I. NUMERICAL METHODS AND SUPPORTING DATA

In this part, we provide some details of the DMRG and VMC methods and also show comparison of the Rényi entropies between the two. We will present additional data analysis in Sec. II E after developing a long-wavelength description of the d-wave metal phase (abbreviated hereafter as “d-metal”).

A. DMRG calculations and observables

We determine the ground state phase diagram of the t-J-K model, Eq. (6) of the main text, by large-scale DMRG calculations. We consider square lattice clusters with total number of sites \( L_x \times L_y \). Here, we study the two-leg ladder system, i.e., \( L_y = 2 \), and use periodic boundary conditions along the \( x \) direction.

Our DMRG calculations generally keep between \( m = 5,000 \) and 20,000 states in each DMRG block. This is found to give excellent convergence in the measurements such as the ground state energy and various correlation functions defined below, with small errors which can be neglected safely for our sizes up to \( L_x = 48 \). The phase boundaries in the \((J/t, K/t)\)-parameter space are determined by extensive scans of the derivatives of the ground state energy and by monitoring the correlation functions. On the other hand, as we will describe later, even with such a large \( m \) we can converge the entanglement entropy only for sizes up to \( L_x = 36 \).

To characterize the ground state properties of the system, as well as properties of the variational wave functions, we calculate the electron Green’s function

\[
G_e(r_i - r_j) = \langle c_{i s}^\dagger c_{j s} \rangle, \tag{1}
\]

where \( e = c \) with \( s = \uparrow, \downarrow \) the electron spin (there is no implied summation over \( s \)). For our electron Green’s function calculations and analysis, we fix \( s \) to one of the two possible flavors of spin, say \( s = \uparrow \), which in the spin-singlet states considered in this work, gives the same Green’s function as the other flavor of spin. The Fourier transform gives us the electron momentum distribution function

\[
\langle c_{i q s}^\dagger c_{j q s} \rangle = \frac{1}{L_x L_y} \sum_{ij} e^{i\mathbf{q} \cdot (r_i - r_j)} \langle c_{i s}^\dagger c_{j s} \rangle. \tag{2}
\]

Similarly, we calculate the electron density-density structure factor in momentum space

\[
\langle \delta n_{\mathbf{q}} \delta n_{-\mathbf{q}} \rangle = \frac{1}{L_x L_y} \sum_{ij} e^{i\mathbf{q} \cdot (r_i - r_j)} \langle (n_i - \rho)(n_j - \rho) \rangle, \tag{3}
\]

where \( n_i = \sum_s c_{is}^\dagger c_{is} \) is the electron number operator at site \( i \) and \( \rho \) is the electron density. To characterize the magnetic properties of the system, we also study the spin structure factor

\[
\langle \mathbf{S}_{\mathbf{q}} \cdot \mathbf{S}_{-\mathbf{q}} \rangle = \frac{1}{L_x L_y} \sum_{ij} e^{i\mathbf{q} \cdot (r_i - r_j)} \langle \mathbf{S}_i \cdot \mathbf{S}_j \rangle, \tag{4}
\]

where the spin operator is defined as \( \mathbf{S}_i = \frac{1}{2} \sum_{a, a'} c_{i a}^\dagger \mathbf{\sigma}_{aa'} c_{i a'} \).

Finally, we measure Cooper pair correlations

\[
G_{\text{Cooper}}[r_i, r_j; r'_k, r'_l] = \langle P[i, j]^\dagger P[k, l] \rangle, \tag{5}
\]

where a Cooper pair operator residing on some nearby sites \( i \) and \( j \) is defined by

\[
P[i, j] = \frac{1}{\sqrt{2}} (c_{ij}^\dagger c_{ij} - c_{ij} c_{ij}^\dagger) = P[j, i]. \tag{6}
\]

We will specifically be interested in diagonal d-wave Cooper pairs as detailed in Sec. II D. Their correlations can be accessed by considering the combination \( G_{\text{Cooper}}[(x, 1), (x + 1, 2); (x', 1), (x' + 1, 2)] - G_{\text{Cooper}}[(x, 1), (x + 1, 2); (x', 2), (x' + 1, 1)] \); it is this specific quantity which we plot below in Fig. 5.

B. Details of the VMC calculations

The central task of the VMC analysis is to construct a pool of variational states for which to collect data that can be compared with the DMRG results. The determinant VMC with the d-metal wave functions is straightforward: Measurements in the VMC simulation of the energy and correlation functions described above are all accomplished by averaging over relative probability amplitudes corresponding to one or two particle hops. These are simply ratios of determinants with one or two changed columns, which can be computed efficiently.
using the previously established methods in Ref. 1. The one exception is the density-density correlator, which merely requires averaging over the product of density operators. These are measured simply by checking for the presence of particles on the relevant sites and therefore involve no relative probability amplitudes.

In principle, one can be quite exhaustive and consider all possible band fillings of the two bands for each of the four partons in the $d$-metal wave function, Eq. (5) of the main text, on the two-leg ladder. Additionally, there exists the freedom to adjust the boundary conditions for each of the partons so long as the overall conditions are periodic for the electron wave function. There are four combinations that accomplish this. Listing the boundary conditions in order of $d_1$, $d_2$, $f_s$, we have: (1) periodic-periodic-periodic, (2) antiperiodic-antiperiodic-periodic, (3) periodic-antiperiodic-antiperiodic, (4) antiperiodic-periodic-antiperiodic. Finally, by reading the boundary conditions in order of $d_1$ and $d_2$ determinants are set to one. For each of these states, there is then the flexibility to adjust these exponents continuously. In attempting to match DMRG results, as in Fig. 5 of the main text, after choosing the correct bare state, we tune the exponents until the features visually match as well as possible. Admittedly, this process is a bit subjective, especially since gaining agreement of some singular features comes at the expense of poorer agreement of other singular features. However, an energetics analysis, with respect to the varied exponents but within a fixed bare state, reveals that the chosen states have energies that are very close to the rather robust energetic minima. Performing energy minimization over all possible orbital fillings usually gives a state that is off by one or two filled orbitals, but again the energies of our chosen states matching the DMRG results are not significantly different from the optimal energy. In Fig. 5 of the main text, the chosen exponents are $p_1 = 0.7$ and $p_2 = -0.4$. Indeed a negative exponent on the $d_2$ determinant is required to achieve a good match with the DMRG. This corresponds to negating the overly strong Pauli repulsion coming from all determinantal factors of the wave function. We can gain further understanding of the properties of our VMC wave functions from the $d$-metal gauge theory described in Sec. II below.

C. Entanglement entropy results

The Rényi entanglement entropies are defined by

$$S_\alpha(\rho_A) = \frac{1}{1-\alpha} \ln \left[ \text{Tr} \left( \rho_A^{\alpha} \right) \right],$$

where $\rho_A$ is the reduced density matrix of a subregion $A$ of the lattice. The limit $\alpha \to 1$ results in the familiar von Neumann entropy $S_1 = -\text{Tr} (\rho_A \ln \rho_A)$.

According to conformal field theory, any given Rényi entropy is expected to scale as

$$S^\mathrm{CFT}_{\alpha}(X, L_x) = \frac{c}{6} \left( 1 + \frac{1}{\alpha} \right) \ln \left[ \frac{L_x}{\pi} \sin \left( \frac{\pi X}{L_x} \right) \right] + c'_{\alpha},$$

for a quasi-1D gapless system in the ground state with periodic boundary conditions in the $\hat{x}$ direction. The subsystem length $X$ represents the number of contiguous rungs contained in the subsystem, the central charge $c$ is equal to the number of gapless modes, and $c'_{\alpha}$ is a non-universal constant.
On the two-leg ladder at $\rho = 1/3$, we expect $c = 2$ gapless modes in the conventional Luttinger liquid phase. In contrast, the bosonization approach to the unconventional $d$-metal phase (see Sec. II) predicts $c = 3$ gapless modes, as there are five partially-filled bands but two are rendered massive in the strong-coupling limit of the gauge theory. Thus, the central charge as determined by measuring the scaling of the entanglement entropy provides a crucial diagnostic for the presence of the non-Fermi liquid $d$-metal phase.

The DMRG has information about the full entanglement spectrum and hence can calculate any $S_x$ including the von Neumann entropy $S_1$, which is the focus of Fig. 7 of the main text. However, calculations of $S_x$ with DMRG are also very difficult to converge for highly entangled gapless systems such as the $d$-metal state, and while we were able to well converge the von Neumann entropy for $L_x = 24$ and 36 length systems, we were unable to obtain convergence for the $L_x = 48$ system. Specifically, for the $L_x = 48$ system at $J/t = 2$ and $K/t = 1.8$ as focused on in the main text, e.g., in Fig. 5, fitting to the von Neumann entropy gives $c = 2.75$ even when keeping $m = 20,000$ states. We estimate that the entropy data near the middle of the sample, $X \simeq L_x/2$, is still a few percent away from full convergence, and, given the sensitivity of the fit parameter $c$ on such inaccuracies, we believe a result $c \simeq 3$ would be obtained in the limit $m \rightarrow \infty$.

There are also finite-size uncertainties associated with the range of $X$ values used in the fits to Eq. (8). For all data in Fig. 7 of the main text, we have restricted the fits to data $X_{\text{min}} = 5 \leq X \leq L_x/2$ in order to focus on the long-distance scaling behavior. Still, at these system sizes, the extracted values of $c$ do depend on $X_{\text{min}}$, and this by itself produces a further uncertainty in $c$ of a few percent.

In light of the abovementioned uncertainties associated with extracting $c$ with the DMRG, and with the goal of further bolstering our arguments for the central charge $c = 3$ within the $d$-metal on larger systems, we have ventured to compute the entanglement entropy in our variational wave functions and make concrete comparisons to the DMRG data. Indeed, by measuring the expectation value of a “swap operator,” the VMC can be used to calculate the Rényi entropies $S_x$ for integer $\alpha \geq 2$. As such, we now focus on the scaling of $S_x$ in both the VMC and DMRG for points within the $d$-metal phase.

Let us consider the characteristic $d$-metal point examined in the main text on the two-leg ladder at $J/t = 2$ and $K/t = 1.8$, with $L_x = 48$. The scaling of the VMC and DMRG Rényi entropies is plotted in Fig. 1. There, we observe a striking agreement in both the overall magnitudes and detailed features of the two data sets. Indeed, the oscillations in $S_2$, which are expected to occur for all but the von Neumann entropy, are in sync with each other. Furthermore, a linear fit gives $c = 2.95$ for the VMC and $c = 2.75$ for the DMRG, in reasonably close agreement with the expected value of $c = 3$ for the $d$-metal phase. For large $X$, the DMRG entropies lag slightly below the VMC entropies: As with the von Neumann entropy data discussed above, the DMRG result did not fully converge despite having kept $m = 20,000$ states. (In order to attempt to average over the relatively large oscillations in $S_2$ for small $X$, these quoted values of $c$ used all available data $X_{\text{min}} = 2 \leq X \leq L_x/2$ in the fits.)

