5.B5 NATURAL CONVECTION AS A HEAT ENGINE: A THEORY FOR CAPE

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1 INTRODUCTION

In this study we look at planetary convection from a large-scale perspective, that is, in statistical equilibrium. We focus on the physical process by which convection adjusts unstable atmospheres. Our hypothesis is that atmospheric convection is a natural heat engine. Our objective is to present a framework usefull for the basic conceptual understanding of the equilibrium state of convecting atmospheres.

2 THEORY

Since the warm moist adiabatic updraft and the cold moist adiabatic downdraft (that includes the falling precipitation) are the most important features of deep convective systems, these systems can be idealized as a Carnot heat engine. Here, it is important to note that *the essential feature of a heat engine is the fact that heat must be absorbed by the working fluid at a higher temperature than heat is rejected.* The flow does not have to be steady, and the working fluid can exchange mass with the environment. Indeed, this is what happens in most engineering heat engines (e.g. gas turbines and internal combustion engines). In order for a real heat engine to approach the Carnot efficiency, the thermodynamic process must be thermally and mechanically reversible which does not occur in natural convection. However, since unstable systems drift towards states of maximum efficiency, nature probably strives towards the Carnot efficiency.

We assume that convection is in quasi-equilibrium, and we follow the convecting air parcel around a streamline (in a closed cycle). Since the closed steady circulation might exist only in a statistical sense, we might imagine that we are following a hypothetical air parcel around an averaged streamline representing a collection of convective cells in statistical equilibrium. The energy cycle of a heat engine can be obtained by integrating Bernoulli's equation and the first law of thermodynamics around a streamline (Emanuel, 1986, 1989), that is

$$Tds - d(\frac{1}{2}|\vec{v}|^2 + c_pT + L_vr + gz) - \vec{F}\cdot\vec{dl} = 0, \quad (1)$$

where T is the absolute temperature, s the specific entropy, $\vec{v}$ the vector velocity, c_p the heat capacity of dry air at constant pressure, L_v the latent heat of vaporization of water, r the water vapor mixing ratio, g the gravity acceleration, z the height above a reference level, $\vec{F}$ the frictional force per unit mass, and $\vec{dl}$ the incremental distance along the streamline.

Integrating equation 1 around a closed cycle, we get

$$\oint Tds - \oint \vec{F}\cdot\vec{dl} = 0, \quad (2)$$

which states that in statistical equilibrium, friction is balanced by the heat input, Tds. Integrating the first law of thermodynamics, we get $\oint Tds = \oint pd\alpha$, where p is the pressure and α is the specific volume. This shows that the first term in equation 2 represents the net work done by the heat engine cycle.

Since the net work done by the heat engine is equal to the total mechanical energy available for convection, we define

$$\boxed{TCAPE \equiv \oint Tds,} \quad (3)$$

where $TCAPE$ is the total convective available potential energy from the heat engine. It is the energy that can be converted to kinetic energy by a heat engine. It includes the available energy that is converted to kinetic energy by both the updraft and the downdraft.

In a thermodynamic diagram, equation 3 integrated around a Carnot cycle represents the area enclosed by the hot and the cold adiabats (s_2 and s_1), and the hot and the cold isotherms (T_h and T_c), respectively at the bottom and the top of a Carnot convective circulation. Thus, the only difference between our definition of total CAPE, and the "standard" meteorological definition, is that instead of using pressure levels as the limit of the integration we use temperature. This is not a problem. We could have obtained the "standard" definition of total CAPE by defining a heat engine cycle between two adiabats and two isobars. However, we argue that there might be no physical reasons for doing that.

It follows from the above that for a general convective heat engine, in statistical equilibrium conditions, we must have

$$TCAPE - \oint \vec{F} \cdot \vec{dl} = 0, \tag{4}$$

which states that in statistical equilibrium, friction is balanced by $TCAPE$. Therefore, *a zero total CAPE value is impossible in convecting atmospheres, except in a hypothetical planet in which the convective motions are inviscid.*

