A wake model for free-streamline flow theory

Part 1. Fully and partially developed wake flows and cavity flows past an oblique flat plate

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A wake model for the free-streamline theory is proposed to treat the two-dimensional flow past an obstacle with a wake or cavity formation. In this model the wake flow is approximately described in the large by an equivalent potential flow such that along the wake boundary the pressure first assumes a prescribed constant under-pressure in a region downstream of the separation points (called the near-wake) and then increases continuously from this under-pressure to the given free-stream value in an infinite wake strip of finite width (the far-wake). Application of this wake model provides a rather smooth continuous transition of the hydrodynamic forces from the fully developed wake flow to the fully wetted flow as the wake disappears. When applied to the wake flow past an inclined flat plate, this model yields the exact solution in a closed form for the whole range of the wake under-pressure coefficient.

1. Introduction

For the physical flow of an incompressible fluid past a bluff body, experimental observations indicate that the flow generally separates from certain points on the obstacle, resulting in wake formation, or, in the case of the cavitating flow of a liquid medium, a vapour-gas cavity occupying a near-wake region. Extending across the separated streamline there is the so-called free shear layer which is known experimentally to be thin and usually quite steady within a certain distance downstream of the separation point. For this reason this part of the wake will be called the 'near-wake', or the 'free-streamline range'. The pressure in such a region is in general approximately constant; this will be called the wake under-pressure, or the cavity pressure in the case of cavity flows.

Further downstream, however, the shear layer gradually becomes broader as the vorticity diffuses and at the same time non-uniformity of the pressure distribution across the layer increases. As a result, these shear layers become unstable and do not continue smoothly far downstream, but roll up to form vortices or become directly the region of turbulent mixing. These vortices mix and diffuse rapidly and are eventually dissipated in the wake. In the case of cavitating flows, rather regular vortex wakes behind the cavity are usually observed also. Thus, after the cavity closes, the flow is rather similar in nature to ordinary wake flows. In such a range, the shape of the free streamline cannot be determined definitely and (with a constant upstream velocity) the wake flow is only stationary in the mean. This part of the wake will be called the 'far-wake', or the 'mixing range'. In between the near- and far-wakes there may exist a transition region in which the separated free streamlines from the two sides of the body re-attach to each other. Along the far-wake the mean pressure increases gradually from the wake under-pressure (or the cavity pressure) and finally recovers the main stream pressure far downstream.

It may be expected on physical grounds that the flow outside the obstacle and the near-wake region may be approximated with good accuracy by a potential flow. Only when the attempt is made to extend this approximate potential flow to large distances from the body (including the far-wake) do the various wake-flow and cavity-flow models arise, such as the Riabouchinsky model (Riabouchinsky 1920), the re-entrant jet model (see, for example, Kreisel 1946; Gilbarg & Serrin 1950), and the wake model proposed independently by Joukowsky (1890), Roshko (1954, 1955) and Eppler (1954). Some of the physical significance of these models has been discussed by Wu (1956*a*). In each of these flow models an artifice of some sort is introduced to admit the under-pressure coefficient as a free parameter, to account for the essential feature of a very complicated process of viscous dissipation in the wake, and to replace the actual wake flow of a real fluid by a simplified model within the framework of an equivalent potential flow. The validity of these flow models will therefore have to be justified by their agreement with experimental observations of the actual flow field near the body as well as the hydrodynamic forces and moments acting on the body.

The purpose of this paper is twofold. First, it serves to introduce a relatively simple wake model which can be readily applied to treat the wake flow or cavity flow past a lifting surface, such as a stalled airfoil or a cavitating hydrofoil. Secondly, it is intended to distinguish between the fully and partially developed wake (or cavity) flows and to recover the limiting case of fully wetted flow as the wake disappears. The wake flow will be called *fully developed* (or *fully cavitating* in case of cavity flows) if the region of the constant-pressure near-wake extends beyond the trailing edge of the plate, and will be called *partially developed* (or *partially cavitating*) if the near-wake region terminates in front of the trailing edge. For brevity these two flow régimes will also be called the full wake flow and the partial wake flow.

In order to develop a theory for the wake flow or cavity flow in which the region of constant pressure may have an arbitrary length, a plane wake-flow model is proposed here using the following physical assumptions.

(i) The entire separated wake is taken to be bounded by two smooth free-streamlines, each of which consist of two different parts. The first part covers the near-wake region which starts from the separation point down to a certain point which is determined from the theory. The pressure along this part assumes a given constant value p_c, the wake under-pressure or the cavity pressure. The value of p_c will be assumed to be less than the free-stream pressure, p_∞, throughout this work. Along the remaining part of the free streamline the pressure is assumed to change continuously from p_c to p_∞ at an infinite distance downstream.

(ii) The only kinematic condition on the free streamlines in the *physical plane* is that they become asymptotically parallel to the main flow at infinity.

(iii) The flow outside the wake is assumed inviscid and irrotational.

(iv) The images of the variable-pressure parts of the two free streamlines in the velocity plane (or the hodograph plane) are assumed to form a branch slit of undetermined shape. This will be referred to as the 'hodograph-slit condition'. Although this assumed flow configuration becomes over-simplified and hence invalid in the far-wake, it is to be expected that the rough approximation of the far-wake will not bear a predominant influence on the actual flow field near the body.

This wake model will now be applied to treat the wake (or cavity) flow past an oblique flat plate for both full and partial wake-flow régimes. The latter case will be treated separately in § 3, in which case the constant-pressure part on the lower free streamline disappears. Furthermore, these analyses may be utilized to treat the wake flow past a plate with a small camber; the resultant flow may be considered as a small perturbation with respect to the basic (non-linear) flow past the flat plate. The treatment of this last problem, however, will be postponed to a future paper.

The present free-streamline theory is applicable to both wake flows in one-phase media (such as in air) and cavity flows in water since the present theoretical results is found to be in good agreement with the experimental observations of Fage & Johansen (1927), which deals with wake flow in air, and the experiments of Parkin (1956), of Silberman (1959), and of Dawson (1959), which are all concerned with cavity flows in water.

2. Fully developed wake flows and cavity flows

2.1. *Analysis of the flow field*

Adopting the present wake model, we consider specifically the steady plane flow of an incompressible fluid, with free-stream velocity U and pressure p_∞, impinging on an oblique flat plate such that the flow separates from the leading edge A and trailing edge B, forming two free streamlines ACI and BC′I which are assumed to become asymptotically parallel to the main stream at downstream infinity I (see figure 1). The shape of ACI and BC′I in the physical plane is otherwise unknown *a priori*. We adopt a Cartesian co-ordinate system (x, y), with the x-axis lying along the plate AB and the origin at the leading edge A. The flow outside the wake is assumed inviscid and irrotational, hence there exists a velocity potential ϕ. As usual, we introduce the complex variable $z = x+iy$, the complex potential $f(z) = \phi+i\psi$, and the complex velocity

$$w(z) = df/dz = u-iv = q\,e^{-i\theta}, \tag{1}$$

where u and v are the x- and y-components of the velocity, $q = |w|$, and θ is the inclination of the velocity vector to the x-axis. Let α be the angle of attack of the plate, then

$$w = w_0 = U\,e^{-i\alpha} \quad \text{at} \quad z = \infty. \tag{2}$$

The kinematic and dynamic conditions on the free streamlines ACI and BC′I are imposed as follows. We assume that the pressure

$$p = p_c \leqslant p_\infty \quad \text{along} \quad \text{AC and BC}', \tag{3}$$

and that p varies continuously and monotonically from p_c to p_∞ along CI and C′I. From (3) and the Bernoulli equation it follows that

$$q = \text{const.} = q_c \quad \text{along AC and BC}', \tag{3a}$$

so that the Bernoulli equation of the external flow may be written

$$p + \tfrac{1}{2}\rho q^2 = p_\infty + \tfrac{1}{2}\rho U^2 = p_c + \tfrac{1}{2}\rho q_c^2. \tag{4}$$