On smaller lattices, for instance $L_x = 36$ at $J/t = 2$ and $K/t = 2$ (parameters chosen so that the ground state is singlet), the DMRG result is further converged and the agreement is even better, as seen in Fig. 2. Here, a scaled-down version of the $d$-metal VMC state was used, with $N_{d1}^{(0)} = 15$ and $N_{d1}^{(1)} = 9$, exponents $p_1 = 0.7$ and $p_2 = -0.4$, and $N_{d1}^{(0)} = N_{d1}^{(1)} = 12$. A linear fit gives $c = 2.95$ for the VMC and $c = 2.97$ for the DMRG, in excellent agreement with $c = 3$. It is remarkable that the VMC trial wave function is able to reproduce the DMRG data so well and that both give the predicted value of $c$ to such high accuracy.
II. LONG-WAVELENGTH DESCRIPTION OF THE d-METAL PHASE

In this part, we develop a long-wavelength theory of the d-metal phase and provide detailed characterization of the main observables, including the electron Green’s function, density and spin correlations, as well as Cooper pair correlations. We illustrate such an analysis with the DMRG and VMC data in the d-metal phase. We conclude with a brief discussion of the instabilities of the d-metal.

A. Gauge theory description and solution by bosonization

We start with the decomposition of the electron operator into three fermionic partons as in Eq. (1) of the main text. We assume that the $d_1$ partons hop preferentially along the ladder direction and partially populate both bonding and antibonding bands, while the $d_2$ partons hop preferentially in the transverse direction and populate only the bonding band. The densities of the $d_1$ and $d_2$ partons are the same and equal to the total electron density (i.e., including spin). We assume that the spinons $f\uparrow$ and $f\downarrow$ populate only the bonding band, and the $\uparrow$ and $\downarrow$ populations are identical so that the state is spin-singlet. The parson occupations are illustrated in Fig. 2 of the main text. The Fermi wavevectors satisfy

$$k^{(0)}_{Fd1} + k^{(\pi)}_{Fd1} = k_{Fd2} = 2k_{FF} = 2\pi\rho,$$  \hspace{1cm} (9)

where $\rho = N_e/(2L_x)$ is the total electron density per site. When discussing the electron distribution function below, it will be convenient to parametrize the $d_1$ Fermi wavevectors as

$$k^{(k_\sigma)}_{Fd1} = \pi\rho + e^{ik\cdot K} , \hspace{1cm} k_y = 0, \pi.$$  \hspace{1cm} (10)

(The same $K$ was already used in the main body.)

We begin with the mean field Hamiltonian in continuum. Linearizing near the Fermi points, the kinetic energy density reads

$$h_{kinetic} = \sum_{k_y=0,\pi} \sum_{P=R/L} P v^{(k_\sigma)}_{d1} d^{(k_\sigma)}_{1P}(-i\partial_x) d^{(\bar{k}_\sigma)}_{1P}$$  \hspace{1cm} \hspace{1cm} (11)

$$+ \sum_{P=R/L} P v^{(\bar{k})}_{d2} d^{(\bar{k})}_{2P}(-i\partial_x) d^{(k)}_{2P}$$  \hspace{1cm} \hspace{1cm} (12)

$$+ \sum_{s=\uparrow,\downarrow} \sum_{P=R/L} P v^{(s)}_{f} f^{(s)}_{1P}(-i\partial_x) f^{(\bar{s})}_{1P}.$$  \hspace{1cm} \hspace{1cm} (13)

Here, superscript $k_\sigma = 0$ or $\pi$ for $d_1$ corresponds to bonding or anti-bonding respectively; $P = R/L = +/-$ refers to right or left moving fields; $v_\sigma$ is the Fermi velocity for fermion “band” $b \in \{ d^{(0)}_1, d^{(\pi)}_1; d_2; f_\uparrow; f_\downarrow \}$.

Going beyond the kinetic energy, we also consider four-fermion interactions, which are required to respect lattice symmetries, time reversal, and spin rotation invariance. First, we have various density-density interactions in terms of the continuum densities $\rho_{bP}$ for fermion species $b$. These are strictly marginal (harmonic in the bosonized treatment) and renormalize band velocities as well as Luttinger parameters. Since the final fixed-point theory will have most general form consistent with the symmetries, we do not spell out the microscopic parton density-density terms.

Next, we have the following four-$d_1$-chargon term:

$$h_{int,4d_1} = w \left[ c_{1R}^{(0)\dagger} d^{(0)}_{1L} \tilde{d}^{(\pi)}_{1L} + H.c. \right] + H.c.$$  \hspace{1cm} (14)

$$= 2w \cos(2\sqrt{2}\varphi_{d1}) .$$  \hspace{1cm} (15)

The last line shows bosonized expression using fields that will be defined below; as we will describe later, this non-harmonic interaction can destabilize the “relative charge sector” “$d1-$”.

We also have the following four-spinon term:

$$h_{int,4f} = -u \left( f_{1L}^\dagger \sigma^a f_R^\dagger \right) \left( f_{1L}^\dagger \sigma^a f_R \right)$$  \hspace{1cm} (16)

$$= \frac{u^2}{8\pi^2} \left[ (\partial_x \varphi_{f\sigma})^2 - (\partial_y \theta_{f\sigma})^2 \right] + u^4 \cos(2\sqrt{2}\theta_{f\sigma}) .$$  \hspace{1cm} (17)

Again, the last line shows bosonized expression using fields that will be defined below; this non-harmonic interaction can destabilize the “spin sector” “$f\sigma$”.

The above exhausts residual four-fermion interactions at generic electron densities and generic $b^{(0/\pi)}_{Fd1}$. On the other hand, at commensurate densities we should also include umklapps. For example, at density $\rho = 1/3$ studied in the DMRG in the main text, there is a single umklapp term

$$h_{int, umkl} = u_{umkl} \left( d_{2R}^\dagger d_{2L} \sum_s f_{sR}^\dagger f_{sL} + H.c. \right)$$  \hspace{1cm} (17)

$$= -4u_{umkl} \cos(2\theta_{f\sigma}) \cos \left( \frac{3\theta_{rot}}{\sqrt{2}} + \frac{\theta_A}{\sqrt{6}} - \frac{2\theta_{f\sigma}}{\sqrt{3}} \right).$$  \hspace{1cm} (18)

If this umklapp term is relevant, it leads to a Mott insulator (CDW) of electrons, while it must be irrelevant in the metallic phase. The absence of any charge ordering in the d-metal phase found in the DMRG at $\rho = 1/3$ will allow us to crudely bound power laws in some correlations and check the overall consistency of the $d$-metal theory, while the umklapp can be always rendered inoperative if we step away from the commensurate density.

The full theory beyond mean field is a parton-gauge theory. Unlike the 2D case, in quasi-1D the parton-gauge theory can be solved by bosonization. More specifically, treating the gauge field fluctuations on long wavelengths amounts to implementing the constraints Eq. (2) of the main text for the coarse-grained densities along the ladder direction. Since the fixed-point Hamiltonian is harmonic in the bosonized fields and the coarse-grained densities are linear functions of the bosonized fields, implementing the constraints becomes simple.

Our bosonization treatment is as follows. We bosonize each fermion band species $b_{Rs/L} (F_b = \{ d^{(0)}_1; d^{(\pi)}_1; d_2; f_\uparrow; f_\downarrow \})$

$$F_{bP} = \eta_b e^{i(\varphi_{bP} + P\theta_{bP})}.$$  \hspace{1cm} (19)
with canonically conjugate boson fields:

\[
[\varphi_b(x), \varphi_b'(x')] = [\theta_b(x), \theta_b'(x')] = 0, \quad (20)
\]
\[
[\varphi_b(x), \theta_b'(x')] = i\pi\delta_{bb'} \Theta(x - x'), \quad (21)
\]
where \(\Theta(x)\) is the Heaviside step function. Here, we have introduced Klein factors, the Majorana fermions \(\{\eta_b, \eta_b'\} = 2\delta_{bb'}\), which assure that the fermion fields with different flavors anti-commute with one another. The slowly varying fermionic densities are simply \(\rho_{\varphi b} \equiv F_{\varphi b} F_{\varphi b'} = \partial_b (P \varphi_b + \theta_b) / (2\pi)\), and hence \(\eta_b \equiv \rho_{\varphi R} + \rho_{\varphi L} = \delta_b \theta_b / \pi\).

In the mean field, we start with five modes, \(\theta_{d1,1/2}, \theta_{d2}, \text{ and } \theta_{f1/2}\). Including gauge fluctuations is simple in the bosonized formulation, they effectively implement the constraints Eq. (2) of the main text for the coarse-grained densities and produce the following pinning of the dual fields (up to fixed constant shifts):

\[
\theta_{d1}^{(0)} + \theta_{d1}^{(\pi)} = \theta_{d2} = \theta_{f1} + \theta_{f1/2}. \quad (22)
\]

Formally, we perform an orthonormal transformation on the \(\theta\) fields:

\[
\theta_{f\sigma} = \frac{1}{\sqrt{2}} \left( \theta_{f1} - \theta_{f1/2} \right), \quad (23)
\]
\[
\theta_{d1-} = \frac{1}{\sqrt{2}} \left( \theta_{d1}^{(0)} - \theta_{d1}^{(\pi)} \right), \quad (24)
\]
\[
\theta_{\rho\text{tot}} = \frac{1}{2\sqrt{2}} \left( \theta_{f1} + \theta_{f1/2} + \theta_{d1}^{(0)} + \theta_{d1}^{(\pi)} + 2\theta_{d2} \right), \quad (25)
\]
\[
\theta_a = \frac{1}{\sqrt{3}} \left( \theta_{d1}^{(0)} + \theta_{d1}^{(\pi)} - \theta_{d2} \right), \quad (26)
\]
\[
\theta_A = \frac{\sqrt{3}}{2\sqrt{2}} \left( \theta_{f1} + \theta_{f1/2} - \theta_{d1}^{(0)} + \theta_{d1}^{(\pi)} + 2\theta_{d2} / 3 \right). \quad (27)
\]

We perform the same transformation on the \(\varphi\) fields, so the new \(\theta\) and \(\varphi\) fields are again canonically conjugate. Gauge field fluctuations render the combinations \(\theta_a\) and \(\theta_A\) in the last two lines massive, effectively leading to the pinnings in Eq. (22), and only the \(\theta_{f\sigma}\), \(\theta_{d1-}\), and \(\theta_{\rho\text{tot}}\) modes remain. Note that the definition of \(\theta_{\rho\text{tot}}\) is fixed once we define the convenient fields \(\theta_{f\rho}\) and \(\theta_{d1-}\) and require orthogonality among these modes and orthogonality to the linear space of equations (22). On the other hand, the definitions of \(\theta_a\) and \(\theta_A\) are somewhat arbitrary (roughly, massiveness of \(\theta_a\) corresponds to gluing \(d_1\) and \(d_2\) parts to form the bosonic chargon \(b\), while massiveness of \(\theta_A\) then glues the \(b\) and \(f\); we can take any orthonormal linear combination of \(\theta_a\) and \(\theta_A\) and such a choice does not affect the \(\theta_{f\rho}\), \(\theta_{d1-}\), and \(\theta_{\rho\text{tot}}\) content of any physical operator below.