Multiplying equation 4 by the convective mass flux, M_c, we get

$$M_c TCAPE - F_d = 0, \tag{5}$$

where $F_d = M_c \oint_c \vec{F} \cdot \vec{dl}$ is the flux of energy mechanically dissipated by the convective system. The convective mass flux is given by $M_c = \rho\sigma w$, where ρ is the updraft air density, w is the magnitude of the mean vertical velocity, and σ is the fractional area covered by updrafts. *At quasi-equilibrium the rate of energy dissipation by convection is equal to the flux of energy available for the heat engine, F_{av}, that is $F_d = F_{av}$.* The flux of energy available for mechanical work can be written as $F_{av} = \epsilon F_{in}$, where ϵ is the thermodynamic efficiency of the heat engine, and F_{in} is the heat input to the heat engine. Thus, at quasi-equilibrium we have that

$$\boxed{TCAPE = (\tfrac{1}{M_c})\epsilon F_{in}.} \tag{6}$$

For a Carnot engine, the thermodynamic efficiency is given by $\epsilon = (\frac{T_h - T_c}{T_h})$, where T_h and T_c are, respectively, the temperatures of the hot and cold reservoirs.

The heat input to the convective heat engine can be written as $F_{in} = M_c(c_p\Delta T + L_v\Delta r)$, where ΔT and Δr are, respectively, the near surface temperature and moisture excess of the convective updrafts over the downdrafts. Thus, using equation 6 it follows that

$$\boxed{TCAPE = \epsilon(c_p\Delta T + L_v\Delta r),} \tag{7}$$

Equation 7 shows that *only a fraction ϵ of the energy released by deep moist convection is available to be transformed into kinetic energy.*

Now, we use dimensional analysis and physical arguments to propose a similarity theory for the fractional area covered by updrafts, σ, valid for both dry and moist convection. In quasi-equilibrium conditions, the entropy excess of the convective updrafts over the downdrafts, must be lost by the emission of infrared radiation to space by the subsiding air parcels (in between the top of convective updrafts and the root of the convective downdrafts). Thus, the magnitude of the mean vertical velocity of the subsiding air parcel is given by

$$w_s = (\frac{\Delta p}{\rho g \tau_r}), \tag{8}$$

where Δp is the pressure thickness (convective mass flux weighted average) of the layer radiating to space (the subsiding layer), g is the gravity acceleration, and τ_r is the radiative timescale. A rough estimate of τ_r can be obtained by considering a slab of atmosphere of pressure thickness Δp and uniform density radiating like a black body (Houghton, 1986), in this case one gets

$$\tau_r = (\frac{c_p \Delta p}{8 g \sigma_r {T_c}^3}), \tag{9}$$

where the parameter σ_r is the Stefan-Boltzmann constant, and T_c is the actual mean temperature of the subsidence region (not the radiative equilibrium temperature, it is the planet's effective temperature).

Using the heat engine framework, we get

$$\epsilon F_{in} = M_c TCAPE. \tag{10}$$

Using the continuity equation, we get

$$\epsilon F_{in} = \rho(\frac{1}{2\mu})(\frac{{w_s}^3}{\sigma^2}), \tag{11}$$

where $(\frac{1}{\mu}) \equiv (\frac{2\oint \vec{F}\cdot\vec{dl}}{w^2})$ is a mechanical dissipation of energy parameter. Substituting equation 8 into equation 11, we get

$$\sigma = \frac{1}{\sqrt{2\mu\epsilon}}(\frac{\Delta p}{\rho g \tau_r})^{(\frac{3}{2})}(\frac{F_{in}}{\rho})^{(-\frac{1}{2})}. \tag{12}$$

Defining $\tau_c \equiv (\frac{\Delta p}{\rho g w})$, we get

$$\boxed{\sigma = \frac{\tau_c}{\tau_r},} \tag{13}$$

where for physically possible solutions we must have $0 < \sigma < 1$. Since in the derivation of the above equation we assumed $(1-\sigma) \approx 1$, it is valid only for $\sigma \ll 1$ (i.e., for $\tau_c \ll \tau_r$). Note that the similarity theory predicts decreases in the fractional area covered by updrafts with decreases in the convective timescale and increases in the radiative timescale.

Based on the above results, we get

$$TCAPE = (\frac{g\,\tau_r}{\Delta p})\epsilon F_{in}. \qquad (14)$$

Equation 14 suggests that increases in the efficiency of the convective heat engine, lead to increases in $TCAPE$. This happens because increases in the thermal efficiency of the convective heat engine lead to increases in the amount of energy that is available to be converted into mechanical work. Thus, the equilibrium state is one of stronger circulations and larger dissipations. Increases in the radiative timescale, or decreases in the pressure thickness of the layer radiating to space (a Chapman layer in a optically thick atmosphere) leads to increases in $TCAPE$. This happens because they both lead to decreases in the fractional area covered by convective updrafts. Thus, larger $TCAPE$ values are necessary to maintain a given convective heat flux.