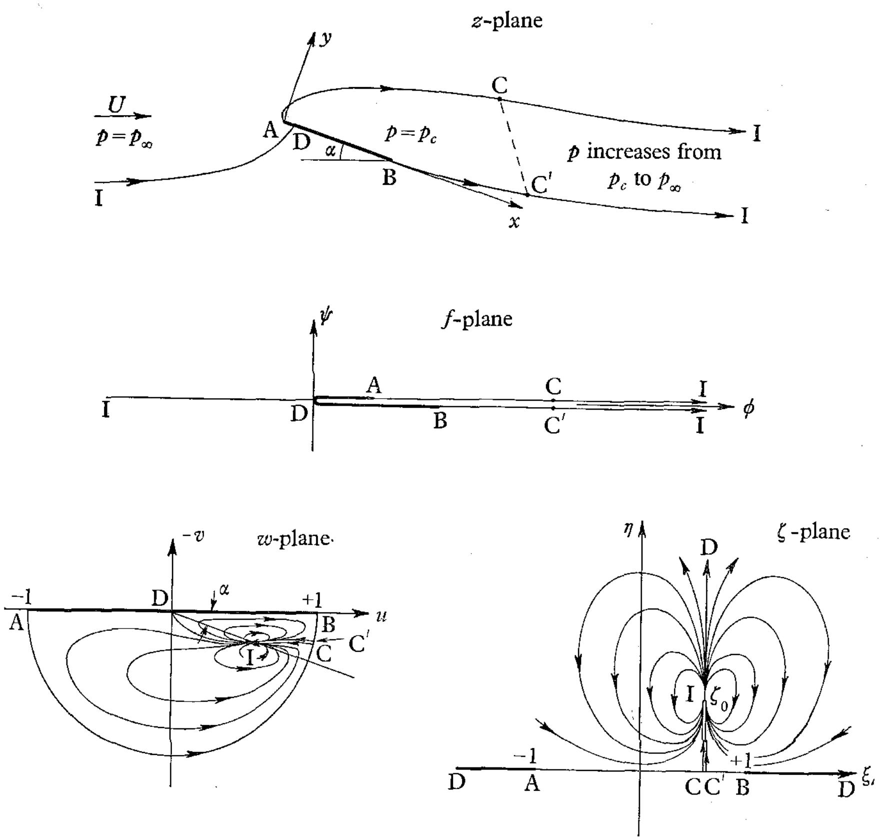

FIGURE 1. The free-streamline model for the fully developed wake flow past an oblique flat plate and its conformal mapping planes.

Since the points C and C′ are located on the two branches of the same stagnation streamline, and since they are the end-points of a constant pressure region, we obviously have

$$\psi_{\mathrm{C}} = \psi_{\mathrm{C'}} = 0, \quad q_{\mathrm{C}} = q_{\mathrm{C'}}. \tag{5a}$$

For complete determination of the points C and C′, we now introduce the assumptions that the potential ϕ and the flow inclination θ have respectively the same values at C and C′:

$$\phi_{\mathrm{C}} = \phi_{\mathrm{C'}}, \quad \theta_{\mathrm{C}} = \theta_{\mathrm{C'}}. \tag{5b}$$

Equations (5a) and (5b) can be combined in the form

$$f_{\mathrm{C}} = f_{\mathrm{C'}}, \quad w_{\mathrm{C}} = w_{\mathrm{C'}}. \tag{6}$$

In the far-wake region bounded by streamlines CI and C′I, the flow is assumed to be dissipated in such a way that p and w on the boundary CI and C′I change monotonically from p_c and w_C, eventually recovering their main-stream values p_∞ and w_0 at infinity downstream. The images of the free streamlines CI and C′I, on which $\psi = 0$, is further assumed to form a branch slit (of undetermined shape) in the hodograph w-plane so that the flow field in the hodograph plane will be simply connected and simply covered, this being the *hodograph-slit condition.* The postulated configuration of the fully developed wake flow (or cavity flow) requires that $\mathrm{Re}\,(z_{C'} - z_B) \geqslant 0$; otherwise the wake flow becomes partially developed. Aside from this phenomenological description of the free streamlines, the details of the flow within the wake (presumably viscous and rotational) are otherwise immaterial in connexion with the exterior flow and hence will not be pursued further in the present work.

It may be pointed out here that in the previous treatment of similar flow problems by Wu (1956*a*) and Mimura (1958), the two conditions in (5*b*) were replaced by $\theta_C = \alpha$ and $\theta_{C'} = \alpha$, and the hodograph-slit condition was reduced to the special form that CI and C′I become straight lines parallel to the main flow. The reasons for adopting the present conditions are: first, this model gives a reasonably good description of the flow outside the wake in comparison with actual flow visualizations; second, use of these conditions provides a rather smooth transition to the partial wake flow; and last, the present flow model results in a somewhat simplified analysis. The validity of the present theory of course may only be justified by its agreement with the experimental results.

For simplicity, both the plate length l and the constant speed q_c on the cavity wall will be normalized to unity so that from (4) we have

$$q_c = 1, \quad U = (1+\sigma)^{-\frac{1}{2}}, \tag{7a}$$

where

$$\sigma = (p_\infty - p_c)/(\tfrac{1}{2}\rho U^2). \tag{7b}$$

The dimensionless parameter σ is usually called the wake under-pressure coefficient or the cavitation number for cavity flows; this parameter characterizes the wake flow. In fact, the different flow régimes of the fully and partially developed flows can also be indicated by different ranges of σ.

Several streamlines $\psi = \text{const.}$ in the w-plane are illustrated in figure 1. Under the normalization $q_c = 1$ and the hodograph-slit condition, the bounding streamline $\psi = 0$ forms the boundary of the semicircle of unit radius in the lower-half w-plane and the slit CIC′; the entire flow is mapped onto the interior of the simply covered semicircle. It is convenient to introduce the parametric ζ-plane, defined by

$$\zeta = \tfrac{1}{2}(w^{-1} + w). \tag{8}$$

By this transformation the entire flow is mapped onto the upper-half ζ-plane, with the point $w_0 = U\,e^{-i\alpha}$ mapped into

$$\zeta_0 = \tfrac{1}{2}(w_0^{-1} + w_0) = \tfrac{1}{2}(U^{-1}e^{i\alpha} + U\,e^{-i\alpha}). \tag{9}$$

Since $\psi = 0$ on the entire real ζ-axis, the complex potential $f(\zeta)$ can be continued analytically into the lower-half ζ-plane by

$$f(\bar{\zeta}) = \overline{f(\zeta)}. \tag{10}$$

Now from the asymptotic behaviour of the streamlines $\psi = \text{const.}$ near the point I, it is evident that $f(\zeta)$ must have a simple pole at $\zeta = \zeta_0$. Otherwise the function $f(\zeta)$ is regular everywhere else in the upper-half ζ-plane. The analytic continuation (10) can be satisfied by placing a simple pole at $\zeta = \bar{\zeta}_0$, which is the reflexion of the first pole in the real ζ-axis. Furthermore, f has only one zero in the entire flow and that $f = 0(\zeta^{-2})$ as $|\zeta| \to \infty$ is obvious from the fact that a small circle (counterclockwise) around D in the f-plane is mapped into a large semicircle (clockwise) in the ζ-plane. Therefore it follows that the solution must be of the form

$$f = \tfrac{1}{4}A(\zeta-\zeta_0)^{-1}(\zeta-\bar{\zeta}_0)^{-1}, \tag{11}$$

where A is a real constant. Since $\psi = 0$ on the lines CI and C′I, we must have $\arg(\zeta-\zeta_0)+\arg(\zeta-\bar{\zeta}_0) = 0$ for ζ lying on these lines; hence CI and C′I are straight lines parallel to the imaginary ζ-axis. In particular, at C and C', we find

$$\zeta_C = \zeta_{C'} = \operatorname{Re}\zeta_0 = \tfrac{1}{2}(U^{-1}+U)\cos\alpha.$$

We shall now assign the range of U for the full wake flow (or the fully cavitating flow) by the condition† that the point C′ be located downstream of the trailing edge B, or, $\zeta_{C'} \leqslant \zeta_B = 1$. The corresponding range of U is determined from this condition and the above equation as

$$U_1 \leqslant U \leqslant 1, \quad U_1 = (1-\sin\alpha)/\cos\alpha = \cos\alpha/(1+\sin\alpha), \tag{12a}$$

in which the upper limit $U \leqslant 1$ follows from the physical condition $\sigma \geqslant 0$ (see equation (7a)). From (7a) and (12a) we find for the full wake flow

$$0 \leqslant \sigma \leqslant \sigma_1, \quad \sigma_1 = U_1^{-2}-1 = 2\tan\alpha\cot(\tfrac{1}{4}\pi-\tfrac{1}{2}\alpha). \tag{12b}$$

Note that $\sigma_1 \sim 2\alpha$ as $\alpha \to 0$, and $\sigma_1 \sim 4(\alpha-\frac{1}{2}\pi)^{-2}$ as $\alpha \to \frac{1}{2}\pi$. Thus for a given α, U has a lower limit $U_1(\alpha)$ and σ has an upper limit $\sigma_1(\alpha)$ for the fully developed wake flow. Let $w = e^{-i\gamma}$ at C and C′, then from (8)

$$\cos\gamma = \zeta_C = \tfrac{1}{2}(U^{-1}+U)\cos\alpha. \tag{13}$$

This equation asserts that $0 \leqslant \gamma \leqslant \alpha$ for U lying in the range given in (12a). Thus the flow inclination γ at C and C′ is always less than its free-stream value α; they are equal only when $U = 1$.