In the fixed-point theory, the “\(f\sigma\)” (spin) sector decouples from the “\(d1-\)” and “\(\rho\text{tot}\)” (charge) sectors:

\[
\mathcal{L} = \mathcal{L}_1[\theta_{f\rho}; \varphi_{f\rho}; \varphi_{f\sigma}] + \mathcal{L}_1[\theta_{d1-}; \varphi_{d1-}; \varphi_{\rho\text{tot}}], \quad (28)
\]
\[
\mathcal{L}_1[\theta; \varphi; g] = \frac{1}{2\pi g} \left[ v(\partial_x \theta)^2 + \frac{1}{v} (\partial_x \theta)^2 \right]. \quad (29)
\]

The last line is a generic Lagrangian for a single mode with velocity \(v\) and Luttinger parameter \(g\) written in Euclidean space-time. In the spin sector, \(g_{f\rho} = 1\) is fixed by the condition of SU(2) spin invariance. The residual interaction Eq. (16) is marginally irrelevant if \(v > 0\).

On the other hand, in the charge sectors, we have most general quadratic Lagrangian

\[
\mathcal{L}[\theta_{d1-}, \varphi_{\rho\text{tot}}] = \frac{1}{2\pi} \left[ \partial_x \Theta^T \cdot A \cdot \partial_x \Theta + \partial_x \Theta^T \cdot B \cdot \partial_x \Theta \right], \quad (29)
\]

where we defined \(\Theta^T \equiv (\theta_{d1-}, \varphi_{\rho\text{tot}})\). In general, \(A\) and \(B\) can be arbitrary positive-definite symmetric matrices.

To get some feel for the charge sectors, we can consider the case where the only interactions between different parton species are due to gauge fluctuations enforcing the constraints Eq. (22). For more generality, we allow density-density interactions within each parton type and encode these in the corresponding renormalized velocities \(v_b\) and Luttinger parameters \(g_b\). The matrices \(A\) and \(B\) are readily evaluated to be

\[
A_{11} = \frac{1}{2} \left( \frac{v_{d1}^{(0)}}{g_{d1}} + \frac{v_{d1}^{(\pi)}}{g_{d1}} \right), \quad (30)
\]
\[
A_{22} = \frac{1}{8} \left( \frac{v_{d1}^{(0)}}{g_{d1}} + \frac{v_{d1}^{(\pi)}}{g_{d1}} + \frac{v_{d2}}{g_{d2}} + \frac{2v_f}{g_f} \right), \quad (31)
\]
\[
A_{12} = A_{21} = \frac{1}{4} \left( \frac{v_{d1}^{(0)}}{g_{d1}} \frac{v_{d1}^{(\pi)}}{g_{d1}} \right), \quad (32)
\]
\[
B_{11} = \frac{1}{2} \left( \frac{v_{d1}^{(0)}}{v_{d1}^{(\pi)}} + \frac{1}{g_{d1} v_{d1}^{(\pi)}} \right), \quad (33)
\]
\[
B_{22} = \frac{1}{8} \left( \frac{v_{d1}^{(0)}}{v_{d1}^{(\pi)}} + \frac{1}{g_{d1} v_{d1}^{(\pi)}} + \frac{4}{g_{d2} v_{d2}} + \frac{2}{g_f v_f} \right), \quad (34)
\]
\[
B_{12} = B_{21} = \frac{1}{4} \left( \frac{1}{g_{d1} v_{d1}^{(\pi)}} - \frac{1}{g_{d1} v_{d1}^{(\pi)}} \right). \quad (35)
\]

As an example, if we take all \(g_b = 1\) but allow general velocities \(v_b\), we can check that the scaling dimension of the interaction Eq. (14) is greater than 2 and hence this interaction is irrelevant. At generic incommensurate electron density, this is the only allowed non-linear interaction other than the spin interaction Eq. (16); the latter can be marginally irrelevant, and hence the \(d\)-metal phase can be stable as a matter of principle.

For illustrations later in the text, we will consider a schematic model where the “\(d1-\)” and “\(\rho\text{tot}\)” modes also decouple, so

\[
\mathcal{L} = \mathcal{L}_1[\theta_{f\rho}; \varphi_{f\rho}; g_{f\rho}] + \mathcal{L}_1[\theta_{d1-}; \varphi_{d1-}; g_{d1-}] + \mathcal{L}_1[\theta_{\rho\text{tot}}; \varphi_{\rho\text{tot}}; g_{\rho\text{tot}}]. \quad (36)
\]
\[
\mathcal{L}_1[\theta; \varphi; g] = \frac{1}{2\pi g} \left[ v(\partial_x \theta)^2 + \frac{1}{v} (\partial_x \theta)^2 \right]. \quad (37)
\]

This situation arises, e.g., in the above case with gauge-only interactions, Eqs. (30)-(35), if we further require \(v_{d1}^{(0)} = v_{d1}^{(\pi)} = v_{d1}\) and \(g_{d1}^{(0)} = g_{d1}^{(\pi)} = g_{d1}\), where we find

\[
g_{d1-} = g_{d1}, \quad (38)
\]
\[
g_{\rho\text{tot}} = 4 \left( \frac{v_{d1}^{(0)}}{g_{d1}} + \frac{2v_{d2}}{g_{d2}} + \frac{v_f}{g_f} \right)^{-1/2} \times \left( \frac{1}{g_{d1} v_{d1}^{(\pi)}} + \frac{2}{g_{d2} v_{d2}} + \frac{1}{g_f v_f} \right)^{-1/2}. \quad (39)
\]
The schematic model holds approximately when the $d_1$ bonding and anti-bonding populations are approximately equal. Furthermore, we think it is valid for the “bare Gutzwiller” wave function, cf. Sec. 1B, since the projection does not know about the intra-species interactions and band velocities; in this case, it is natural to set all bare Luttinger parameters equal to $g_0 = 1$ and all velocities equal, and we obtain $g_{d1} = g_{pot} = 1$. If we require all bare $g_0 = 1$ but allow general velocities, Eq. (39) would give $g_{pot} \lesssim 1$.

Here we note that the $d$-metal phase found in the DMRG is roughly consistent with the approximation of decoupled “$d1$--” and “$pot$” modes with $g_{pot}$ significantly larger than 1 (see our discussion of observables below). Also, as described in Sec. 1B, our optimal VMC wave functions have powers on the determinants and are significantly away from the bare Gutzwiller states. We can crudely model the effect of adding power on the $d_2$ determinant, $\psi_{d2}(\{R_1\}) \rightarrow |\psi_{d2}(\{R_1\})|^{p-1} |\psi_{d2}(\{R_1\})|$, by including the Luttinger parameter $g_{d2}$ in the bare Lagrangian for the $\theta_{d2}$ mode. The Slater determinant $\psi_{d2}$ fills only one band, and a crude guess from studies with such 1D Jastrow-Luttinger wave functions is that $g_{d2} = 1/p_2$ for $p_2 > 0$. In particular, $g_{d2} \rightarrow \infty$ for $p_2 \rightarrow 0$. In this limit, $g_{pot}$ in Eq. (39) can be as large as 2 even when $g_{d1} = g_1 = 1$. Below, we will use values $g_{d1} = 1$ and $g_{pot} = 2$ to illustrate such a “dressed Gutzwiller” wave function. Our VMC measurements with the bare and dressed Gutzwiller wave functions support the above conjectures for the effective Luttinger parameters in both cases.

Related to the above discussion, we remark that the DMRG does not find any translational symmetry breaking that could be driven by the umklapp Eq. (18) at density $\rho = 1/3$. In the approximation of decoupled “$d1$--” and “$pot$” modes, the irrelevance of the umklapp requires $g_{pot} > 4/3$. This is consistent with the large effective $g_{pot}$ deduced from the DMRG correlations and from the optimal VMC wave functions, see Sec. II E below.

Having discussed the general structure of the $d$-metal fixed point and some approximate models, we now turn to the characterization of the phase in terms of electron Green’s function, density and spin correlations, as well as Cooper pair correlations that are measured in the DMRG.

**B. Electron Green’s function**

We begin with the electron Green’s function $G_e(x,y)$, $e = e_\uparrow \uparrow/4$. In the mean field, we have

$$G_e^{mf}(x,y) = G_{d1}(x,y) G_{d2}(x) G_f(x),$$

$$G_e^{mf}(x,k_y) \sim \frac{\sin(k_{F1d1}x)}{x^3} \sin(k_{F2d2}x) \sin(k_{FJ}x).$$

In the first line, we used the fact that the $d_2$ and $f_J$ partons populate only the bonding band; hence the $y$-dependence comes only from the $d_1$ parton. In the second line, it is convenient to work with bonding or anti-bonding components, $k_y = 0$ or $\pi$. The general structure of the oscillating contributions is obtained by expanding the sines in the last equation,

$$G_e^{mf}(x,k_y) \approx \sum_{\alpha,\beta,\gamma=\pm} iA^{(k_y)}_{\alpha\beta\gamma} C^{(k_y)}_{\alpha\beta\gamma},$$

$$Q^{(k_y)}_{\alpha\beta\gamma} = \alpha k_{F1d1} + \beta k_{F2d2} + \gamma k_{FJ}$$

$$= \alpha e^{ik_y} k_c + (\alpha + 2\beta + \gamma) \pi \rho,$$

where we allowed general amplitudes $A^{(k_y)}_{\alpha\beta\gamma}$ and scaling dimensions $\Delta^{(k_y)}_{\alpha\beta\gamma}$. Contribution at the wavevector $Q^{(k_y)}_{\alpha\beta\gamma}$ comes from the following combination of long-wavelength parton fields,

$$d_{1\alpha}(k_y) d_{2\beta} f_{\gamma s} \sim e^{i\Phi^{(k_y)}_{\alpha\beta\gamma,s}} + \Phi^{(k_y)}_{\alpha\beta\gamma,s},$$

where $\alpha, \beta, \gamma = R/L = +/-; s = \uparrow \downarrow$; and

$$\Phi^{(k_y)}_{\alpha\beta\gamma,s} = \dot{\varphi}_{(k_y)} + k_{d1} \varphi_{d1} + k_{d2} \varphi_{d2} + k_{FJ} \varphi_{fJ}$$

$$- \frac{e^{ik_y}}{\sqrt{2}} \varphi_{d1} + \sqrt{2} \varphi_{d2} + \varphi_{fJ},$$

$$\Theta^{(k_y)}_{\alpha\beta\gamma; s} = \theta_{d1} \alpha + \beta \theta_{d2} + \gamma \theta_{fJ}$$

$$= \frac{e^{ik_y}}{\sqrt{2}} \theta_{d1} + \frac{\alpha + 2\beta + \gamma}{\sqrt{2}} \theta_{d2} + \frac{\gamma}{\sqrt{2}} \theta_{fJ} + \frac{\alpha - \beta}{\sqrt{3}} \theta_{a} + \frac{-\alpha - 2\beta + 3\gamma}{\sqrt{6}} \theta_{A}.$$

The last line contains $\theta_{a}$ and $\theta_{A}$ that are pinned upon including gauge fluctuations, and from here on we often drop these fields when they only modify overall complex phases of the operator expressions. For ease of reference, Table I lists different cases where we fix $\alpha = +$. We will consider these cases and entries in the table after some more general discussion. We are primarily interested in the scaling dimensions $\Delta^{(k_y)}_{\alpha\beta\gamma}$.