For tropical values of the various parameters ($\epsilon = 0.10$, $F_{in} = 150$, $\Delta p = 4 \times 10^4$, $\rho = 1$, $g = 9.8$, $\tau_r = 7.5 \times 10^5$, and $\mu = 0.5$, all in SI units) equation 12 gives $\sigma = 1 \times 10^{-4}$. This is a reasonable value, since only about 3% of the tropics is covered by active deep convection, and only about 1% of the fractional area covered by deep convection might be covered by undiluted updrafts (Cotton and Anthes, 1989). With the above parameters, equation 14 gives $TCAPE = 2900JKg^{-1}$, which is of the order of magnitude of the observed value.

Alternatively, for an atmosphere radiating as a black body from a layer of temperature T_c, one can use equation 9, getting

$$\boxed{TCAPE = (\frac{c_p}{8\sigma_r T_c{}^3})\epsilon F_{in}.} \qquad (15)$$

In this case, one should also use equations 9 and 12, getting

$$\boxed{\sigma = \frac{1}{\sqrt{2}\,\mu\,\epsilon}(\frac{8\,\sigma_r\,T_c^3}{\rho\,c_p})^{(\frac{3}{2})}(\frac{F_{in}}{\rho})^{(-\frac{1}{2})}.} \qquad (16)$$

Substituting $TCAPE$ by $(\frac{1}{2\mu})w^2$ into equation 15, we get an expression for the magnitude of the mean convective velocity

$$\boxed{w = \sqrt{(\frac{c_p}{8\sigma_r T_c{}^3})2\mu\epsilon F_{in}}.} \qquad (17)$$

Equation 15 shows that increases in the atmosphere's heat capacity lead to linear increases in the equilibrium $TCAPE$ value. It also shows that decreases in the atmosphere's "effective temperature" (T_c) lead to both, a linear increase in the equilibrium $TCAPE$ value through increases in the heat engine efficiency, and a strong nonlinear increase through decreases in the emission of thermal radiation to space. This happens because they both lead to decreases in the fractional area covered by convective updrafts.

In a typical tropical storm system the updrafts are about $2\,K$ warmer than the downdrafts (Zipser, 1977). Thus, assuming $\epsilon = 0.1$, $\Delta T = 2\,K$, $\Delta r = 0.002$ (from the assumption that the surface temperature is $299\,K$, and that both the updrafts and the downdrafts are saturated) we get $TCAPE = 700JKg^{-1}$. In a typical midlatitude storm system, $\Delta T = 4\,K$ (Newton, 1966), in this case we get $TCAPE = 1650Jkg^{-1}$. These results are of the same order of magnitude as the $TCAPE$ values predicted by equation 14.

3 CONCLUSIONS

We postulate that, from the large-scale perspective, planetary convection works as a heat engine. The mechanical work produced by the heat engine is used to force convective motions that are, then, mechanically dissipated into heat. On the large-scale, the convective motions are just strong enough so that the net work done by the heat engine is used exclusively to overcome the mechanical dissipation opposing the convective circulations. The volume integral of the work produced by the planetary heat engine provides a measure of the quasi-equilibrium amount of total CAPE present on the planet's atmosphere. This CAPE value is a fundamental global number qualifying the state of the planet in quasi-equilibrium conditions.

4 BIBLIOGRAPHY

Emanuel, K. A., 1986. An Air-Sea Interaction Theory for Tropical Cyclones. Part I: Steady-State Maintenance. *J. Atmos. Sci.*, **43**, 585-604.

Emanuel, K. A., 1989. Polar Lows as Artic Hurricanes. *Tellus*, **41A**, 1-17.

Cotton, W. R., and R. A. Anthes, 1989. Storm and Cloud Dynamics. *Academic Press*, New York. 883 pp.

Houghton, J. T., 1986. The Physics of Atmospheres. *Cambridge Univ. Press*, Cambridge. 271 pp.

Newton, C. W., 1966. Circulations in Large Sheared Cumulunimbus. *Tellus*, **18**, 699-712.

Zipser, E. J., 1977. Mesoscale and Convective-Scale Downdrafts as Components of Squall-Line Circulation. *Mon. Wea. Rev.*, **105**, 1568-1589.