Combining (8), (9), and (11), we obtain

$$f = \frac{Aw^2}{(w-w_0)(w-\bar{w}_0)(w-w_0^{-1})(w-\bar{w}_0^{-1})}, \tag{14}$$

where w_0 is given by (2). We see here that the present model yields the complex potential as a one-valued analytic function of w in a closed form.

The physical z-plane is determined by integration as

$$z(w) = \int_{-1}^{w}\frac{1}{w}\frac{df}{dw}dw = \left[\frac{f}{w}\right]_{-1}^{w} + \int_{-1}^{w}\frac{f}{w^2}dw, \tag{15a}$$

which gives

$$\begin{aligned} z(w)+a = \{f(w)/w\} + iB\{&(\bar{w}_0^{-1}-\bar{w}_0)[w_0^{-1}\log(w-w_0)-w_0\log(w-w_0^{-1})] \\ &-(w_0^{-1}-w_0)[\bar{w}_0^{-1}\log(w-\bar{w}_0)-\bar{w}_0\log(w-\bar{w}_0^{-1})]\}, \end{aligned} \tag{15b}$$

† For further discussion of this condition, see the remarks following equation (17a).

where the constant B is related to A by

$$A/B = 2(U^{-1}-U)\sin\alpha[(U^{-1}+U)^2-(2\cos\alpha)^2]. \tag{15c}$$

The real constant a in (15*b*) is of such value that $z = 0$ at the point A. This result shows that $z(w)$ has a simple pole and a logarithmic singularity at the points w_0, $\overline{w}_0$, $1/w_0$, $1/\overline{w}_0$. The logarithmic singularities of $z(w)$ are admissible since the flow does not cover the entire z-plane due to the infinitely long wake. In order that $z(w)$ will be single-valued in the flow region, two branch cuts are introduced in the w-plane, one from w_0 along IC to $1/\overline{w}_0$, the other being the reflexion of the first cut into the real axis. Now since the plate has unit length, $z(1)-z(-1) = 1$. The result of this calculation gives

$$A = [(U^{-1}+U)^2-(2\cos\alpha)^2]/K, \tag{16a}$$

$$K = 2\frac{(U^{-1}+U)^2+(2\cos\alpha)^2}{(U^{-1}+U)^2-(2\cos\alpha)^2}+\frac{\pi(U^{-1}+U)}{2\sin\alpha} +\frac{(U^{-1}+U)^2-(2\cos\alpha)^2}{(U^{-1}-U)\sin\alpha}\tan^{-1}\left(\frac{U^{-1}-U}{2\sin\alpha}\right). \tag{16b}$$

This relation determines the coefficient A, and therefore completes the solution. It is noted that A and B are positive real constants.

When the point w moves along the cut from C′ to I and back to C, the function $\log(w-w_0)$ increases by $2\pi i$ while the other functions in (15*b*) resume their original values. Hence

$$z_C-z_{C'} = (2\pi B/w_0)(\overline{w}_0-\overline{w}_0^{-1}) = -2\pi B(U^{-2}-\cos 2\alpha - i\sin 2\alpha), \tag{17a}$$

which shows that $x_C < x_{C'}$, that is, the projection of the point C on the plate is always upstream of C′. Consequently, as the cavitation number increases (U decreases), the point C will pass over the trailing edge B before the point C′ reaches B at $U = U_1$ (see equation (12)). It will be seen later that in order to have a smooth transition to the partial wake-flow régime treated below, we should adopt for the range of fully developed wake flow, instead of (12), the condition $\operatorname{Re} z_C \geqslant \operatorname{Re} z_B$, and then consider a different flow régime after x_C becomes equal to x_B and before the point C′ reaches the trailing edge B. However, it is found that the hydrodynamic forces in the present case given below are continuous even for $U < U_1$, although the flow configuration for $U < U_1$ is no longer the full wake flow under consideration. From the numerical results it will be shown that the transition to the partial wake-flow case can be interpolated very smoothly, especially when the incidence angle α is small. Therefore, for practical purposes, condition (12) may be used as the approximate range of U in the full wake-flow régime.

The distance between the points C and C′ in the direction normal to the main flow is

$$h = \operatorname{Im}[(z_C-z_{C'})e^{-i\alpha}] = 2\pi B(1+U^{-2})\sin\alpha. \tag{17b}$$

This quantity may be compared with the lateral spacing of the Kármán vortex street behind the oblique plate.

2.2. *Lift and drag*

With q_c normalized to unity, the pressure difference $(p-p_c)$ may be written, from (4), as

$$p-p_c = \tfrac{1}{2}\rho(1-q^2) = \tfrac{1}{2}\rho(1-w\overline{w}). \tag{18}$$

Let the x- and y-components of the hydrodynamic force acting on the plate be denoted by X and Y, then

$$X+iY = i\int_{\mathrm{A}}^{\mathrm{B}}(p-p_c)\,dz = \tfrac{1}{2}i\rho\int_{\mathrm{CABC}'}(1-w\overline{w})\,dz, \tag{19}$$

where in the last step the contour of integration is extended to CABC′ since $w\overline{w} = 1$ on AC and BC′. The first term of the last integral is simply

$$X_1+iY_1 = \tfrac{1}{2}i\rho(z_{\mathrm{C}'}-z_{\mathrm{C}}) = -(i\pi\rho B/w_0)(\overline{w}_0-\overline{w}_0^{-1})$$

by using (17*a*). The complex conjugate of the second term in (19) is

$$X_2-iY_2 = \tfrac{1}{2}i\rho\int_{\mathrm{CABC}'}w\overline{w}\frac{d\overline{z}}{df}df = \tfrac{1}{2}i\rho\int_{\mathrm{CABC}'}w\,df = \tfrac{1}{2}i\rho\oint f\,dw,$$

since f is purely real on CABC′. The last step is obtained by integration by parts and by making use of condition (6); the corresponding contour in the w-plane is counterclockwise around the unit semicircle. Now the integrand is analytic, and regular everywhere inside the contour except at the simple pole $w = w_0$. Hence by applying the theorem of residues,

$$X_2-iY_2 = i\pi\rho B w_0(\overline{w}_0-\overline{w}_0^{-1}).$$

Combining X_1+iY_1 and X_2+iY_2 to obtain $X+iY$, we find that

$$X = 0, \quad Y = \pi\rho B(U^{-2}-U^2). \tag{20}$$

Therefore the hydrodynamic force of magnitude Y acts normal to the plate.

The normal force coefficient is defined, as usual, by $C_N = Y/(\tfrac{1}{2}\rho U^2 l)$, where l is the plate length (which is set to unity presently). Hence from (20), (15*c*) and (16), we obtain

$$C_N = \pi(U^{-1}+U)/(KU^2\sin\alpha), \tag{21}$$

where K is given by (16*b*). The lift and drag coefficients are of course

$$C_L = C_N\cos\alpha, \quad C_D = C_N\sin\alpha. \tag{22}$$

The coefficients C_L and C_D are plotted versus the under-pressure coefficient σ for several values of α in figures 2 and 3. In figures 4 and 5 the coefficients C_L and C_D are also plotted versus σ on a log-scale in order to incorporate the values of C_L and C_D in the partial wake-flow case. Fortunately there are several experimental results available for comparison with the theory. The experimental measurements of C_L and C_D for a cavitated flat plate in a high-speed closed water tunnel reported by Parkin (1956) are shown in figures 2 and 3. Also included in these figures are the experimental data of C_L and C_D for a cavitating flat plate in a free-jet water tunnel presented by Silberman (1959). A third set of data are taken from Dawson's experiments (1959) which were carried out in a free-surface water tunnel. All these results are reproduced here with σ equal to the cavitation number based on the measured cavity pressure and without the correction of the tunnel boundary effect. The present theory overpredicts slightly these data at small cavitation numbers, but the general trend of agreement between the theory

