambda_k(j)$ denote the kth eigenvalue of the boundary value problem,

$$\int_\Omega \sum_{|\alpha| \leqq j} D^\alpha y(x) D^\alpha \varphi(x) \, dx = \lambda \int_\Omega y(x)\varphi(x) \, dx \quad \text{for all } \varphi \in H^j(\Omega), \tag{4.3}$$

where the λ_k are arranged in order of increasing magnitude and repeated according to their multiplicity. We remark that it may be shown that $\lambda_k(j) \sim k^{2j/N}$ as $j < k \to \infty$, cf. [11].

To begin we prove some denseness results. These results extend Theorems 3.2 and 3.3 of [21]. Using Theorem 2.1 of [21], we have the next theorem.

THEOREM 4.1. *Let $d \equiv 2m - 1$, where m is a positive integer, $\Omega \subset R \equiv \bigtimes_{i=1}^N [a_i, b_i]$ be a closed, bounded subset of R^N, and $C \equiv \{\rho_k\}_{k=1}^\infty \subset \mathscr{P}(\mathrm{R})$ be a quasi-uniform sequence of partitions of R such that $\bar{\rho}_k \to 0$ as $k \to \infty$. If there exists $\gamma \in C^p(\Omega)$ with $\gamma(x) > 0$ for all $x \in \operatorname{int} \Omega$ and $D^\alpha\gamma(x) = 0$, $0 \leqq |\alpha| \leqq p - 1$ for all $x \in \partial\Omega$, $-1 \leqq z_i \leqq 2m - 2$ and $0 \leqq p \leqq \min_{1 \leqq i \leqq N} (z_i + 1)$, then $\bigcup_{\rho \in C} \gamma(\otimes_{i=1}^N S(2m-1, \Delta_i, z_i))$ is dense in $H^p_p(\Omega)$.*

THEOREM 4.2. *Let $d \equiv 2m$, where m is a positive integer, $\Omega \subset R \equiv \bigtimes_{i=1}^N [a_i, b_i]$ be a closed, bounded subset of R^N, and $C \equiv \{\rho_k\}_{k=1}^\infty \subset \mathscr{P}(R)$ be a quasi-uniform sequence of partitions of R such that $\bar{\rho}_k \to 0$ as $k \to \infty$. If there exists $\gamma \in C^p(\Omega)$ with $\gamma(x) > 0$ for all $x \in \operatorname{int} \Omega$ and $D^\alpha\gamma(x) = 0$, $0 \leqq |\alpha| \leqq p - 1$ for all $x \in \partial\Omega$, $-1 \leqq z_i$*

$\leqq 2m-1$ *and* $0 \leqq p \leqq \min_{1\leqq i\leqq N}(z_i+1)$, *then* $\bigcup_{\rho\in C}\gamma(\otimes_{i=1}^N S(2m,\Delta_i,z_i))$ *is dense in* $H_p^p(\Omega)$.

Now, using Theorem 4.1 of [21], we have the following two fundamental results.

THEOREM 4.3. *Let* $d \equiv 2m-1$, *where* m *is a positive integer*, $\Omega \subset R \ \bigtimes_{i=1}^N[a_i, b_i]$, *a closed, bounded subset of* $\mathbf{R}^N$, *and* $\theta \in C^1(\mathbf{R}^N)$ *be such that* $\partial\Omega \equiv \{x\in\mathbf{R}^N|\theta(x) = 0\}$, $\theta(x) > 0$ *for all* $x \in \operatorname{int}\Omega$, *and* $\sum_{i=1}^N |D_i\theta(x)| \neq 0$ *for all* $x \in \partial\Omega$. *Let* $C \subset \mathscr{P}(R)$ *be a quasi-uniform collection of partitions of* R. *If* $\theta \in C^r(\Omega)$, *there exists a positive constant*, K, *such that*

$$\lambda_t^{-1/2}(r-p) \leqq E\left(r,p,\theta^p\left(\bigotimes_{i=1}^N S(2m-1,\Delta_i,z_i)\right)\right) \leqq K(\bar{\rho})^{r-2p}, \tag{4.4}$$

where $t \equiv 1 + \dim(D^\alpha\theta^p(\otimes_{i=1}^N S(2m-1,\ \Delta_i,\ z_i)))$, $|\alpha| = p$ *for all* $1 \leqq m$, $\rho\in C$, $-1 \leqq z_i \leqq 2m-2$, $2p \leqq r \leqq 2m+p$ *and* $0 \leqq p \leqq \min_{1\leqq i\leqq N}(z_i+1)$.

