The response of a nonlinear string to deterministic loading has been analyzed by several authors. G. F. Carrier; E. W. Lee; and the author have analyzed the response of a nonlinear string to deterministic loading. This paper is devoted to the study of a nonlinear string subjected to a random excitation. It is shown that, as in the case of deterministic loading, the additional stress induced by the stretching of the string reduces the mean squared deflection at every point compared with that for the equivalent linear string.

**Formulation**

Consider an elastic string, clamped at its ends subjected to a random loading. The nomenclature used is as follows:

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>( u )</td>
<td>transverse deflection of string</td>
</tr>
<tr>
<td>( T_0 )</td>
<td>initial tension in string</td>
</tr>
<tr>
<td>( \rho )</td>
<td>linear density of string</td>
</tr>
<tr>
<td>( \rho \beta )</td>
<td>viscous damping coefficient</td>
</tr>
<tr>
<td>( A )</td>
<td>cross-sectional area of string</td>
</tr>
<tr>
<td>( E )</td>
<td>elastic modulus of string</td>
</tr>
<tr>
<td>( L )</td>
<td>length of string</td>
</tr>
<tr>
<td>( f(x, t) )</td>
<td>loading on string</td>
</tr>
</tbody>
</table>

If the deflections and slopes are considered to be moderately small (consistent with the string remaining elastic) the equation of motion of the string is

\[
\rho \frac{\partial^2 u}{\partial t^2} + \beta \frac{\partial u}{\partial t} = \left[ T_0 + \frac{AE}{2L} \int_0^L \left( \frac{\partial u}{\partial x} \right)^2 \, dx \right] \frac{\partial^2 u}{\partial x^2} + f(x, t) \tag{1}
\]

subject to the boundary conditions that

\[
u(0, t) = u(L, t) = 0 \tag{2} \]

In the analysis that follows it will be assumed that \( f(x, t) \) is a randomly varying force with a clipped Gaussian white spectrum and has the following cross-correlation:

\[
f(x, t) f(\eta, t) = 4D \delta(x - \eta) \sin \omega_\tau \tag{3}
\]

where

\[
\tau = t - \eta
\]

Since the eigenmodes for equation (1) are sine waves, let

\[
\begin{align*}
\nu(x, t) &= \sum_{i=1}^N a_i(t) \sin \frac{i\pi x}{L} \\
\nu(x, t) &= \sum_{i=1}^N b_i(t) \sin \frac{i\pi x}{L}
\end{align*} \tag{4}
\]

where \( N \) is chosen such that \( \omega_N < \omega < \omega_{N+1} \).

As a consequence of the assumed form of the cross-correlation (3), the \( a_i \) in (4) are uncorrelated and the power spectrum \( g_i(\omega) \) of \( a_i \) is given by

\[
g_i(\omega) = \frac{2AD}{L^2} \frac{\omega_i}{2\pi} 0 < \omega < \omega_i \tag{5}
\]

Substituting (4) into (1)

\[
\frac{\partial^2 b_i}{\partial t^2} + \beta \frac{\partial b_i}{\partial t} + \left( \frac{i\pi}{L} \right)^2 \frac{T_0}{\rho} \left[ 1 + \frac{AE}{2LT_0} \times \int_0^L \left( \sum_{i=1}^N b_i \sin \frac{i\pi x}{L} \right)^2 \, dx \right] b_i = a_i(t)/\rho \tag{6}
\]

Using the orthogonality of the circular functions, equation (6) becomes

\[
\frac{\partial^2 b_i}{\partial t^2} + \beta \frac{\partial b_i}{\partial t} + \left( \frac{i\pi}{L} \right)^2 T_0 \rho \left[ 1 + \alpha \sum_{i=1}^N i^2 b_i \right] b_i = a_i(t)/\rho \tag{7}
\]

where

\[
\alpha = \frac{AE}{4TL} \left( \frac{\pi}{L} \right)^2 \tag{8}
\]

**Equivalent Linear Form of Equation (7)**

The method of equivalent linearization has been applied successfully in the past to nonlinear differential equations with harmonic excitation. It will now be shown that, by a slightly different approach to the problem, this method may be applied to equation (7).