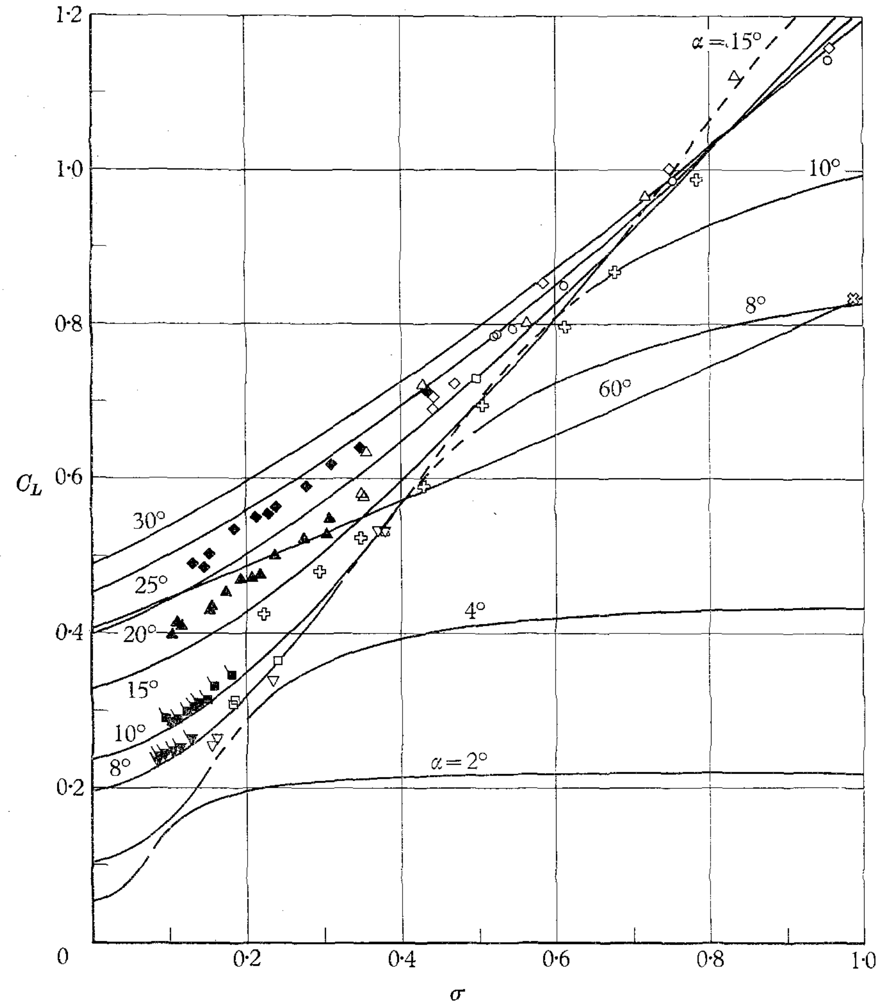

FIGURE 2. Variation of C_L with the wake under-pressure coefficient σ (or the cavitation number) for the flat plate. Parkin's experiments were performed in a high-speed closed water tunnel, Silberman's experiments in a free-jet water tunnel, and Dawson's experiments in a free-surface water tunnel, all data being reproduced here with σ equal to the cavitation number based on the measured cavity pressure and without the correction of the tunnel boundary effect. ——, Present theory. Parkin's data: $\bigtriangledown$, $\alpha = 8°$; $\square$, $\alpha = 10°$; ✣, $\alpha = 15°$; $\triangle$, $\alpha = 20°$; $\diamond$, $\alpha = 25°$; $\circ$, $\alpha = 30°$; ⌘, $\alpha = 60°$. ▲, ◆, Silberman's data. ▼, ■, Dawson's data.

and experiments may be considered to be good. In figure 4 several values of C_L derived from the experiments made by Fage & Johansen (1927) for the wake flow of air past a flat plate are included; these data are obtained with σ based on the measured constant base pressure and without wall-effect corrections. A comparison shows that the present theory is in excellent agreement with these experimental results.

In the limit, as $U \to 1$ (or $\sigma \to 0$), we find from (21) that

$$C_N = 2\pi \sin\alpha/(4+\pi\sin\alpha),$$

which is the familiar classical result of Kirchhoff for the infinite cavity flow past an inclined lamina. When the plate is set normal to the flow, $\alpha = \frac{1}{2}\pi$, we have

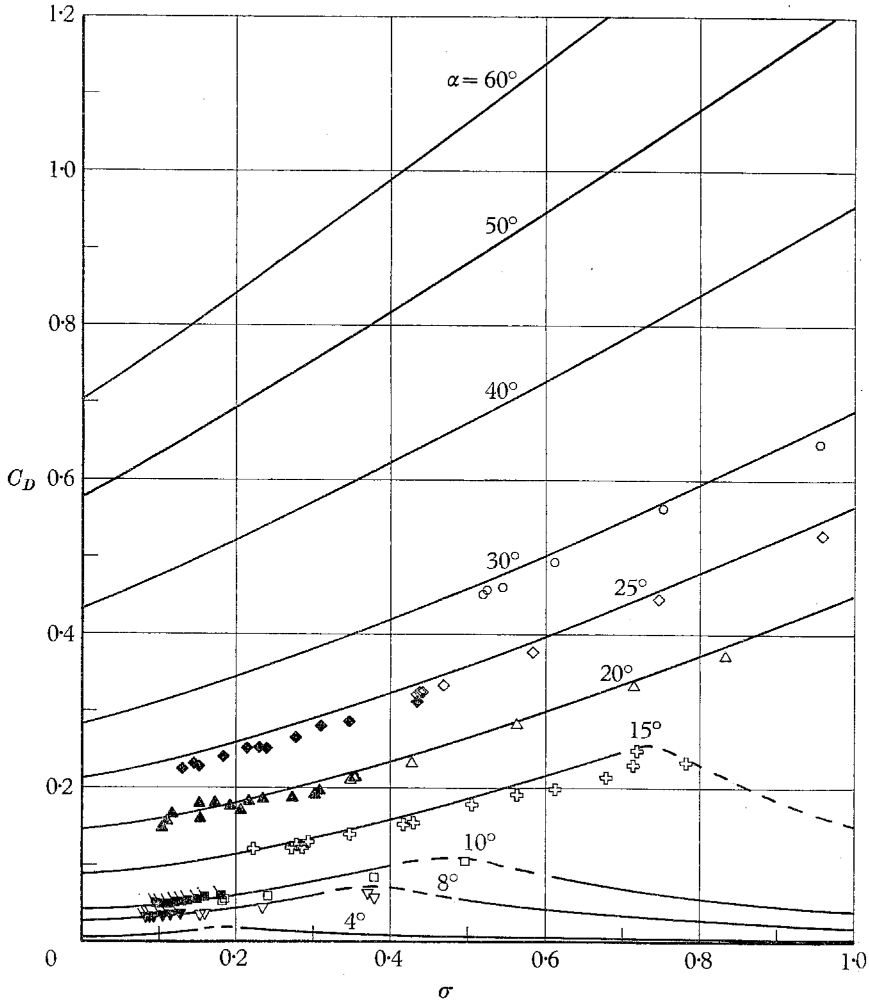

Figure 3. Variation of C_D with σ for the flat plate (same legend as figure 2).

by symmetry that $\theta_C = \theta_{C'} = \frac{1}{2}\pi$ which are the conditions adopted by Roshko (1954) in proposing his model. For $\alpha = \frac{1}{2}\pi$, we deduce from (21) that

$$C_D = C_N = \tfrac{1}{2}\pi\{U^3(1+U^2)^{-1} + U^2(1-U^2)^{-1}[\tfrac{1}{2}\pi - (1+U^2)\tan^{-1}U]\}^{-1}, \quad (23)$$

which is the result of Roshko (1954).

When the point C′ approaches B, so that the region of constant pressure is limited to the space above the plate, one derives for $U = U_1$ (see equation (12)) and for α small the result

$$C_L \approx C_N \approx \pi\alpha. \tag{24}$$

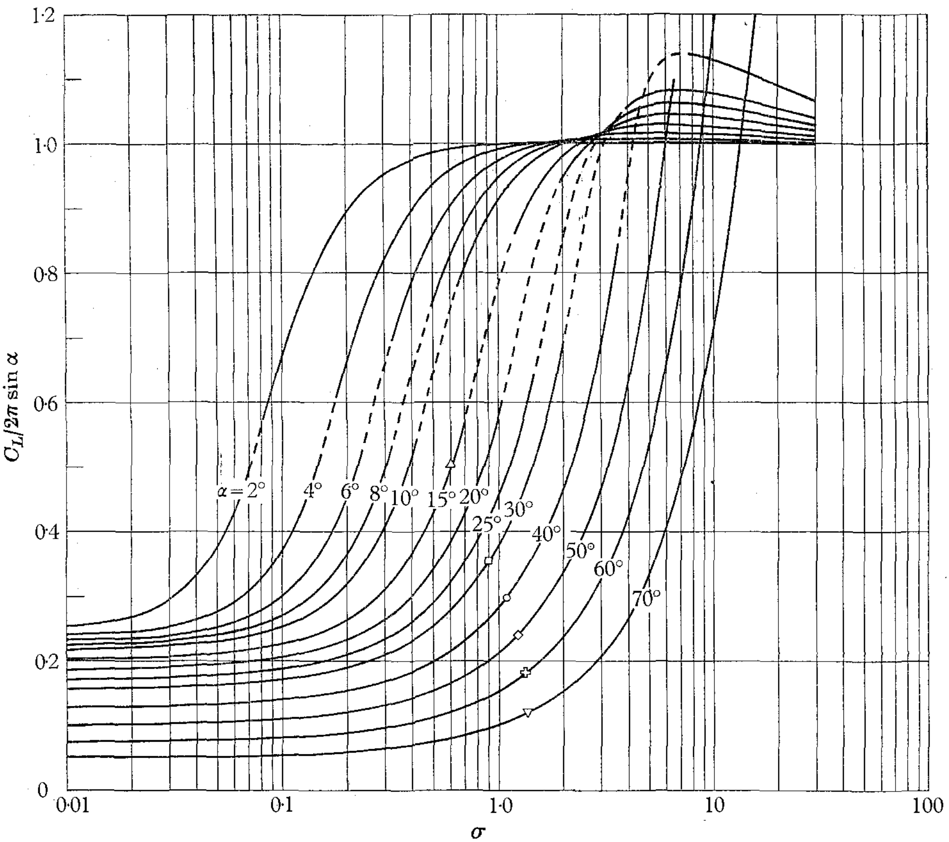

FIGURE 4. Variation of C_L over an extended range of σ to cover both the fully and partially developed wake flows past the flat plate, the transition between these two régimes of wake flows being appropriately faired-in with dashed lines. The experiments of Fage & Johansen were carried out in a wind tunnel, the results being reproduced with the wake under-pressure coefficient based on the measured constant base pressure and without the correction of the tunnel wall effect. ——, Present theory. Fage & Johansen data: △, $\alpha = 15°$; □, $\alpha = 30°$; ○, $\alpha = 40°$; ◇, $\alpha = 50°$; ✣, $\alpha = 60°$; ▽, $\alpha = 70°$.