First, we note that the spin sector “$f_J$” is decoupled from the rest and has $g_{fJ} = 1$; the corresponding fields contribute 1/4 to the scaling dimension of the electron correlation for any wavevector $Q^{(k_y)}_{\alpha\beta\gamma}$. Thus, the scaling dimension depends only on the “$d1$--” and “$pot$” content of both $\Phi^{(k_y)}_{\alpha\beta\gamma,s}$ and $\Theta^{(k_y)}_{\alpha\beta\gamma,s}$ fields. We expect that the main difference among the cases comes from the $\theta_{pot}$ content, since its coefficient $(\alpha + 2\beta + \gamma)/(2\sqrt{2})$ can vary in magnitude from 0 for $\alpha = \gamma = -\beta$ to $\sqrt{2}$ for $\alpha = \gamma = \beta$; this content is listed in one of the columns in Table I. Note, however, that in general the relative signs of the “$d1$--” and “$pot$” components are also important and affect the scaling dimensions, thus making the scaling dimensions also depend on $k_y$; this is indeed what we find from fitting the DMRG data (see Sec. II E). It is only in the case when the “$d1$--” and “$pot$” decouple that the scaling dimensions do not depend on the signs of the coefficients.

Second, by using the general result $\Delta^{(k_y)}(\sum_{\alpha,\beta,\gamma} \Phi^{(k_y)}_{\alpha\beta\gamma,s}) \geq \frac{1}{2} |\sum_{\alpha,\beta,\gamma} a_j b_j|$ valid for any canonically conjugate set $\varphi_{d1}$, $\varphi_{d2}$, $\varphi_{fJ}$, and any coefficients $a_j$, $b_j$, we can obtain an exact general bound

$$\Delta^{(k_y)} \geq \frac{1}{4} + \frac{|2\alpha + 2\beta + \gamma|}{4},$$

obtained by expanding the sines in the last equation.
which is also listed in Table I.

Third, we can calculate the scaling dimensions in the approximation that the “$d1−$” and “$ρtot$” modes decouple:

\[
\begin{align*}
\Delta^{(k_y)}_{\alpha \beta \gamma} &= \Delta_{++-} + \frac{(\alpha + 2\beta + \gamma)^2}{32}g_{\text{tot}}, \quad (51) \\
\Delta_{+++} &= \frac{1}{4} + \frac{1}{8} \left( \frac{1}{g_{d1−}} + \frac{1}{g_{d1−}} \right) + \frac{1}{2g_{\text{tot}}}. \quad (52)
\end{align*}
\]

This illustrates the preceding discussion of the dependence of the scaling dimensions on the $\theta_{\text{tot}}$ content. By setting $g_{d1−} = g_{\text{tot}} = 1$, we obtain scaling dimensions in the bare Gutzwiller wave function also listed in the table. On the other hand, by setting $g_{d1−} = 1$ and $g_{\text{tot}} = 2$, we obtain scaling dimensions in the dressed Gutzwiller wave function (tentatively corresponding to powers on the determinants $p_1 = 1$ and $p_2 = 0$); these dimensions are listed in the last column in Table I. Power law fits for the electron Green’s function measured in the VMC are consistent with these predictions.

Let us now consider different entries in Table I. We can roughly understand the trends in the scaling dimensions by appealing to so-called “Amperean” rules for interactions mediated by gauge fields: \textsuperscript{13-15} Parallel currents attract and hence processes containing such currents are enhanced, while anti-parallel currents repel and hence such processes are suppressed. The splitting of the electron into three partons leads to two gauge fields. We again use the picture where we first break the electron into the spinon $f$ and chargon $b$, and then break the chargon $b$ into the partons $d_1$ and $d_2$. We can then think of the two gauge fields as follows. The first gauge field works to glue the $d_3$ and $d_2$ together to form the chargon; the two partons carry opposite gauge charges with respect to this gauge field, hence processes that contain the $d_1$ and $d_2$ moving in opposite (same) directions are enhanced (suppressed). The enhanced combinations produce the main features in the boson distribution function in the so-called DBL[2,1] phase of Ref. 3 at wavevectors $(k_F, d - k_F, k_y)$ marked in the top panel in Fig. 3. The second gauge field works to glue the chargons and spinon to form the electron; the $d_{1/2}$ and $f$ carry opposite gauge charges with respect to this gauge field, and hence processes that contain $d_{1/2}$ and $f$ moving in opposite (same) directions are enhanced (suppressed).

The precise mathematics in $(1 + 1)D$ is that the gauge field fluctuations pin the fields $\theta_e$ and $\theta_A$ and hence can reduce the fluctuating content in Eq. (47), particularly for the Amperean-enhanced combinations. The first two rows in Table I correspond to oppositely moving $d_1$ and $d_2$ partons. From the exact lower bound, we see that these entries can be potentially more enhanced than the other two rows. In the first row, the spinon moves parallel to the $d_1$ and anti-parallel to the $d_2$, while in the second row the situation is interchanged; both situations produce the same exact lower bound, but the first one is apparently more enhanced in the approximate model with decoupled “$d1−$” and “$ρtot$” modes. The third row in Table I has the $d_1$ and $d_2$ partons moving in the same direction, but the corresponding Amperean suppression is somewhat compensated by the spinon moving in the opposite direction to the two; in the approximate model with decoupled “$d1−$” and “$ρtot$”, the second and third row have the same scaling dimension. The last row in Table I has all partons moving in the same direction; this combination is suppressed by all gauge field fluctuations, and we see that it has the largest scaling dimension, which is always larger than the mean field value of 3/2.

We conclude with a simple understanding of the main landscapes in the electron distribution function, as measured in the DMRG and VMC in Fig. 5(a) of the main text. This discussion also re-iterates the origin of the wavevectors in the first two rows in Table I. We can take the chargon distribution function from the earlier DBL[2,1] study.\textsuperscript{3} This has dominant features at wavevectors obtained by combining oppositely moving $d_1$ and $d_2$ partons. For simplicity, we approximate the dominant features by $\delta$-functions as shown in the top panel in Fig. 3. For the spinons $f$, we take a simple step distribution shown in the middle panel in Fig. 3. The electron distribution function is a convolution of the chargon and spinon distribution functions. The result is shown in the bottom panel in Fig. 3. The inner steps arise from combinations where $f$ is moving parallel to the $d_1$ and anti-parallel to the $d_2$ and correspond precisely to the first row in Table I, while the outer steps arise from combinations where $f$ is moving anti-parallel to the $d_1$ and parallel to the $d_2$ and correspond to the second row in Table I. We can see from Table I that the inner singularities are likely stronger than the outer ones, particularly if $g_{\text{tot}}$ is large, and this is consistent with the DMRG findings in the main text.

---

**Table I:** Analysis of different contributions, Eq. (44), to the electron operator. We list the wavevector $Q_{\alpha\beta\gamma}$, the coefficient of $\theta_{\text{tot}}$ in Eq. (47), which strongly affects the scaling dimension $\Delta^{(k_y)}_{\alpha\beta\gamma}$, exact lower bound on $\Delta^{(k_y)}_{\alpha\beta\gamma}$, the scaling dimension $\Delta^{(k_y)}_{\alpha\beta\gamma}$ in the approximation of decoupled “$d1−$” and “$\rho\text{tot}$” modes, with $\Delta_{++-}$ given in Eq. (52); specialization to $g_{d1−} = g_{\text{tot}} = 1$ appropriate for the bare Gutzwiller wave function; and specialization to $g_{d1−} = g_{\text{tot}} = 1$, $g_{\text{tot}} = 2$ appropriate for the dressed Gutzwiller wave function that roughly captures the DMRG structure factors (see text for details). Note that in general the scaling dimensions also depend on $k_y$.

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$\beta$</th>
<th>$\gamma$</th>
<th>$Q_{\alpha\beta\gamma}$, Eq. (43)</th>
<th>$\theta_{\text{tot}}$ coeff.</th>
<th>$\Delta^{(k_y)}_{\alpha\beta\gamma}$, exact bound</th>
<th>$\Delta^{(k_y)}_{\alpha\beta\gamma}$, Eq. (50), crude model</th>
<th>$\Delta^{(k_y)}_{\alpha\beta\gamma}$, bare Gutzwiller</th>
<th>$\Delta^{(k_y)}_{\alpha\beta\gamma}$, dressed Gutzwiller</th>
</tr>
</thead>
<tbody>
<tr>
<td>+</td>
<td>+</td>
<td>+</td>
<td>$e^{ik_y} K_{\alpha\beta\gamma}$</td>
<td>0</td>
<td>$\ge 1/2$</td>
<td>$\Delta_{++-}$</td>
<td>1</td>
<td>3/4</td>
</tr>
<tr>
<td>+</td>
<td>-</td>
<td>-</td>
<td>$-2\pi p + e^{ik_y} K_{\alpha\beta\gamma}$</td>
<td>$-\frac{1}{2}$</td>
<td>$\ge 1/2$</td>
<td>$\Delta_{++-} + g_{\text{tot}}/8$</td>
<td>9/8</td>
<td>1</td>
</tr>
<tr>
<td>+</td>
<td>+</td>
<td>-</td>
<td>$2\pi p + e^{ik_y} K_{\alpha\beta\gamma}$</td>
<td>$\frac{1}{2}$</td>
<td>$\ge 1$</td>
<td>$\Delta_{++-} + g_{\text{tot}}/8$</td>
<td>9/8</td>
<td>1</td>
</tr>
<tr>
<td>+</td>
<td>+</td>
<td>+</td>
<td>$4\pi p + e^{ik_y} K_{\alpha\beta\gamma}$</td>
<td>$\sqrt{2}$</td>
<td>$\ge 3/2$</td>
<td>$\Delta_{++-} + g_{\text{tot}}/2$</td>
<td>3/2</td>
<td>7/4</td>
</tr>
</tbody>
</table>
operator that have the right symmetry properties:

\[
n_{(2k_Fd_2,0)} \equiv d_{2L}^* d_{2R} \sim i e^{i \sqrt{2} \theta_{\text{tot}}}, \\
n_{(2k_Fd_1,0)} \equiv d_{1L}^* d_{1R} \sim i e^{i (\pm \sqrt{2} \theta_{d_1} - \frac{\theta_{\text{tot}}}{\sqrt{2}})} , \\
n_{(k_{FD_1}^{(0)},k_{FD_1})} \equiv d_{1L}^* d_{1R} + d_{1L}^* d_{1R} \sim -i \eta_1^0 \eta_1^1 \sin(\sqrt{2} \phi_{d_1}) e^{i \frac{\theta_{\text{tot}}}{\sqrt{2}}}, \\
n_{(k_{FD_1}^{(0)}-k_{FD_1}),\pi} \equiv d_{1L}^* d_{1R} + d_{1L}^* d_{1R} \sim -i \eta_1^0 \eta_1^1 (\pi) \sin(\sqrt{2} \phi_{d_1}) e^{i \sqrt{2} \theta_{d_1} - \frac{\theta_{\text{tot}}}{\sqrt{2}}}, \\
n_{(2k_Ff,0)} \equiv \sum_s f_{sL} s_{sR} \sim i \cos(\sqrt{2} \theta_{f}) e^{i \frac{\theta_{\text{tot}}}{\sqrt{2}}}. 
\]

In the second line, ± refers to $e^{ik_y}$. Note also that at $\rho = 1/3$ we have $2k_{Fd_2} = -2k_{Ff}$, and the product $n_{(2k_Fd_2,0)}n_{(2k_Ff,0)}$ gives the umklapp term Eq. (18); however, at any other density these are distinct wavevectors. Combinations similar to those in Eqs. (55) and (57) but with minus sign between the terms contribute to a current operator rather than the density fluctuation and are not spelled out here.