Rewriting equation (7) in the form

\[
\frac{\partial^2 b_i}{\partial t^2} + \beta \frac{\partial b_i}{\partial t} + \left( \frac{i\pi}{L} \right)^2 T_0 \rho \left[ 1 + \alpha \sum_{i=1}^N i^2 b_i \right] b_i = \frac{a_i(t)}{\rho} \tag{9}
\]
\[
\frac{\partial^2 b_n}{\partial t^2} + \beta \frac{\partial b_n}{\partial t} + \left( \frac{n\pi}{L} \right)^2 \frac{T_0}{\rho} K_n b_n + \epsilon(b_1, b_2, \ldots b_N) = \frac{a_n}{\rho} \quad (9)
\]

If in (9), \( \epsilon(b_1, b_2, \ldots b_N) \) is neglected, (9) is linear and can be solved readily. The smaller \( \epsilon(b_1, b_2, \ldots b_N) \) is, the smaller the error in neglecting it. The logical choice of \( K_n \) is, therefore, that value of \( K_n \) which minimizes \( \epsilon(b_1, b_2, \ldots b_N) \).

The choice of minimization procedure is somewhat arbitrary, but, as with the analogy with Galerkin's method, and for mathematical expediency, it is desirable to use the minimization of the mean squared error

\[
\bar{e}^2 = \frac{1}{2T} \int_{-T}^T \left[ \left( \frac{n\pi}{L} \right)^2 \frac{T_0}{\rho} b_n - K_n b_n \right]^2 \, dt \quad (10)
\]

If the process is an ergodic one, then equation (10) may be replaced by the stochastic average. Further, since the \( a_i \) are uncorrelated

\[
\bar{e}^2 = \left[ \prod_{i=1}^N (2\pi)^{1/2} \sigma_i \right]^{-1} \int_{-\infty}^\infty \left[ \left( \frac{n\pi}{L} \right)^2 \frac{T_0}{\rho} b_n - K_n b_n \right]^2 \, \exp \left( -b_i^2/2\sigma_i^2 \right) \, db_n \quad (11)
\]

The power spectrum of \( b_n \) is therefore given by

\[
P_n(\omega) = 2\pi \left\{ \left( \frac{n\pi}{L} \right)^2 \frac{T_0}{\rho} - K_n \right\} \left[ 1 + \alpha \sum_{i=1}^N \sigma_i^2 \right] \left[ \exp \left( -b_i^2/2\sigma_i^2 \right) \right] \quad (12)
\]

The mean squared value of \( b_n \) is readily computed from (17). Thus

\[
\frac{\partial^2 b_n}{\partial t^2} + \beta \frac{\partial b_n}{\partial t} + \left( \frac{n\pi}{L} \right)^2 \frac{T_0}{\rho} K_n b_n + \epsilon(b_1, b_2, \ldots b_N) = \frac{a_n}{\rho} \quad (9)
\]

If in (9), \( \epsilon(b_1, b_2, \ldots b_N) \) is neglected, (9) is linear and can be solved readily. The smaller \( \epsilon(b_1, b_2, \ldots b_N) \) is, the smaller the error in neglecting it. The logical choice of \( K_n \) is, therefore, that value of \( K_n \) which minimizes \( \epsilon(b_1, b_2, \ldots b_N) \).

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\]

where \( \exp \left( -b_i^2/2\sigma_i^2 \right) \) is the first probability density function of \( b_i \).