Therefore, as the full wake flow is at the transition to partial wake flow, the lift coefficient on the plate held at a small incidence angle is approximately half the aerodynamic value $2\pi\alpha$.

2.3. *Pressure distribution and the free-streamline configuration*

The pressure distribution on the wetted and separated sides of the plate is readily determined from (4) and (15), which provide a parametric representation of $p(x, y)$. On the separated side $p = p_c$, hence

$$C_p \equiv (p - p_\infty)/(\tfrac{1}{2}\rho U^2) = -\sigma. \tag{25a}$$

On the wetted side w is real, $-1 < w < 1$, hence, from (4) and (7),

$$C_p = 1-(1+\sigma)\,w^2 \quad \text{for} \quad -1 < w < 1; \tag{25b}$$

and, from (15), for $-1 < w < 1$,

$$x(w) = AU^2\left[I(-1)+wI(w)+\int_{-1}^{w} I(u)\,du\right], \tag{26a}$$

where $$I(w) = (U^2+w^2-2wU\cos\alpha)^{-1}(1+w^2U^2-2wU\cos\alpha)^{-1}. \tag{26b}$$

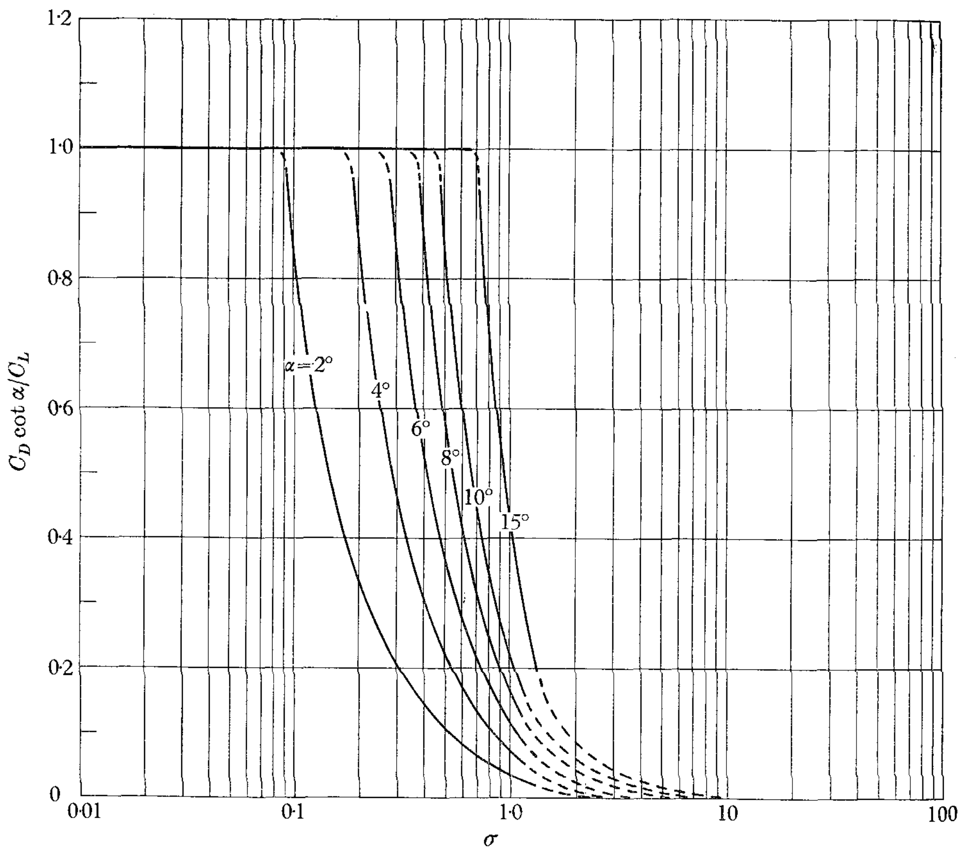

FIGURE 5. Variation of C_D over an extended range of σ for the flat plate. The dashed lines for $\sigma > 1$ are not the theoretical results, but are shown to indicate what would be expected on physical grounds.

The above parametric solution $C_p(w)$ and $x(w)$ is shown in figure 6*a–d* for $\alpha = 90°$ (*a*), 69·85° (*b*), 49·85° (*c*), 29·85° (*d*), with the respective under-pressure coefficient $\sigma = 1{\cdot}380$, 1·360, 1·230 and 0·924. The corresponding experimental results were obtained by Fage & Johansen (1927); the mean 'base pressure coefficient' $\sigma_{\exp}$ was observed experimentally for the wake flow in air. However, no correction due to the tunnel wall effect was made for these data. A comparison shows that the present theory and the experiments are in excellent agreement.

The shapes of the free streamlines AC and BC′ can be determined from (15) as follows. Along AC, $w = e^{-i\theta}$, $\gamma \leqslant \theta \leqslant \pi$, γ being given by (13), we have

$$z-z_A = \frac{AU^2}{(1+U^2+2U\cos\alpha)^2} + \frac{AU^2 e^{i\theta}}{[1+U^2-2U\cos(\theta-\alpha)][1+U^2-2U\cos(\theta+\alpha)]}$$
$$+iAU^2\int_\theta^\pi e^{i\theta}[1+U^2-2U\cos(\theta-\alpha)]^{-1}[1+U^2-2U\cos(\theta+\alpha)]^{-1}\,d\theta. \quad (27a)$$

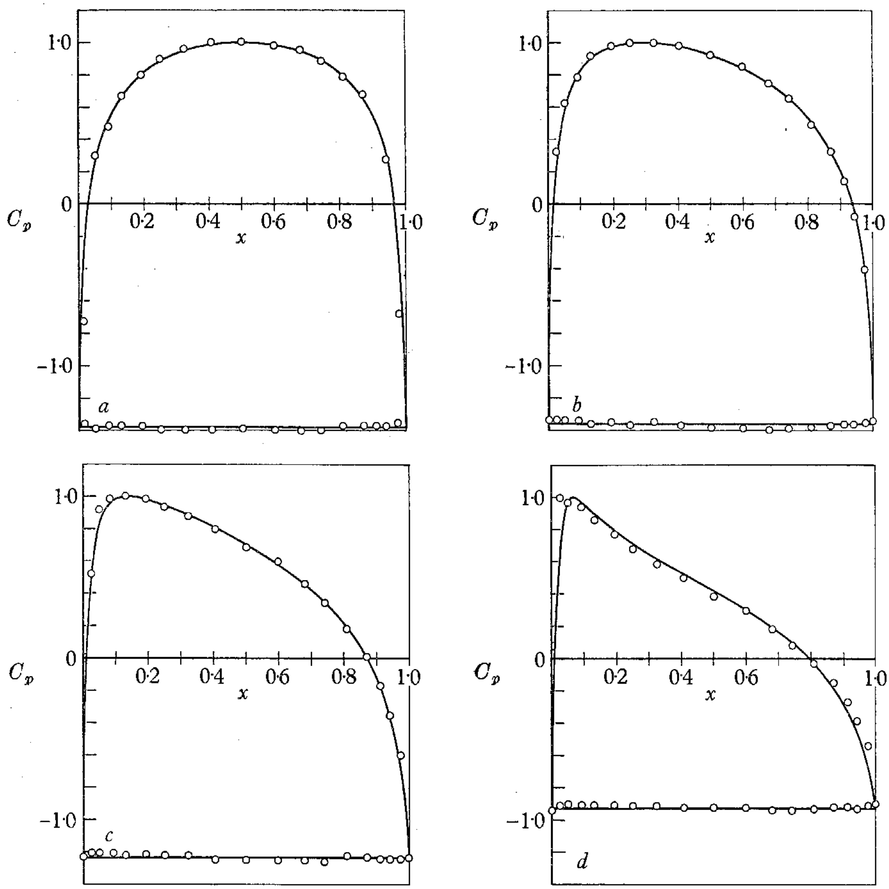

FIGURE 6. Pressure distributions on the wetted and separated sides of the oblique plate. The experimental results of Fage & Johansen are reproduced from graph reading of the original paper, as the tabulated data are not available. ——, Present theory. ○, Fage & Johansen data: (*a*) $\alpha = 90°$; $\sigma = 1{\cdot}380$; (*b*) $\alpha = 69{\cdot}85°$, $\sigma = 1{\cdot}360$; (*c*) $\alpha = 49{\cdot}85°$, $\sigma = 1{\cdot}230$; (*d*) $\alpha = 29{\cdot}85°$, $\sigma = 0{\cdot}924$.