Proof. The left-hand inequality of (4.4) follows as in Theorem 3.3. To prove the right-hand inequality, we note that if $u \in H_p^r(\Omega)$, there exists a sequence $\{v_k\}_{k=1}^\infty \subset C_p^r(\Omega)$ such that $\|u-v_k\|_{H^r(\Omega)} \to 0$ as $k\to\infty$ and hence for all $s\in\gamma(\otimes_{i=1}^N S(2m-1,\Delta_i,z_i))$,

$$\begin{aligned}\|u-s\|_{H^p(\Omega)}/\|u\|_{H^r(\Omega)} \leqq{}& \|u-v_k\|_{H^p(\Omega)}/\|u\|_{H^r(\Omega)}\\ &+ \|v_k-s\|_{H^p(\Omega)}/\|u\|_{H^r(\Omega)}.\end{aligned} \tag{4.5}$$

Since the first term on the right-hand side of (4.5) goes to 0 as $k\to\infty$, the required result follows by bounding the second term on the right-hand side of (4.5). This is accomplished by using the result of Theorem 4.1 of [21] as was done in the proof of Theorem 5.1 of [10].

THEOREM 4.4. *Let* $d \equiv 2m$, *where* m *is a positive integer*, $\Omega \subset R \equiv \times_{i=1}^N[a_i, b_i]$, *a closed, bounded subset of* R^N, *and* $\theta\in C^1(R^N)$ *be such that* $\partial\Omega \equiv \{x\in R^N|\theta(x) = 0\}$, $\theta(x) > 0$ *for all* $x\in\operatorname{int}\Omega$, *and* $\sum_{i=1}^N|D_i\theta(x)| \neq 0$ *for all* $x\in\partial\Omega$. *Let* $C \subset \mathscr{P}(R)$ *be a quasi-uniform collection of partitions of* R. *If* $\theta\in C^r(\Omega)$, *there exists a positive constant*, K, *such that*

$$\lambda_t^{-1/2}(r-p) \leqq E\left(r,p,\theta^p\left(\bigotimes_{i=1}^N S(2m,\Delta_i,z_i)\right)\right) \leqq K(\bar{\rho})^{r-2p}, \tag{4.6}$$

where $t \equiv 1+\dim(D^\alpha\theta^p(\otimes_{i=1}^N S(2m,\Delta_i,z_i)))$, $|\alpha| = p$ *for all* $1 \leqq m$, $\rho\in C$, $-1 \leqq z_i \leqq 2m-1$, $2p \leqq r \leqq 2m+p+1$ *and* $0 \leqq p \leqq \min_{1\leqq i\leqq N}(z_i+1)$.

We remark that inequalities (4.4) and (4.6) do *not* show that the subspaces $\theta^p(\otimes_{i=1}^N S(d,\Delta_i,z_i))$ are optimal. However, this may be due to our method of proof.

A bounded, open set $\Omega \subset R^N$ is said to be a *Lipschitz domain* if and only if there exists a finite open covering $\{\mathscr{U}_i\}_{i=1}^n$ of the boundary $\partial\Omega$, finitely many finite cones $\{\gamma_i\}_{i=1}^n$, and a positive number ε such that every point of $\partial\Omega$ is the center of a sphere of radius ε entirely contained in one of the sets $\mathscr{U}_i$ and every point of $\mathscr{U}_i \cap \Omega$ is the vertex of a translate of γ_i, $1 \leqq i \leqq n$, contained entirely in Ω.

Interpreting $\otimes_{i=1}^N S(2m-1,\Delta_i,z_i)$ and $\otimes_{i=1}^N S(2m,\Delta_i,z_i)$ as the restriction to Ω of the appropriate spline functions defined on R, we have the following theorem.

THEOREM 4.5. *Let $\Omega \subset R \equiv \times_{i=1}^{N}[a_i, b_i]$ be a Lipschitz domain, m be a positive integer, and C be a quasi-uniform set of partitions of R. There exists a positive constant, K, such that*

$$\lambda^{-1/2}(r-p) \leqq E\left(r, p, \bigotimes_{i=1}^{N} S(2m-1, \Delta_i z_i)\right) \leqq K(\bar{\rho})^{r-p}, \tag{4.7}$$

where $t \equiv 1 + \prod_{i=1}^{N}\{(M_i+1)(2m-z_i+1)+z_i+1\}$ for all $1 \leqq m, \rho \in C, -1 \leqq z_i \leqq 2m-2, 0 \leqq r \leqq 2m, 0 \leqq p \leqq \min_{1 \leqq i \leqq N} \min(z_i+1, r)$ and

$$\lambda_t^{-1/2}(r-p) \leqq E\left(r, p, \bigotimes_{i=1}^{N} S(2m, \Delta_i, z^i)\right) \leqq K(\bar{\rho})^{r-p}, \tag{4.8}$$

where $t \equiv 1 + \prod_{i=1}^{N}\{(M_i+1)(2m-z_i+2)+z_i+1\}$ for all $1 \leqq m$, $\rho \in C$, $-1 \leqq z_i \leqq 2m-1$, $0 \leqq r \leqq 2m+1$ and $0 \leqq p \leqq \min_{1 \leqq i \leqq N} \min(z_i+1, r)$.

Proof. Inequalities (4.7) and (4.8) follow directly from Theorems 3.3 and 3.4 and the Calderon extension theorem, cf. [15, Theorem 3.4.3].

As in § 3, the results of Theorem 4.5 show that subspaces of spline functions are optimal in the sense that no other subspaces of the same dimension give asymptotically smaller approximation errors under the stated hypotheses.

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