If in (9), \( \epsilon(b_1, b_2, \ldots b_N) \) is neglected, \( b_n \) is Gaussian since \( a_n \) is Gaussian. Thus

\[
p(b_i) = \frac{1}{(2\pi)^{1/2} \sigma_i} \exp \left( -b_i^2/2\sigma_i^2 \right) \quad (12)
\]

Minimizing \( \bar{e}^2 \) with respect to \( K_n \):

\[
K_n = \left[ \int_{-\infty}^\infty \cdots \right]^{-1} \int_{-\infty}^\infty \cdots \left[ \left( \frac{n\pi}{L} \right)^2 \frac{T_0}{\rho} b_n - K_n b_n \right]^2 \, \exp \left( -b_i^2/2\sigma_i^2 \right) \, db_n \quad (11)
\]

Evaluating the integrals:

\[
K_n = 1 + \alpha \left[ \sum_{i=1}^N i^2 \sigma_i^2 + 2n^2 \right] \sigma_i^2 \quad (15)
\]

If in equation (9) \( \epsilon \) is neglected, then

\[
\frac{\partial^2 b_n}{\partial t^2} + \beta \frac{\partial b_n}{\partial t} + \left( \frac{n\pi}{L} \right)^2 \frac{T_0}{\rho} K_n b_n = \frac{a_n(t)}{\rho} \quad (16)
\]

The power spectrum of \( b_n \) is therefore given by

\[
P_n(\omega) = g_n(\omega)^{-1} \left[ \left( \frac{n\pi}{L} \right)^2 \frac{T_0}{\rho} - K_n \right]^{-1} \left[ 1 + \alpha \sum_{i=1}^N \sigma_i^2 \right] \quad (17)
\]

where \( g_n(\omega) \) is given by equation (5).
The positive root being taken in (25) since \( y_\star \) by definition is positive. Multiplying both sides of (25) by 4\( z \), and summing on \( n \)

\[
4zS = -N(1 + zS) + N[(1 + zS)^2 + 8z]^{1/2}
\]  
(26)

Rearranging terms and squaring

\[
1 + z\left(1 + \frac{4}{N}\right)S = (1 + zS)^2 + 8z
\]
(27)

Solving for \( zS \):

\[
zS = \left[-1 + (1 + 4(N + 2)z)^{1/2}\right]^{-1} \left[2\left(1 + \frac{2}{N}\right)\right]^{-1}
\]
(28)

Since \( zS \) is positive, the positive radical is taken.

Substituting (28) into (25), \( y_\star \) may be calculated. It will be observed from (25) that \( y_\star \) is independent of \( n \), thus there is equipartition of energy among the modes.

### Calculation of Mean Squared Deflection \( \bar{u^2} \)

Using equation (4)

\[
\bar{u^2} = \left[\sum_{i=1}^{N} b_i \sin^2 \frac{i \pi x}{L}\right]
\]
(29)

\[
\bar{u^2} = \sum_{i=1}^{N} b_i^2 \sin^2 \frac{i \pi x}{L}
\]
(29)

Owing to the particular cross correlation function (3), the \( b_i \) are uncorrelated: thus

\[
b_i b_j = 0 \quad i \neq j
\]
(31)

Equation (30) therefore becomes

\[
\bar{u^2} = \sum_{i=1}^{N} b_i^2 \sin^2 \frac{i \pi x}{L}
\]
(30)

From equations (23) and (25)

\[
\bar{b^2} = \sigma_i^2 = y_i \sigma_{i^2}^2
\]

\[
= \sigma_{i^2}^2 \left[-(1 + zS) + [(1 + zS)^2 + 8z]^{1/2}\right]^{-1}
\]
(33)

Substituting (33) into (32)

\[
\bar{U^2} = \sigma_i \bar{u^2} \left[-(1 + zS) \right.

\[+ [(1 + zS)^2 + 8z]^{1/2}\] \left[ight]^{-1} \sum_{i=1}^{N} \frac{1}{i^2} \sin^2 \frac{i \pi x}{L}
\]
(34)

In the linearized problem given by setting \( \alpha = 0 \) in (7), the mean squared deflection \( \bar{U^2} \) is given by

\[
\bar{U^2} = \sigma_i \bar{u^2} \sum_{i=1}^{N} \frac{1}{i^2} \sin^2 \frac{i \pi x}{L}
\]
(35)

Thus

\[
\bar{U^2} = \left[-(1 + zS) + [(1 + zS)^2 + 8z]^{1/2}\right]^{-1}
\]
(36)

where \( zS \) is given by equation (28).