Along BC′, $w = e^{-i\theta}$, $0 \leqslant \theta \leqslant \gamma$, we obtain

$$z-z_B = \frac{AU^2 e^{i\theta}}{[1+U^2-2U\cos(\theta-\alpha)][1+U^2-2U\cos(\theta+\alpha)]} - \frac{AU^2}{(1+U^2-2U\cos\alpha)^2}$$
$$-iAU^2\int_0^\theta e^{i\theta}[1+U^2-2U\cos(\theta-\alpha)]^{-1}[1+U^2-2U\cos(\theta+\alpha)]^{-1}\,d\theta. \quad (27b)$$

The shapes of the free streamlines AC and BC′ can be calculated from these equations for given α and σ.

In particular, the transverse distance h between the points C and C′ in the direction normal to the main flow, given by (17*b*), can be expressed by using (15*c*), (16) and (21) as

$$h/l = \sigma^{-1}C_N \sin\alpha = C_D/\sigma, \tag{27c}$$

where the plate length l is restored for completeness. This simple result can also be derived by a momentum consideration applied to the far-wake ICC′I. In figure 7 this value of h/l is plotted versus σ for several values of α. Also shown in

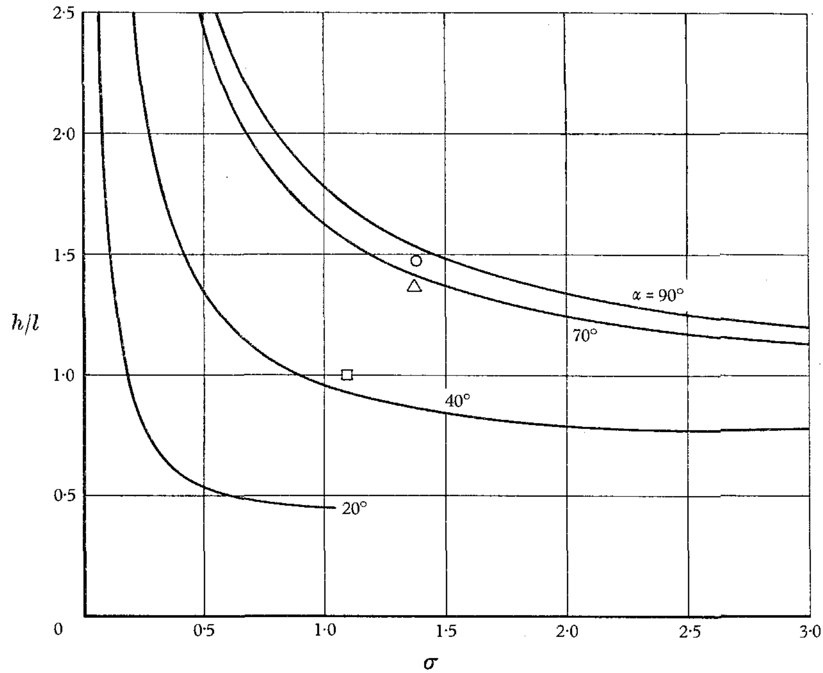

FIGURE 7. Variation of the asymptotic width of the wake with σ. ——, Present theory. Fage & Johansen data: ○, $\alpha = 90°$; △, $\alpha = 70°$; □, $\alpha = 40°$.

figure 7 are a few values of h/l calculated by Fage & Johansen (1927), using Kármán's stability relation $h = 0{\cdot}281a$ and the measured values of the vortex spacing a. The present theoretical result compares favourably well with such estimates, although this flow model is not expected to reproduce any details of the far-wake flow. In the actual measurements of h/l, however, Fage & Johansen reported that h/l increases towards the downstream as the vortices diffuse.

3. Partially developed wave flows and cavity flows

3.1. *The flow model; analysis of the flow field*

As described in § 1, the partially developed wake flow is defined by a configuration in which the near-wake of constant pressure p_c covers only a part of the suction side of the lifting plate, starting from the leading edge A and terminating at a certain point C upstream of the trailing edge B (see figure 8). The pressure in the wake further downstream increases continuously and recovers at infinity downstream its upstream value p_∞. In order to describe this type of flow to a good approximation and to render the flow exterior to the wake subject to simple analysis, the following model is proposed.

The stagnation streamline ID is split into two branches; one of them follows the lower surface to the trailing edge B and then forms a free streamline BI, extending to downstream infinity I; the other branch separates from the leading

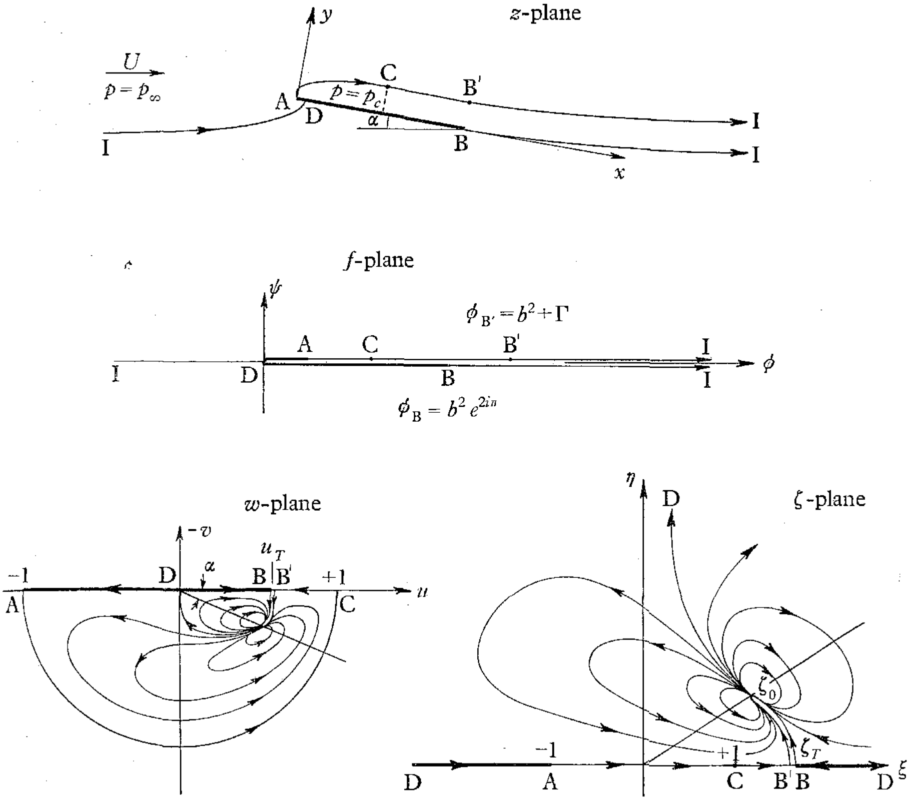

FIGURE 8. The free-streamline model for the partially developed wake flow past an oblique flat plate and its conformal mapping planes.

edge A to form another free streamline ACB′I such that $p = p_c$ on AC, with $x_C \leqslant x_B$, and that p increases monotonically along CB′I. The streamline CB′ is assumed to be parallel to the plate. The point B′ is defined by

$$\psi_B = \psi_{B'} = 0 \quad \text{and} \quad \mathrm{Re}\,(z_{B'} - z_B) = 0. \tag{28a}$$

Furthermore, we assume that the complex velocity w takes the same value at B and B′,

$$w_{B'} = w_B (= u_T \text{ say}). \tag{28b}$$

This condition implies that the velocity at B′ is parallel to the plate, and that the pressure at B′ and B are equal. Physically it is conceivable that, if the wake at the trailing edge is narrow, the pressure cannot vary appreciably across the wake. This condition will be imposed even though the wake may be moderately thick. However, for $\alpha > 45°$, the condition (28b) would be expected to lose gradually its physical significance, since the non-uniformity of the pressure distribution across the wake and over the suction side of the flat plate will extend over such a wide region that the flow outside the wake can no longer be approximated by this simple description. After all, from the physical point of view, the partial wake flow would be of interest only for small and moderate values of α. For this reason, our present treatment will be limited to the range $0 < \alpha < 45^\circ$.