We can similarly obtain fermionic bilinear contributions to the electron spin operator, e.g.,

\[
S_{(2k_Ff,0)}^z \equiv \sum_s s_{sL} s_{sR} \sim -\sin(\sqrt{2} \theta_{f}) e^{i \frac{\theta_{\text{tot}}}{\sqrt{2}}}. 
\]

In the bosonized expressions above, we have omitted $\theta_d$ and $\theta_A$ as these are pinned by the gauge field fluctuations; the discussion of the power law correlations in the $d$-metal phase only depends on the displayed fluctuating field content.

Finally, we have bilinears carrying zero momentum contributing to both charge and spin, e.g.,

\[
S_{(0,0)}^z \equiv \sum_{s,p} s_{sL}^p s_{sR} = \frac{\sqrt{3}}{\pi} \partial_s \theta_\sigma; 
\]

such contributions have scaling dimension 1.

Let us first note some general relations among scaling dimensions for various density and spin correlations:

\[
\Delta[n_{(2k_Ff,0),\pi}] = \Delta[S_{(2k_Ff,0)}^z] = \frac{1}{2} + \frac{\Delta[n_{(2k_Fd_2,0)}]}{4}, \\
\Delta[n_{(k_{FD_1}^{(0)}-k_{FD_1}),\pi}] \geq 1. 
\]

Thus, the scaling dimension of $n_{(k_{FD_1}^{(0)}-k_{FD_1},\pi)}$ is always larger than the mean field value of 1.

Next, we can get some quantitative feel by using the approximation of decoupled "$d_1$" and "$\rho$tot" modes; the scal-

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**FIG. 3:** The electronic $d$-metal is obtained by putting the chargons into the DBL[2,1] (Bose-metal) state of Ref. 3 and spinons into the Fermi sea (spin liquid) state. Top panel: Crude approximation to the chargon distribution function where we replace the peaks in Fig. 8(a) of Ref. 3 with delta-functions; the wavevectors are the step singularities correspond to the first two rows in Table I. Note that this sketch does not contain potentially important wavevectors listed in the third row in Table I.

**C. Density and spin correlations**

We now discuss the electron density and spin correlations. From the various ways of writing the electron number in terms of partons, cf. Eq. (2) of the main text, we can immediately obtain fermionic bilinear contributions to the electron number.

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**SUPPLEMENTARY INFORMATION**

We now discuss the electron density and spin correlations. From the various ways of writing the electron number in terms of partons, cf. Eq. (2) of the main text, we can immediately obtain fermionic bilinear contributions to the electron number.
ing dimensions in this approximation are:
\[ \Delta [n(2k_{Fz},0)] = \frac{g_{\text{tot}}}{2}, \]

\[ \Delta [n(2k_{Fz},0)] = \frac{g_{d1}}{2} + \frac{g_{\text{tot}}}{8}, \]

\[ \Delta [n(k^{(0)}_{Fd},k^{(\pi)}_{Fd},\pi)] = \frac{1}{2g_{d1}} + \frac{g_{\text{tot}}}{8}, \]

\[ \Delta [n(k^{(0)}_{Fd},-k^{(\pi)}_{Fd},\pi)] = \frac{1}{2} \left( \frac{g_{d1}}{g_{d1} + g_{d1}} \right), \]

\[ \Delta [n(2k_{Fz},0)] = \Delta [S^z(2k_{Fz},0)] = \frac{1}{2} + \frac{g_{\text{tot}}}{8}. \]

As before, we obtain results for the bare Gutzwiller wave function by setting \( g_{d1} = g_{\text{tot}} = 1 \). In this case, the smallest scaling dimension is \( \Delta [n(2k_{Fz},0)] = \frac{1}{2} \) corresponding to slow \( |x|^{-1} \) power law. We also have \( \Delta [n(k^{(0)}_{Fd},0)] = \Delta [n(k^{(0)}_{Fd}+k^{(\pi)}_{Fd},\pi)] = \Delta [n(2k_{Fz},0)] = \Delta [S^z(2k_{Fz},0)] = \frac{3}{2} \) corresponding to \( |x|^{-5/4} \) power law, which is also enhanced over the mean field \( |x|^{-2} \). The enhancement of the correlations comes from the reduction of the fluctuating \( \theta \) content, which is the \((1+1)D\) realization of the “Amperean” enhancement\(^3\)–\(^5\) of correlations for processes that contain parallel gauge charge currents. On the other hand, \( \Delta [n(k^{(0)}_{Fd},-k^{(\pi)}_{Fd},\pi)] = 1 \) and is not enhanced over the mean field, since the particle and hole partons are on the same side of the Fermi sea and create anti-parallel currents. We performed measurements in sample bare Gutzwiller wave functions and verified the dominant \( |x|^{-1} \) and \( |x|^{-5/4} \) power laws for different oscillating components in the density structure factor.

We remark that the d-metal realized in the DMRG appears to have larger \( g_{\text{tot}} \) and hence \( n(2k_{Fz},0) \) can be suppressed compared to \( n(2k_{Fz},0) \). In fact, from our earlier discussion of the umklapp term Eq. (18) at density 1/3 and absence of charge order in the d-metal phase found in the DMRG, we can conclude that \( \Delta [h_{\text{umkl}}] > 2 \), and hence have an exact bound:
\[ \Delta [n(2k_{Fz},0)] = \frac{4}{9} \Delta [h_{\text{umkl}}] = \frac{2}{9} > \frac{2}{3}. \]

In the approximation of decoupled “d1–” and “\( \rho \text{tot} \)” modes, this corresponds to \( g_{\text{tot}} > 4/3 \), and in this case \( n(2k_{Fz},0) \) already has larger scaling dimension than \( n(2k_{Fz},0) \) (if we assume \( g_{d1} = 1 \)). Let us also quote the numbers for our earlier dressed Gutzwiller example where we set \( g_{d1} = 1 \) and \( g_{\text{tot}} = 2 \): We get \( \Delta [n(2k_{Fz},0)] = 1 \) that is larger than \( \Delta [n(2k_{Fz},0)] = \Delta [n(k^{(0)}_{Fd}+k^{(\pi)}_{Fd},\pi)] = \Delta [n(2k_{Fz},0)] = \Delta [S^z(2k_{Fz},0)] = 3/4, \) so the latter singularities are more strong. The DMRG indeed finds visible features in the density structure factors at wavevectors \((2k_{Fz},0)\), which we track to identify the \( d_1 \) bonding/antibonding orbital populations. On the other hand, at density 1/3, \( 2k_{Fz} = -2k_{Fz} = \pi/3 \) and these wavevectors are not visible in the density structure factors in much of the data. In the above approximation, we can make the \((2k^{(0)}_{Fd},0)\) to be dominant if we also assume \( g_{d1} < 1 \). This is consistent with the condition that the interaction \( h_{\text{int},4d_1} \) in Eq. (14) is irrelevant.

Furthermore, inspired by the DMRG observation of a singularity in the spin structure factor at wavevector \((2K_z,0)\), we have also considered four-fermion contributions to the spin operator, since, in the strongly coupled theory, these can be comparable in prominence to the fermionic bilinears. We have identified two interesting terms, which can be constructed by combining the already exhibited bilinears,
\[ S^z_{(2k_{Fz},0)} \sim AS^z_{(-2k_{Fz},0)}n_{(2k_{Fz},0)} + BS^z_{(2k_{Fz},0)}n_{(-2k_{Fz},0)} \sim \sin(\sqrt{2}\theta_{F^z})e^{i\sqrt{2}d_1-}, \]
\[ S^z_{(0,\pi)} \sim S^z_{(-2k_{Fz},0)}n_{(k^{(0)}_{Fd}+k^{(\pi)}_{Fd},\pi)} + H.c. \sim i\eta_{(0)}(\pi)\cos \left[ \frac{2(\sqrt{2}\theta_A - \theta_a)}{\sqrt{3}} \right] \\
\times \sin(\sqrt{2}\theta_{F^z}) \sin(\sqrt{2}\rho_{d1-}). \]

Note that the first term can also be constructed by combining \( c_{(k^{(0)}_{Fd})}^+ \) and \( c_{(k_{Fd})} \), and is a kind of “electron 2k_{Fz}” from the dominant feature observed in the electron distribution function. Similarly, the second term can be constructed by combining, e.g., \( c_{(k^{(0)}_{Fd})}^+ \) and \( c_{(k_{Fd})} \). Writing out those electron bilinears gives six-parton terms, which turn out to be equivalent to the above four-parton terms upon considering the pinning of \( \theta_A \) and \( \theta_a \). In the last equation, we have also carefully kept track of these pinned fields, since the very presence or absence of this contribution can depend on the pinning values. Both the DMRG and VMC have a visible feature in the spin structure factor at the wavevector \((0,\pi)\), cf. Fig. 5(c) of the main text, and hence we conjecture that the pinning of the fields \( \theta_A \) and \( \theta_a \) is such as to give nonzero cosine for their combination exhibited above. This knowledge will be useful when discussing Cooper pair correlations in the next section (note that the precise pinning values of the \( \theta_A \) and \( \theta_a \) fields were not important in the observables discussed earlier).

In the same spirit, we can also construct four-parton contributions to the electron density, \( n_{(2K_z,0)} \) and \( n_{(0,\pi)} \). We expect the former to be present generically with properties similar to \( S^z_{(2K_z,0)} \). On the other hand, the expression for the latter contains sine of the \( \theta_A \) and \( \theta_a \) combination in Eq. (71), and we conjecture that the pinning is such that the \( n_{(0,\pi)} \) term vanishes. This is consistent with the absence of any feature in the DMRG and VMC density structure factor at \((0,\pi)\), compare Figs. 5(b) and (c) of the main text.

From the bosonized expressions, we can see that the scaling dimension of \( S^z_{(0,\pi)} \) is related to that of the allowed interaction \( h_{\text{int},4d_1} \), Eq. (14):
\[ \Delta [S^z_{(0,\pi)}] = \frac{1}{2} + \frac{\Delta [h_{\text{int},4d_1}]}{4}. \]

Since the stability of the d-metal requires \( \Delta [h_{\text{int},4d_1}] > 2 \), we conclude that \( \Delta [S^z_{(0,\pi)}] > 1 \). This is indeed what we found from the DMRG measurements. In the approximation with
decoupled “d1−” and “ρtot” modes, we find
\[
\Delta[S^\pi_{(2k_x,0)}] = 1 - g_{d1-}, \quad (74)
\]
\[
\Delta[S^\pi_{(0,\pi)}] = 1 - 1 - 2g_{d1-}. \quad (75)
\]

To summarize, many such qualitative and semi-quantitative considerations of the observables, relations among exponents, and stability to perturbations are internally consistent, giving us more confidence that the phase found in the DMRG is indeed the electronic d-metal.

D. Cooper pair correlation

To complete the discussion of observables, we also consider electron Cooper pair operators. Besides being prominent observables themselves, these are needed, e.g., in Sec. II F for the detailed description of the phases proximate to the d-metal when the electron operator gets gapped. We consider only spin singlets and can start with microscopic Cooper operators defined as in Eq. (6) on some nearby pair of sites. We can write each electron operator in terms of the partons and obtain contributions containing six continuum parton fields (two for each species d1, d2, and f). We examined all such six-fermion terms, and below we present most interesting ones bearing in mind the application to proximate phases. Rather than doing direct expansion, we can use symmetry arguments to identify which microscopic Cooper pairs receive particular contributions.