If in (36) \( z \ll 1 \), (36) reduces to

\[
\bar{U^2} \approx (1 + zS)^{-1} \leq 1
\]
(37)

Further, if in (28) \( N \) is large,

\[
zS \approx 1/\sqrt{-1 + (1 + 4Nz)^{1/2}}
\]
(38)

Using (38), equation (37) becomes

\[
\frac{\bar{U^2}}{\bar{u^2}} \approx 2[1 + (1 + 4Nz)^{1/2}]^{-1} \leq 1
\]
(39)

Thus the mean squared deflection in the nonlinear case is always less than the mean squared deflection in the equivalent linear string.

### Conclusions

From the foregoing analysis it is clearly seen that the effect of the stress induced by the stretching of the string is to reduce the mean squared deflection at every point in the string compared with the corresponding mean squared deflection in the equivalent linear string.

### APPENDIX

(a) Evaluation of integral (18). Let

\[
\left(\frac{n \pi}{L}\right)^2 \frac{T_n}{\rho} K_n = \tilde{\omega}_n^2
\]

Thus

\[
\sigma_n^2 = 2Nd/\rho L \int_0^\infty \left[\tilde{\omega}_n^2 - \omega^2 \right] + \left[|\beta \omega^2| \right]^{-1} d\omega
\]
(40)

Using integral 211.16c of "Integraltaffel, Ester Teil, Unbestimmte Integral," by W. Grobner and N. Hofreiter (Springer, 1949), equation (40) becomes

\[
\sigma_n^2 = 2D/\rho L \omega \left[\frac{\sin \phi_0 \left(\tilde{\omega}_n^2 - \omega_0^2\right) + \sin \phi_0 \left(\tilde{\omega}_n^2 - \omega_0^2\right)}{\omega_0^2 - \omega_0^2} \right]^{-1/2}
\]
(41)

where

\[
\tan \phi_0 = \frac{\beta (\tilde{\omega}_n^2 - \omega_0^2)}{\omega_0^2 - \omega_0^2}
\]
(42)

If \( \beta \) is small compared to \( \tilde{\omega}_n \), equation (41) reduces to

\[
\sigma_n^2 \approx 2D/\rho L \omega_0^2 \beta \left[\tan^{-1} \left(\tilde{\omega}_n^2 - \omega_0^2\right) \right]^{-1/2}
\]
(43)

For \( \beta/\tilde{\omega}_n \) very small, the second term may be neglected entirely.

Thus

\[
\sigma_n^2 \approx 2D/\rho L \omega_0^2 \beta \left[\tan^{-1} \left(\tilde{\omega}_n^2 - \omega_0^2\right) \right]^{-1/2}
\]
(44)

If \( \omega_0 > \tilde{\omega}_n \)

\[
\sigma_n^2 \approx 2D/\rho L \tilde{\omega}_n^2 \beta
\]
(45)

If \( \tilde{\omega}_n > \omega_0 \)

\[
\sigma_n^2 \approx 2D/\rho L \omega_0^2 \beta \left[\omega_0^2 - \omega_0^2\right]^{-1/2}
\]
(46)

This justifies the approximate representation of \( u \) in equation (4).

(b) Justification for neglecting term \( z \sum_{N+1}^{n} \delta^i (\sigma_i^2/\sigma_n^2) \).

As is shown in (46), for \( \tilde{\omega}_n > \omega_0 \)

\[
\sigma_i^2 \approx 2D\omega_0 [p L \pi \tilde{\omega}_n (\tilde{\omega}_n^2 - \omega_0^2)]^{-1}
\]
(47)
For $\alpha$ small

$$\sum_{N+1}^\infty \frac{\sigma^2_t}{\sigma_{1,0}^2} \leq \frac{\beta}{2\pi \omega_0 K_{N+1}^{1/4}} \ln \left[ \frac{N + 2}{N} \right]$$

If $\beta$ is small and $N$ is large, this ratio becomes very small and may be neglected, thus justifying the treatment given in the paper.