Conditions (28a, b) define the point B′ and replace the assumptions (5a, b) for the case of full wake flows. The streamlines B′I and BI will again be assumed to form a slit in the hodograph plane (the hodograph-slit condition). As in the previous case, the plate length l and the constant speed q_c on AC are again both normalized to unity. Since p increases monotonically along CB′I, the pressure p_T at B and B′ is greater than p_c, and hence obviously $0 < u_T < 1$.

It should be noted that, if conditions (28a, b) are to be fulfilled, a circulation around the wake must be introduced, and consequently the potential f will not have the same value at B and B′. In fact, we must have $f_{B'} - f_B = \Gamma$, where Γ is the circulation around BDACB′. The existence of a circulation is an essential feature of the partial wake flow as compared with the previous case; without the circulation, the transition to the fully wetted flow would be greatly impaired.

Under the normalization $q_c = 1$ and the hodograph-slit condition, the flow is mapped conformally into the interior of a simply-covered semicircle of unit radius in the lower-half w-plane as shown in figure 8. The illustration is self-explanatory.

By the transformation (8) the entire flow is mapped into the upper-half ζ-plane, with the point $w_0 = U e^{-i\alpha}$ mapped into ζ_0 given by (9); the boundary of the semicircle in the w-plane is mapped into the entire real ζ-axis. From the configuration of the streamlines near $\zeta = \zeta_0$, it is again evident that f must have a simple pole at $\zeta = \zeta_0$. Furthermore, in order to satisfy (28a, b) a vortex must be introduced at $\zeta = \zeta_0$. From this singular behaviour of f and the property (10) it follows that the solution $f(\zeta)$ must be of the form

$$2\pi f(\zeta) = \frac{Q}{\zeta - \zeta_0} + \frac{\bar{Q}}{\zeta - \bar{\zeta}_0} - i\Gamma \log \frac{\zeta - \zeta_0}{\zeta - \bar{\zeta}_0}, \tag{29}$$

where Q (complex in general) is the strength of the simple pole and Γ (real) is the circulation about $\zeta = \zeta_0$. From the local conformal behaviour of $f(\zeta)$ near $\zeta = \infty$, as was explained for (11), we must again require that $f = O(\zeta^{-2})$ as $|\zeta| \to \infty$. Expanding the right-hand side of (29) for large ζ and equating the coefficient of ζ^{-1} to zero, we obtain

$$Q + \bar{Q} = -i\Gamma(\zeta_0 - \bar{\zeta}_0) = (U^{-1} - U)\,\Gamma \sin\alpha, \tag{30}$$

where use has been made of (9). Let the value of ζ at B and B′ be ζ_T, then from (8) the value of w at B and B′ is

$$u_T = \zeta_T - (\zeta_T^2 - 1)^{\frac{1}{2}}. \tag{31}$$

Obviously we must have $\zeta_T > 1$ so that $0 < u_T < 1$ for the partial wake flow. The streamlines BI and B′I form a slit which is perpendicular to the real ζ-axis at ζ_T. Then, since a small semicircle around B in the f-plane is mapped into a small quarter-circle around ζ_T in the ζ-plane, it follows that $df/d\zeta = 0$ at $\zeta = \zeta_T$. From this condition and (30) we obtain

$$\begin{aligned}\zeta_T &= (Q\bar{\zeta}_0 - \bar{Q}\zeta_0)/(Q - \bar{Q}) = \tfrac{1}{2}(\zeta_0 + \bar{\zeta}_0) + \tfrac{1}{2}(Q + \bar{Q})\,(Q - \bar{Q})^{-1}\,(\bar{\zeta}_0 - \zeta_0) \\ &= \tfrac{1}{2}(U^{-1} + U)\cos\alpha - \tfrac{1}{2}i(Q + \bar{Q})\,(Q - \bar{Q})^{-1}\,(U^{-1} - U)\sin\alpha.\end{aligned} \tag{32}$$

The physical plane z is determined by the integration

$$z = \int_{-1}^{w} \frac{1}{w}\frac{df}{dw}\,dw. \tag{33a}$$

Carrying out the integration and making use of (30), we obtain

$$2\pi(z+a) = \frac{2Q}{(w-w_0)(w-w_0^{-1})} + \frac{2\bar{Q}}{(w-\bar{w}_0)(w-\bar{w}_0^{-1})} + \frac{b-i\Gamma}{w_0}\log(w-w_0)$$
$$-w_0(b+i\Gamma)\log(w-w_0^{-1}) + \frac{\bar{b}+i\Gamma}{\bar{w}_0}\log(w-\bar{w}_0) - \bar{w}_0(\bar{b}-i\Gamma)\log(w-\bar{w}_0^{-1}), \tag{33b}$$

where
$$b = -2Q/(w_0^{-1}-w_0) = -2Q/(U^{-1}e^{i\alpha}-Ue^{-i\alpha}), \tag{33c}$$

and the real constant $2\pi a$ is determined such that $z = 0$ at the point A. The function $z(w)$ has a simple pole and a logarithmic singularity at the points w_0, $\bar{w}_0$, $1/w_0$, $1/\bar{w}_0$. In order than $z(w)$ be single-valued in the flow field, two branch-cuts are introduced in the w-plane, one from w_0 along IB and its image path (reflected into the real w-axis) to $w = \bar{w}_0$, the other being the image of the first cut into the unit circle $w\bar{w} = 1$. As the point w traces along the cut from B, around the point w_0 and ends up at B′, the function $\log(w-w_0)$ increases by $2\pi i$, whereas the other functions in (33b) are unaltered. Hence from (33)

$$z_{B'} - z_B = i(b-i\Gamma)/w_0.$$

But condition (28a) requires that $(z_{B'}-z_B)$ be purely imaginary, say

$$z_{B'} - z_B = i\beta\Gamma, \tag{34}$$

where β is a real constant. By comparison we have

$$b = (\beta w_0 + i)\Gamma. \tag{35}$$

From (30), (33c) and (35) we can solve for β, Q and B, giving

$$\beta = 2U\sin\alpha/(1-U^2\cos 2\alpha), \tag{36}$$

$$Q = \tfrac{1}{2}\Gamma\{(U^{-1}-U)\sin\alpha - i\cos\alpha(U^{-1}-U+2\beta U^2\sin\alpha)\}. \tag{37}$$

Substituting this equation in (32), we obtain

$$\zeta_T = \tfrac{1}{2}(U^{-1}+U)\cos\alpha + \frac{(1-U^2)^2\sin^2\alpha}{2U\cos\alpha(1-U^2+2\beta U^3\sin\alpha)}, \tag{38}$$

which is determinate for given U and α. Now application of the condition $\zeta_T > 1$ to (38) for the partial wake flow will lead to a permissible range of U for each α, say $0 \leqslant U < U_p(\alpha)$ such that $\zeta_T(U_p, \alpha) = 1$. However, it can be verified that $U_p(\alpha)$ is approximately equal to $U_1(\alpha)$ defined by (12) for moderate and small values of α. In fact, it is readily shown that

$$\zeta_T(U_1, \alpha) = 1 + \tfrac{1}{2}\alpha^3 + O(\alpha^4) \quad \text{as} \quad \alpha \to 0.$$

Therefore, the difference between U_1 and U_p will not be pursued further, and $0 < U < U_1$ will be used as the approximate range of U for the partial wake flow.

Upon substitution of (35) to (37) into (33), we obtain

$$2\pi(z+a)/\Gamma = 2U(1-U^2)\sin\alpha[1+w^2-2wU\cos\alpha(1-\beta U\sin\alpha)]\,I(w)$$
$$+\beta\log[(w-w_0)(w-\bar{w}_0)] - w_0(\beta w_0+2i)\log(w-w_0^{-1})$$
$$-\bar{w}_0(\beta\bar{w}_0-2i)\log(w-\bar{w}_0^{-1}), \tag{39}$$

where $I(w)$ is given by (26 *b*). Finally the circulation strength Γ is determined by the scale of the plate length such that $z_B - z_A = z(u_T) - z(-1) = 1$. The result of this calculation yields

$$\begin{aligned}\pi/\Gamma = U(1-U^2)\sin\alpha\{[1+u_T^2-2u_T U\cos\alpha(1-\beta U\sin\alpha)]\,I(u_T)\\ -2[1+U\cos\alpha(1-\beta U\sin\alpha)\,I(-1)]\}+\tfrac{1}{2}\beta\log\frac{u_T^2+U^2-2u_T U\cos\alpha}{1+u_T^2U^2-2u_T U\cos\alpha}\\ +2U\cos\alpha(1-\beta U\sin\alpha)\tan^{-1}\left[\frac{U(1+u_T)\sin\alpha}{1-u_T U^2+U(1-u_T)\cos\alpha}\right],\end{aligned} \tag{40}$$

in which β is given by (36), $I(w)$ by (26 *b*), and u_T by (31) and (38). This equation determines the circulation Γ in terms of U and α.