The most important such six-fermion terms are
\[
\mathcal{P}^{(c)}_{(0,\pi)} \equiv d_{2R}d_{2L}d_{1R}d_{1L}f_L^1f_L^1 + (R \leftrightarrow L) \quad (76)
\]
\[
\sim \eta_1^{(0)}(\pi)\eta_1^{(0)}(\pi)\sin \left[\frac{2(\sqrt{2}\theta_A - \theta_\sigma)}{\sqrt{3}}\right] e^{\pm 2\sqrt{2}\phi_{\rho_{tot}}},
\]
\[
\mathcal{P}^{(o)}_{(0,\pi)} \equiv -i\left[d_{2R}d_{2L}d_{1R}d_{1L}f_L^1f_L^1 - (R \leftrightarrow L)\right] \quad (77)
\]
\[
\sim \eta_1^{(0)}(\pi)\eta_1^{(0)}(\pi)\cos \left[\frac{2(\sqrt{2}\theta_A - \theta_\sigma)}{\sqrt{3}}\right] e^{\pm 2\sqrt{2}\phi_{\rho_{tot}}}.\]

These terms carry momentum (0, π), i.e., they are translationally invariant along the ladder and are odd under the leg interchange. They are defined to be invariant under time reversal, as is appropriate for the singlet Cooper pairs. The “(c)” “[o]” combination is even [odd] under mirror \((x, y) \rightarrow (-x, y)\); the numerical constant is sine [cosine] of the particular combination of the pinned fields \(\theta_A\) and \(\theta_\sigma\). We expect that one or the other combination is nonzero, but which one depends on the detailed pinning specifying the d-metal phase (see below). From the identified symmetry properties, we can see that the combination \(\mathcal{P}^{(c)}_{(0,\pi)}\) contributes to leg-bond Cooper pairs that are anti-symmetric under the leg interchange:
\[
P[(x, 1), (x + 1, 1)] - P[(x, 2), (x + 1, 2)] \sim \mathcal{P}^{(c)}_{(0,\pi)}(x) + \ldots. \quad (78)
\]

On the other hand, the combination \(\mathcal{P}^{(o)}_{(0,\pi)}\) contributes to diagonal Cooper pairs anti-symmetric under the leg interchange:
\[
P[(x, 1), (x + 1, 2)] - P[(x, 2), (x + 1, 1)] \sim \mathcal{P}^{(o)}_{(0,\pi)}(x) + \ldots \quad (79)
\]

The \(\mathcal{P}^{(o)}_{(0,\pi)}\) terms have the smallest content of fluctuating fields in the d-metal — only the \(\phi_{\rho_{tot}}\) part that is necessary to encode the electrical charge of the Cooper pairs. In the approximation of decoupled “d1−” and “ρtot” modes, the scaling dimension is \(2/g_{d1-}\) (equal to 2 in the bare Gutzwiller and 1 in the dressed Gutzwiller wave functions). By measuring whether the leg-bond or diagonal Cooper pair shows power law behavior, we can constrain the appropriate pinning of the fields \(\theta_A\) and \(\theta_\sigma\). As we will present in Sec. II E, both the DMRG and VMC find that it is the diagonal Cooper pairs that have dominant power law correlations in real space. Hence we conclude that the pinning is such that \(\mathcal{P}^{(o)}_{(0,\pi)}\) is nonzero while \(\mathcal{P}^{(c)}_{(0,\pi)}\) is zero, which is also consistent with the presence of \(S^\pi_{(0,\pi)}, \text{Eq. (72)}, \text{and the absence of } n_{(0,\pi)} \text{ features in the DMRG and VMC. The specific diagonal Cooper pairs Eq. (79) are natural in the model with electronic ring exchange: e.g., they arise when solving the ring Hamiltonian for two electrons on a single placket. They can be viewed as having a d-wave character, which is one of the motivations for naming our non-Fermi-liquid phase as “d-wave metal.”

We also mention the following six-fermion terms
\[
\mathcal{P}^{(0,0)} \equiv d_{2R}d_{2L}d_{1R}d_{1L}f_L^1f_L^1(f_{L1}f_{L1} - f_{F1}f_{F1}) \quad (80)
\]
\[
\sim -\eta_1^{(0)}\eta_1^{(0)}e^{\pm 2\sqrt{2}\phi_{\rho_{tot}}} \cos(\sqrt{2}\theta_{f_{tot}}), \quad (81)
\]

where ± refers to \(e^{\pm ik_\sigma}\). Such terms carry zero momentum, i.e., they are translationally invariant along the ladder and are even under the leg interchange. They are also even under the mirror \((x, y) \rightarrow (-x, y)\) and contribute, e.g., to a rung Cooper pair \(P[(x, 1), (x, 2)]\) (as well as leg-bond or diagonal Cooper pairs symmetric under the leg interchange). In some loose sense, they can be viewed as “s-wave” Cooper pairs, while the \(\mathcal{P}^{(0,0)}\) ones are “d-wave”, the precise distinction lying in the transformation properties under the discrete ladder symmetries. Because of the additional fluctuating field content, we expect the s-wave ones to have larger scaling dimension [equal to \(2/g_{d1-} + 1/(2g_{d1-}) + 1/2\) in the decoupled “d1−” and “ρtot” approximation] and to be less visible in the d-metal than the d-wave ones. However, the s-wave can become comparably prominent in a spin gap phase discussed in Sec. II F.

E. Sample of the DMRG power law fits

Guided by the long-wavelength description of the d-metal, we performed detailed fits for the power laws in various correlations at dominant wavevectors. Here we highlight some results at the DMRG point \(J/t = 2, K/t = 1.8\), presented in Fig. 5 in the main text. To remind the readers, the best d-metal candidate for this \(48 \times 2\) ladder with 32 electrons has
\[2k_{F \delta}^{(0)} = 21 \cdot 2\pi/48, \ 2k_{F \delta}^{(\pi)} = 11 \cdot 2\pi/48, \ \text{cf. Fig. 2 of the main text.}\]

The main features in the electron distribution function occur at the wavevectors listed in the first three rows of Table I: \(K_e = 5\pi/48, 2\pi\rho + K_e = 37\pi/48, \) and \(2\pi\rho - K_e = 27\pi/48, \) for either \(k_y = 0 \) or \(\pi.\) We fit the electron Green’s function to an expression

\[G_e(x, k_y) = \sum_a C_a^{(k_y)} \sin(Q_a x) / \left[ (L_x/\pi) \sin(\pi x/L_x) \right]^{2\Delta_a^{(k_y)}}, \quad (82)\]

The dominant oscillation is at the wavevector \(K_e,\) which is readily recognized in the electron momentum distribution function in Fig. 5(a) of the main text. As illustrated in Fig. 4, including only this wavevector already captures the overall real-space dependence and gives \(2\Delta_a^{(0)} \approx 1.3\) and \(2\Delta_a^{(\pi)} \approx 1.5\) for the bonding and antibonding electrons respectively. Fits including the other two wavevectors (not shown) capture also finer features without affecting much the estimates of \(\Delta_a^{(0/\pi)}\). We thus confirm the general expectation that the exponents can be different in the bonding and antibonding electron distribution functions. Figure 4 also shows the electron Green’s function measured in the VMC wave function. Similar fits in this case (not shown) give roughly equal values \(2\Delta_a^{(0/\pi)} \approx 2\Delta_a^{(\pi)} \approx 1.25,\) which is also expected in the approximate gauge theory treatment with decoupled “d1" and “ptot". Our trial wave functions and the approximate treatment of the gauge theory are not qualitatively accurate in this respect. The structure of the gauge theory is qualitatively accurate—it is only that we do not know numerical values of the couplings in the full theory with coupled “d1" and “ptot" modes. Nevertheless, we see that the approximate treatment provides a reasonable semi-analytical guide.

Turning to the spin and density correlations, we fit these to an expression similar to Eq. (82), but with cosines instead of sines. It is simple to fit the spin correlations at \(k_y = \pi,\) since there is only one feature, Eq. (72). We estimate \(2\Delta_0^0 \approx 2\Delta_0^\pi \approx 2.2,\) which satisfies the d-metal stability requirement discussed after Eq. (73) and gives us an estimate \(g_{d1} \approx 0.85\) in the approximation with decoupled “d1” and “ptot”. On the other hand, the spin correlations at \(k_y = 0\) have three important wavevectors: \(0, 2\pi/3,\) and \(2K_e = 10\pi/48.\) The power law for the zero-momentum component is fixed at \(x^{-2}.\) Fitting power law decays at the other two wavevectors allows us to estimate \(g_{ptot} \approx 3.5 - 4\) and \(g_{d1} \approx 3.5 - 4\) consistent with the previous estimate. The large value of \(g_{ptot}\) explains the weakness of the feature at \(2\pi/3.\) As discussed in Sec. I B, our matching VMC state has small negative power \(p_2 = -0.4\) on the \(d_2\) determinant, which can indeed reproduce such large \(g_{ptot}.\) On the other hand, the value of \(g_{d1} < 1\) implies that the component at \(2K_e\) has power law decay slower than \(x^{-2}\) in real space and singularity in the structure factor that is stronger than slope change; the singularity at this wavevector can be noticed already in the DMRG data in Fig. 5 of the main text, while the VMC has harder time reproducing this.

It is also simple to fit the density correlations at \(k_y = \pi,\) where we have two wavevectors \(k_{F\delta}^{(0)} + k_{F\delta}^{(\pi)} = 2\pi/3\) and \(k_{F\delta}^{(0)} - k_{F\delta}^{(\pi)} = 10\pi/48.\) The power law fits give \(2\Delta_0^0 = 2\Delta_0^\pi \approx 2.3\) and \(2\Delta_1^0 = 2\Delta_1^\pi \approx 2.1.\) The latter is consistent with the general bound in Eq. (63) and with \(g_{d1} \approx 0.85\) at \(1/2\) of the previous estimate. We expect that all these components have similar scaling dimension of order 1. We can indeed get nice fits, but because of the many parameters it is difficult to give accurate individual exponents; nevertheless, all are consistent with the previous estimates of \(g_{d1}\) and \(g_{ptot}.

Finally, we also measured the diagonal d-wave Cooper pair correlations in the DMRG and VMC. For the \(48 \times 2\) sample above, we found it difficult to fully converge the pair correlations in the DMRG (the electron Green’s function shown earlier converged more readily). Nevertheless, we observed very clearly a slow power law decay in this Cooper channel, fitting roughly \(1/x^{1.15}\) in the DMRG and \(1/x^{0.85}\) in the VMC. For illustration, in Fig. 5 we show the measurements in the smaller \(36 \times 2\) sample at \(J/t = 2\) and \(K/t = 2,\) where the DMRG data is better converged, with the remaining uncertainty less than 5% (the same sample was used in the Rényi entropy comparisons in Fig. 2). We see good match between the DMRG and VMC results, with power law fits giving roughly \(1/x^{0.9}\) in the DMRG and \(1/x^{0.3}\) in the VMC. Comparing with the predictions in Sec. II D, the exponents are again consistent with \(g_{ptot} = 2.1.\) All other Cooper pair correlators, including also the leg-bond d-wave Eq. (78),
FIG. 5: DMRG and VMC diagonal d-wave Cooper pair correlations [see expression below Eq. (6)] in the system of length $L_x = 36$ at $J/t = 2$ and $K'/t = 2$ (same as in Fig. 2). We show the DMRG data for 8 and $K'/t = 2$ (same as in Fig. 2). We show the DMRG data for 8 and 15, where the convergence error is estimated to be less than 5% having kept $m = 8,500$ states in the DMRG. Thin line shows fit of the DMRG data to the functional form of the phase found in the DMRG as the metal theory. This provides strong support for the identified correlations match well with those expected in the superconductivity in this strange metal phase.

To conclude, many detailed properties of the various measured correlations match well with those expected in the d-metal phase. This provides strong support for the identification of the phase found in the DMRG as the d-metal.

F. Instabilities of the d-metal

With all observables at hand, we can now consider phases that would be obtained out of the d-metal when one or both of the interactions $h_{int,4f}$ and $h_{int,4f}$, Eqs. (14) and (16) respectively, become relevant. Since the electron operator gets gapped out, a detailed description requires consideration of the Cooper pair operators. In the d-metal phase and in all phases below, the Cooper pair $\mathcal{P}(0,\pi) \sim e^{i\theta_{d,t}/\sqrt{2}}\rho_t$ discussed in Sec. II D is always prominent, and only one more Cooper pair will come to prominence in one of the phases. On the other hand, clear distinctions between possible phases are already apparent with the density and spin observables. There are three cases:

1) If the interaction $h_{int,4f}$ is relevant, it pins $\theta_{d,t}$ and gaps the spin. Assuming $h_{int,4d_t}$ remains irrelevant, we have two gapless modes “$d_1$ -” and “$\rho_{tot}$” and the full theory is similar to the DBL [2, 3] theory in Ref. 3. A prominent density observable is $n(2k_F,0) \sim e^{i\theta_{d,t}/\sqrt{2}}$, whose wavevector $(2\pi,0)$ is determined by the electron density. Loosely speaking, this phase can be thought of as a two-leg analog of a “pseudogap,” i.e., a nonsuperconducting quantum fluid with a spin gap.

2) If the interaction $h_{int,4t}$ is relevant, it pins $\phi_{d,1}$ - wave. Assuming $h_{int,4f}$ remains irrelevant, we have two gapless modes “$f\sigma$” and “$\rho_{tot}$”, with one Luttinger parameter in the latter sector. In this case, the spin correlations remain gapless. Depending on the precise pinning of $\phi_{d,1}$ - determined by the sign of $w$ in Eq. (14), we get dominant density correlations $n(p_{F,t} + p_{F,d,1} + p_{F,d,1}^*) \sim \sin(\sqrt{2}\phi_{d,1} - )e^{i\theta_{d,t}/\sqrt{2}}$ or current correlations $j(p_{F,t} + p_{F,d,1} + p_{F,d,1}^*) \sim \sin(\sqrt{2}\phi_{d,1} - )e^{i\theta_{d,t}/\sqrt{2}}$; the wavevector is $(2\pi,0)$ and is distinct from the case 1). This phase can be viewed as a phase where the bosonic chargons form a paired-boson state, while the spin remains gapless, and is akin to a non-Fermi liquid termed “orthogonal metal” discussed recently in Ref. 16.

3) Finally, if both $h_{int,4d_d}$ and $h_{int,4f}$ are relevant, we have pinning of the fields $\theta_{d,t}$ and $\phi_{d,1} -$ as in the cases 1) and 2) above. Only one gapless mode “$\rho_{tot}$” remains. We have presence of both $(2\pi,0)$ and $(2\pi,\pi)$ wavevectors in the density (or appropriate current depending on the sign of $w$) correlation with the same power law. Furthermore, the Cooper pair $\mathcal{P}(0,\pi)$, Eq. (80), at wavevector $(0,0)$ now becomes as prominent as the $\mathcal{P}(0,\pi)$ at $(0,\pi)$. This phase is the two-leg analog of a superconductor.

Since the DMRG did not observe strong density correlations at the wavevector $2\pi$ along the ladder or any other signatures of instability, we conclude that our d-metal phase is stable in the $t$-, $J$-, $K$- model. In future work, it would be interesting to modify the model to further explore the above proximate phases.

III. ELIMINATING CONVENTIONAL LUTTINGER LIQUID SCENARIOS IN FAVOR OF THE d-METAL

In light of the remarkable success to date of describing 1D and quasi-1D interacting quantum systems with conventional Luttinger liquid theory, it is natural to ask whether the results in our putative d-metal phase can be reproduced with such a conventional weak-coupling approach. Clearly, since the number of gapless modes in the putative d-metal ($c = 3$) is larger than in the conventional one-band metal ($c = 2$), the former cannot be understood as an instability of the latter. However, there are more complicated scenarios that one may envision that involve strong Fermi surface renormalization, as well as electron pairing, but that still lie within the conventional Luttinger framework and still assume a free electron starting point.

For instance, one could first imagine the $K$ term renormalizing the free electron band structure such that the antibonding band eventually gets populated (as if $K$ had the effect of sim-
ply renormalizing the interchain hopping $t_\perp$ towards zero—this is admittedly somewhat natural given that $H_K$ conserves the number of electrons in each chain). If we denote a conventional Luttinger liquid with $\alpha$ gapless charge modes and $\beta$ gapless spin modes as $\text{CoS}\beta$ (see Ref. 18), then this free electron state would be some C2S2 metal with $c = 4$ gapless modes, say a charge ($\rho$) and spin ($\sigma$) mode for each band $(0/\pi)$: $\theta_\rho/\theta_\pi$. In principle, a spin gap in the antibonding band could be opened through relevance of a term involving only a cosine of the $\theta_{\pi}^\dagger$ field, giving a C2S1 metal with $c = 3$ gapless modes. However, this possibility can be immediately ruled out in our putative $d$-metal region by noting that the DMRG state unambiguously has a critical Green’s function for the antibonding electrons: See the sharp step in $\langle c_{q_x}^\dagger c_{q_y}^\dagger \rangle$ at $q = (q_x, q_y) = (K_c, \pi)$ in Fig. 5(a) of the main text, as well as the discussion in Sec. II E of the slow power law decay of the Green’s function in real space. Pinning of the $\theta_{\pi}^\dagger$ field, on the other hand, directly implies that the electron Green’s function would decay exponentially at $q_\theta = \pi$, in clear contradiction with our DMRG data.

Other aspects of the DMRG data are also markedly inconsistent with this C2S1 scenario. For example, throughout the $d$-metal phase at, say, $J/t = 2$ (to avoid small polarization observed at smaller $J/t$), we observe an enhanced feature in the spin-spin structure factor $(S_{q_x} \cdot S_{-q_x})$ at $q = (2\pi, 0, 0)$ for all $K/t$. The location of this feature is fixed by the electron density and is readily explainable by our $d$-metal theory [see Eq. (60)]. In the C2S1 state discussed above, however, the only singularity in $(S_{q_x} \cdot S_{-q_x})$ at $q_y = 0$ would be at $q_x = 2k_F^{(0)}$, where $\pm k_F^{(0)}$ denotes the Fermi points for the bonding electrons assumed to be gapless. This wavevector is not fixed by the electron density and is more akin to our observed feature at $q_y = 2K_c$. Hence, presence of the feature at $q = (2\pi, 0, 0)$ in $(S_{q_x} \cdot S_{-q_x})$ [see Fig. 5(c) in the main text] is not consistent with a conventional C2S1. The C2S1 state also fails to explain things like why $\langle c_{q_x}^\dagger c_{q_y}^\dagger \rangle$ has substantial weight on the other hand, directly implies that the electron Green’s function it- self gives reason to at least anticipate that the $t$-$J$-$K$ model Hamiltonian may harbor the non-Fermi liquid $d$-metal phase.

IV. MOTIVATION FOR THE $t$-$J$-$K$ MODEL TO REALIZE ELECTRONIC $d$-METAL

The purpose of this section is to give some analytical intuition why the electronic ring Hamiltonian likes the particular $d$-metal phase. Of course, we cannot rule out every single weak-coupling scenario, including even more complicated and contrived ones, but the above two possibilities would be the most natural in our view, and they are clearly not working. On the other hand, we stress that our $d$-metal framework can basically describe the entire DMRG data set in a very natural, unified fashion, giving us a high degree of confidence that our novel non-perturbative $d$-metal theory is indeed correct. In addition, as discussed in the next section, the structure of the $d$-metal gauge theory itself gives us reason to at least anticipate that the $t$-$J$-$K$ model Hamiltonian may harbor the non-Fermi liquid $d$-metal phase.
and \(|\chi_d|^2\) for the \(d\)-spinons comes from adding the two terms in Eq. (83) and summing over the spin labels, while we have the minus sign for the \(d\)-partons from the Fermi statistics. If \(d_1\) and \(d_2\) partons hop preferentially in the \(\hat{x}\) (respectively \(\hat{y}\)) direction, then \(|\chi_{d_1,x}|^2 > |\chi_{d_1,y}|^2\) (respectively \(|\chi_{d_2,x}|^2 < |\chi_{d_2,y}|^2\)). In this case, the expectation value of the ring term is negative, so for the positive \(K\) in our model we obtain low ring energy; increasing the anisotropy in the \(d_1\) and \(d_2\) hoppings lowers the ring energy, while the optimal anisotropy is determined from the competition with the electronic kinetic and spin exchange energies. This is our crude mean field argument for the ring terms prefer the particular fractionalized state and why the \(d_1\) and \(d_2\) partons develop strong anisotropies for large \(K\).

We can also provide a more constructive motivation for the electronic ring model as a candidate for realizing the \(d\)-metal phase. This “reverse engineering” argument starts with an effective lattice gauge theory for the \(d\)-metal phase. This “reverse engineering” argument starts with an effective lattice gauge theory for the \(d\)-metal phase. This “reverse engineering” argument starts with an effective lattice gauge theory for the \(d\)-metal phase. This “reverse engineering” argument starts with an effective lattice gauge theory for the \(d\)-metal phase.

In the gauge theory Hamiltonian, the parts are coupled to the gauge field \(A\) with opposite gauge charges, so that the physical electron operator is gauge neutral. We have also generalized the boson ring term with coupling \(K^{(b)}\) to a gauge-invariant form that also respects the square lattice symmetries. The \(h\) and \(\kappa\) terms are standard for the lattice gauge field dynamics.