It is of interest to note the limiting case of the fully wetted flow.† From (36), (38) and (31) we deduce immediately that, as $U \to 0$,

$$\beta \approx 2U\sin\alpha, \quad \zeta_T \approx (1+U^2\cos 2\alpha)/2U\cos\alpha, \quad u_T \approx U\cos\alpha, \tag{41a}$$

and hence, from (40), that

$$\Gamma \approx \pi U\sin\alpha\{1-(U\sin\alpha)^2\log(U\sin\alpha)^2+O(U^2)\} \quad \text{as} \quad U\to 0. \tag{41b}$$

Furthermore, it is seen from (34) that $z_{B'} - z_B \to 0$ like U^2 as $U \to 0$, and from (39) that $z_C = z(1) \to 0$ as $U \to 0$. Thus as $U \to 0$ (or rather $U/q_c \to 0$ as $q_c \to \infty$ for fixed U), the constant-pressure region vanishes and the thickness of the wake reduces to zero, the flow thereby becoming fully wetted. The results that $u_T = U\cos\alpha$, and $\Gamma/(\text{chord}) = \pi U\sin\alpha$ are of course both well known in airfoil theory.

3.2. *Lift and drag in the partial wake flow*

The calculation of the hydrodynamic forces on the inclined plate in a partial wake flow is less straightforward than in the full wake-flow case, since if the forces are to be determined by integration of the pressure difference across the plate, the pressure on the suction side of the plate is now subject to certain arbitrariness in interpretation. However, in view of the physical significance of the condition (28 *b*), we shall assume that the hydrodynamic force acting on the plate is equal to that on the closed body B′CADBB′, with its base BB′ exposed to a uniform base pressure p_T. This assumption enables us to calculate the force directly from the exterior potential flow without considering the viscous flow of the real fluid within the wake. The force so determined may be conjectured to include the effects due to cavity formation near the leading edge and the equivalent dissipation in this potential flow model.

For the present purpose the Bernoulli equation may be written

$$p+\tfrac{1}{2}\rho w\bar{w} = p_T+\tfrac{1}{2}\rho u_T^2. \tag{42}$$

The hydrodynamic force acting on the plate is then given by

$$X+iY = i\oint_C (p-p_T)\,dz = \tfrac{1}{2}i\rho\oint_C (u_T^2-w\bar{w})\,dz, \tag{43}$$

† The fully wetted flow past a flat plate can physically be realized only when the leading edge is sufficiently round.

where the contour C denotes the path B′CAB. This is the pressure integral on the closed body B′CABB′ since $p = p_T$ on BB′. The first term of this integral is simply

$$X_1+iY_1 = \tfrac{1}{2}i\rho u_T^2 \oint_C dz = \tfrac{1}{2}i\rho u_T^2(z_B - z_{B'}) = \tfrac{1}{2}\rho u_T^2 \beta\Gamma,$$

using (34). The complex conjugate of the second term in (43) is

$$X_2 - iY_2 = \tfrac{1}{2}i\rho \oint_C w\bar{w}\frac{d\bar{z}}{df}\,df = -\tfrac{1}{2}i\rho \oint_{C_w} w\frac{df}{dw}\,dw,$$

where the contour C_w is counterclockwise around the unit semicircle in the w-plane. Now, from the previous solution (29) and (8), it is seen that the above integrand $w\,df/dw$ is an analytic function of w, whose only singularity within the contour C_w is a double pole at $w = w_0$, at which the residue is found to be $-(b+i\Gamma)\,w_0/2\pi$. Hence, by the theorem of residues,

$$X_2 - iY_2 = -\tfrac{1}{2}\rho(b+i\Gamma)\,w_0 = -\rho\Gamma w_0(i+\tfrac{1}{2}\beta w_0),$$

where use has been made of (35). Combining X_1+iY_1 and X_2+iY_2 to obtain $X+iY$, we find that

$$X+iY = \rho U\Gamma\, e^{i\alpha}[i+\tfrac{1}{2}\beta U(u_T^2 U^{-2} e^{-i\alpha} - e^{i\alpha})]. \tag{44}$$

It is noted from the above result that the force component parallel to the plate, X, generally does not vanish in the partial wake flow. In particular, when $U \to 0$, use of the limiting values (41 a, b) in (44) yields

$$X = -\rho U\Gamma \sin\alpha,$$

which is known as the leading-edge suction in airfoil theory. (That the tangential force component X is in general not zero perhaps cannot be explained entirely within the framework of potential theory. This is partly because the approximated mechanism of dissipation takes place over a portion of the plate. In the real physical case, the flow pattern is of course very complex.)

Finally, resolving the force into a lift L and a drag D, we obtain

$$D+iL = (X+iY)\,e^{-i\alpha} = \rho U\Gamma[i+\tfrac{1}{2}\beta U(u_T^2 U^{-2} e^{-i\alpha} - e^{+i\alpha})], \tag{45}$$

and hence

$$C_L = L/\tfrac{1}{2}\rho U^2 \text{ (chord)} = (2/U)\Gamma[1-\tfrac{1}{2}\beta U(1+u_T^2 U^{-2})\sin\alpha], \tag{46}$$

$$C_D = D/\tfrac{1}{2}\rho U^2 \text{ (chord)} = \beta\Gamma(u_T^2 U^{-2} - 1)\cos\alpha, \tag{47}$$

where β is given by (36), u_T by (31) and (38), and Γ by (40). Near the transition between the full wake and partial wake flows, we let $U = U_1$ (see equation (12)) and consider small values of α. For this case we derive from (12), (36), (31) and (38) that

$$U_1 \approx 1-\alpha+\tfrac{1}{2}\alpha^2, \quad \beta \approx 1-\alpha+\tfrac{3}{2}\alpha^2, \quad u_T \approx 1-\alpha^{\frac{3}{2}}-\tfrac{3}{4}\alpha^{\frac{5}{2}}, \tag{48}$$

and, from (40), $\Gamma \approx \frac{1}{2}\pi\alpha$. Substituting these values into (46) and (47), we obtain

$$C_L \approx \pi\alpha, \quad C_D \approx \alpha C_L, \quad \text{for} \quad U = U_1\,(\alpha \ll 1). \tag{49}$$

On the other hand, in the limiting case of fully wetted flow, $U \to 0$, we substitute (41) into (46) and (47), giving

$$C_L \approx 2\pi\sin\alpha[1-(U\sin\alpha)^2\log{(U\sin\alpha)^2}+O(U^2)], \tag{50a}$$

$$C_D \approx -2\pi U^2 \sin^4\alpha\cos\alpha. \tag{50b}$$

Equation (49) coincides with (24) and (22) which are the upper limits of C_L and C_D in the full wake flow for $\alpha \ll 1$; this indicates that the transition from the full wake to partial wake flow is smooth for small and moderate values of incidence angle α. Equation (50 a) shows that for small U, C_L is slightly greater than the classical aerodynamic value $2\pi \sin\alpha$ and eventually tends to $2\pi \sin\alpha$ as $U \to 0$. This is known as the leading-edge bubble effect which produces a small positive camber over the original flat-plate airfoil. For small U, (50 b) shows that C_D attains a negative value, which is very small for small α and is of smaller order than the classical leading-edge suction. This rather unfavourable result may be attributed to the over-simplification of this partial wake-flow model.

Equations (46) and (47) are plotted in figures 2 to 5 for a range of α from 2° to 40°. For moderate values of α, the result shows that the transition from the full wake to partial wake flow becomes increasingly less smooth with increasing α; a smooth curve in the transition region is appropriately faired-in with dashed lines. Furthermore, the drag has been found to become negative (but small in magnitude) beyond a certain range of the cavitation number σ; this part of the curve is shown by the dotted lines. In spite of these rough approximations, the present wake-flow model is seen able to account for the salient features of the wake flow, as the incorporated experimental results clearly indicate.

Concluding remarks

It may be mentioned that in dealing with the plane cavity flows past a thin body at a small angle of attack, Tulin (1955) proposed a linearized theory which has stimulated numerous research activities. A survey of the literature in this field has been given by Parkin (1959). Among these works it suffices to cite a few which are relevant to our present consideration. The supercavitating flat plate was first treated by Tulin (1955); the case of arbitrary profile was considered by Wu (1956 b). The linearized theory has been extended independently by Acosta (1955) and by Geurst & Timman (1956) to deal with the partially cavitating flat plate, and by Geurst (1960) to consider the partially cavitating plate of arbitrary profile. Another linearized theory based on a different cavity-flow model has been proposed by Fabula (1958), which is also reviewed in Parkin (1959). For further discussion of these theoretical results and a comparison between the linearized and non-linear theories and the experiments, reference may be made to Parkin (1959, 1961) since such a task is beyond the scope of the present paper.

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