For large \(h\), the electric fields are pinned at \(E_{rr'} = 0\) and the partons are confined. We can identify the sector \(E_{rr'} = 0\) with the physical Hilbert space of electrons. Starting with this limit and working perturbatively in \(|t^{(f)}|^2\), \(|t^{(b)}|^2\), and \(K^{(b)}\), we obtain an effective Hamiltonian for electrons that contains the following terms:

\[H = \sum_{(rr')} E_{rr'}^2 - \kappa \sum_{(rr')} \cos(\nabla \cdot A) - \sum_{(rr')} \left( t_{rr'}^{(f)} e^{-iA_{rr'}} f_{r\alpha}^+ f_{r'\alpha} + \text{H.c.} \right) - \sum_{(rr')} \left( t_{rr'}^{(b)} e^{iA_{rr'}} b_{r\alpha}^+ b_{r'\alpha} + \text{H.c.} \right) + K^{(b)} \sum_{(rr')} \left[ (e^{iA_{rr'+\hat{x}}} e^{iA_{rr'+\hat{y}}}) b_{r\alpha}^+ b_{r'\alpha} + \text{H.c.} \right]. \]

(85)

The electron hopping is obtained from second-order processes hopping both chargon and spinon, while the spin-spin interaction is obtained from second-order processes exchanging only spinons. The electron ring term is obtained from third-order processes involving hopping spinons on opposite edges of a square and chargon ring exchange on the square [for simplicity, we have used isotropic spinon hopping and \(|\chi_d|^2\) for the \(d\)-spinons comes from adding the two terms in Eq. (83) and summing over the spin labels, while we have the minus sign for the \(d\)-partons from the Fermi statistics. If \(d_1\) and \(d_2\) partons hop preferentially in the \(\hat{x}\) (respectively \(\hat{y}\)) direction, then \(|\chi_{d_1,x}|^2 > |\chi_{d_1,y}|^2\) (respectively \(|\chi_{d_2,x}|^2 < |\chi_{d_2,y}|^2\)). In this case, the expectation value of the ring term is negative, so for the positive \(K\) in our model we obtain low ring energy; increasing the anisotropy in the \(d_1\) and \(d_2\) hoppings lowers the ring energy, while the optimal anisotropy is determined from the competition with the electronic kinetic and spin exchange energies. This is our crude mean field argument for the ring terms prefer the particular fractionalized state and why the \(d_1\) and \(d_2\) partons develop strong anisotropies for large \(K\). We can also provide a more constructive motivation for the electronic ring model as a candidate for realizing the \(d\)-metal phase. This “reverse engineering” argument starts with an effective lattice gauge theory for the \(d\)-metal phase. This “reverse engineering” argument starts with an effective lattice gauge theory for the \(d\)-metal phase.
model studied in the main text, where we allow ourselves to vary independently the electron hopping and ring-exchange amplitudes, as well as the antiferromagnetic exchange coupling. In the end, it is the detailed numerical study that establishes the phase diagram of this Hamiltonian.

We conclude by mentioning that the d-wave correlations present in our two-leg d-wave metal are of the \(d_{xy}\) variety. These are built in by taking the ring term to operate on elementary plaquettes of the square lattice: \((r, r + \hat{x}, r + \hat{y}, r + \hat{y})\). Looking forward, both when going to more legs with the DMRG and when studying two dimensions directly with the VMC, it will be interesting to consider a d-wave metal of the \(d_{x^2-y^2}\) variety, a phase which is potentially more relevant to the cuprates. A promising model to realize this phase would include, instead of that considered in this work, a ring-exchange term operating on all plaquettes \((r, r + \hat{x} + \hat{y}, r + 2\hat{y}, r - \hat{x} + \hat{y})\). Addressing the applicability of such models to real cuprate materials, as well investigating incipient \(d_{x^2-y^2}\)-wave superconductivity out of the putative \(d_{x^2-y^2}\)-wave metal, are very exciting future problems.

V. POSSIBLE MICROSCOPIC ORIGIN OF THE ELECTRONIC RING TERMS BY PROJECTING THE COULOMB INTERACTION INTO A TIGHT-BINDING BAND

Here we show how the electronic ring terms can appear by projecting the Coulomb interaction into a tight-binding band. Our goal is to emphasize that such terms are rather simple and natural, even if they do not look familiar. This analysis can also motivate more realistic estimates of the ring couplings from ab-initio calculations, which would be interesting to pursue for the cuprates and other strongly correlated materials.

We start with a tight-binding model for electrons moving in a periodic ionic potential,

\[
H_{\text{kin}} = -\sum_{\langle ij \rangle} \left( t_{ij} c_i^\dagger c_j + \text{H.c.} \right),
\]

where \(c_i^\dagger\) creates an electron with spin \(\alpha\) in a Wannier orbital \(w_i(r)\) localized near an ion \(R_i\) (here and below, summation over repeated spin indices is implied). In the context of a particular material, the Wannier orbitals might be obtained, e.g., from the Bloch states of a narrow band in an LDA-type band structure that crosses the Fermi energy. Next we want to include the electron-electron interaction. We assume spin-independent pair-wise interaction such as screened Coulomb repulsion \(u_{\text{Coulomb}}(r, r')\), which could be obtained from the bare Coulomb within an RPA approach, integrating out the filled and empty LDA bands. Projecting the interaction into the tight-binding band, we obtain

\[
H_{\text{int}} = \frac{1}{2} \sum_{ijkl} \langle ij|\hat{U}|kl\rangle c_i^\dagger c_j c_k^\dagger c_l + \text{H.c.},
\]

with

\[
\langle ij|\hat{U}|kl\rangle = \int_{r, r'} w_i^*(r) w_j^*(r') u_{\text{Coulomb}}(r, r') w_k(r) w_l(r').
\]

At this stage, it is customary to focus on terms that do not change the electron number on each site:

\[
H^{(0)}_{\text{int}} = \frac{1}{2} \sum_{ij} \langle ij|\hat{U}|ij\rangle n_i(n_i - 1) + \frac{1}{2} \sum_{ij \neq j} \langle ij|\hat{U}|ij\rangle n_i n_j - \frac{1}{2} \sum_{i \neq j} \langle ij|\hat{U}|ji\rangle \left( 2S_i \cdot S_j + \frac{1}{2} \right).
\]

The first and second terms are the familiar on-site (Hubbard) and inter-site repulsion. The last term is the inter-site spin interaction, where we used \((c_{i\alpha}^\dagger c_{i\beta})(c_{j\beta}^\dagger c_{j\alpha}) = 2S_i \cdot S_j + 1/2\). Note that in such a derivation, the spin exchanges can come out as ferromagnetic. On the other hand, in materials with Mott physics, effective antiferromagnetic exchanges arise from a further interplay of the hopping \(t\) and Hubbard repulsion \(U\) and dominate over the bare ferromagnetic couplings in \(H^{(0)}_{\text{int}}\). This is reminiscent of the Hartree-Fock overestimation of the ferromagnetic tendencies in an electron gas due to neglect of the electronic correlations. While we will not belabor this concern further, it is good to keep in mind that one needs to look at \(H_{\text{int}}\) as a whole, a warning that applies equally to our schematic discussion below.

We now want to stress that keeping only the terms in \(H^{(0)}_{\text{int}}\) is not complete. There are more four-fermion terms that arise at the same level of treatment but that do not preserve the number of electrons on each site. Let us assume for simplicity that our electronic orbitals are peaked on sites of a square lattice and respect the symmetries of the lattice. Such a microscopic model is not applicable to the cuprates but can serve as a good illustration. Consider a square placket formed by sites 1, 2, 3, and 4 referring to ions \(R_1, R_2 = R_1 + \hat{x}, R_3 = R_1 + \hat{x} + \hat{y},\) and \(R_4 = R_1 + \hat{y}\) respectively. Among various terms in \(H_{\text{int}}\), we find

\[
\langle 1, 3|\hat{U}|2, 4 \rangle \left( e_{1\alpha}^\dagger e_{3\beta}^\dagger e_{2\beta} c_{2\alpha} + e_{1\alpha}^\dagger e_{3\beta}^\dagger e_{2\beta} c_{4\alpha} \right) + \text{H.c.},
\]

where we used \(\langle 1, 3|\hat{U}|4, 2 \rangle = \langle 1, 3|\hat{U}|2, 4 \rangle\) from the assumed orbital symmetries. The pair-hopping term in the brackets is precisely the ring term in Eq. (83), with the coupling

\[
K = \langle 1, 3|\hat{U}|2, 4 \rangle
= \int_{r, r'} w_1^*(r) w_3^*(r') u_{\text{Coulomb}}(r, r') w_2(r) w_4(r').
\]

The \(w_{1,2,3,4}\) orbitals are peaked on ions \(R_{1,2,3,4}\) respectively. Assuming fairly localized orbitals, the main contribution in the above integral will come from configurations where \(r\) is somewhere between sites \(R_1\) and \(R_2\), while \(r'\) is somewhere between sites \(R_3\) and \(R_4\). Then we could approximate \(u_{\text{Coulomb}}(r, r') \approx u_{\text{Coulomb}}(a) f_{12}(r) f_{34}(r')\), \(u_{\text{Coulomb}}(a) \approx u_{\text{Coulomb}}(|R_1 + R_2|/2, |R_3 + R_4|/2)\), and obtain:

\[
K \approx u_{\text{Coulomb}}(a) \int_{r} w_1^*(r) w_2(r) f_{12}(r) \int_{r'} w_3^*(r') w_4(r') f_{34}(r')
= u_{\text{Coulomb}}(a) \int_{r} w_1^*(r) w_2(r) f_{12}(r)^2.
\]
Here \( f_{34}(r) \) is an \( O(1) \) function peaked between the sites \( \mathbf{R}_1 \) and \( \mathbf{R}_2 \), and similarly for \( f_{34}(r') \). We introduced these functions ad-hoc to implement an observation that \( O(1) \) variations of \( u_{\text{Coul}}(r, r') \) with \( r \) or \( r' \) will eliminate cancellations that lead to orthogonality of the Wannier orbitals; instead, we anticipate obtaining factors like \( \int_{\mathbf{R}} w_{1}^{*}(\mathbf{r}) w_{2}(\mathbf{r}) f_{12}(\mathbf{r}) \), which is on the order of the overlap of the non-orthogonal atomic orbitals. It would clearly be desirable to perform such calculations more accurately in realistic contexts. Here we want to point out that the above estimate gives a positive value for \( K \), a key assumption we have made in our electronic ring model and presumably necessary for the \( d \)-metal. More realistic calculations would hopefully give a reliable estimate of the sign of \( K \) as well as its magnitude.

Because of the overlap integrals, the ring term will be significantly smaller than the on-site Hubbard repulsion. However, we treat the latter by prohibiting double occupancy, and then the relevant energy scales to compare with are some effective hopping and antiferromagnetic spin exchange couplings. Note that the hopping amplitudes themselves are set by overlaps between atomic orbitals times typical ionic potentials, so it is not inconceivable to estimate the ring terms as \( K \sim t^2 / u_{\text{Coul}} \), which can be comparable to the spin exchange couplings.

It is important to note that the electron ring terms in our work are different from four-spin ring exchange terms that arise at order \( t^4 / U^3 \) in effective spin models for so-called weak Mott insulators.\(^8\)\(^2\) Our electron ring terms are also four-site terms but move two charges from one diagonal of a square to the other previously unoccupied diagonal. Thus they are four-fermion rather than four-spin (eight-fermion) terms and can arise more directly from the Coulomb interaction.

Finally, as noted in the previous section, to search for seeds of \( d_{x^2-y^2} \) pair correlations in the context of the cuprates, we would want to consider electron ring terms that act on four sites \( \mathbf{R}_1, \mathbf{R}_3 + \hat{x} + \hat{y}, \mathbf{R}_1 + 2\hat{y} \) and \( \mathbf{R}_1 - \hat{x} + \hat{y} \). More reliable estimates of such terms would be highly desirable, as well as estimates of all other terms in the projected Coulomb interaction Eq. (89).