ON THE THEORETICAL REQUIREMENTS FOR A PERIODIC BEHAVIOUR OF THE UNIVERSE

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Abstract

This article investigates, on the basis of relativistic mechanics and relativistic thermodynamics, the theoretical requirements for nonstatic models of the universe to exhibit a behaviour which is periodic in time. The discussion is limited to models which can be regarded from a large-scale point of view as filled with a uniform distribution of fluid, and the time coordinate is purposely chosen so as to agree with the proper time that would be used by local observers at rest in this fluid. It is first shown that the analytical requirements which would correspond to the continuous expansion and contraction of such models between definite maximum and minimum limits, together with the thermodynamic requirement that the expansion and contraction must be reversible, could not both be simultaneously satisfied with any fluid for filling the model which has reasonable properties. Attention is then turned to models of the universe having line elements which would be quasi-periodic in the time. A wide range of possibilities is found for models which would expand from zero proper volume to a maximum and then return, the treatment failing however to provide the analytical conditions for a minimum at the lower limit. It is pointed out, nevertheless, that from a physical point of view contraction to zero proper volume could only be followed by renewed expansion, so that we might expect for such models a continued series of expansions and contractions. Furthermore, it is found possible to satisfy the analytical requirements for such quasi-periodic behaviour by models which would expand and contract at a finite rate reversibly without increase in entropy, so that we might then expect a continued series of *identical* expansions and contractions. It is pointed out that Einstein's recent model of a nonstatic universe filled with incoherent matter, exerting no pressure and unaccompanied by radiation, would satisfy the thermodynamic as well as the analytical requirements for such a series of identical repetitions, and models filled solely with black-body radiation and with an equilibrium mixture of radiation and perfect gas are discussed which also satisfy these requirements. In conclusion some remarks are made concerning the bearing of such findings on the behaviour of the actual universe.

§1. Introduction

THE problem of the ultimate fate of our physical surroundings has long been the subject of human thought and speculation. *A priori*, three general types of future behaviour of the universe as a whole would appear to be possible. First, the universe might permanently remain in an approximately steady state; second, it might be subject to progressive change in some particular direction; and third, it might undergo a periodic change in properties, returning at definite intervals at least approximately to an earlier condition.[1]

[1] Different combinations of these general types of behaviour would of course also be possible. Thus we might have periodic happenings taking place simultaneously in different parts of the universe so related as to assure a steady state for the universe looked at as a whole. Or we might have a quasi-periodic behaviour superimposed on a general progressive change in a particular direction.

The possibility for the universe to be in an approximately steady state has usually been regarded as untenable, on account of the implications of the second law of thermodynamics and the actual observation of such important unidirectional phenomena as the flow of radiation from the stars. But interesting suggestions as to compensatory processes, which might really be maintaining the universe in a steady state, have sometimes been made.[2]

The second possibility of a progressive change in the properties of the universe in a particular direction, that of maximum entropy, has usually been regarded in the past as a necessary consequence of the second law of thermodynamics and has furnished what may be called the orthodox view of physics. This view might carry with it, however, the unsatisfactory implication that the universe was created with a supply of available energy at a finite time in the past. Moreover, the apparent inevitability of the view as a necessary consequence of the second law is diminished both by the possibility of fluctuations away from the maximum of entropy discovered many years ago by Boltzmann,[3] and by the previously unsuspected possibility for thermodynamic changes to take place at a finite rate without increase in entropy, which I have recently shown[4,5] to be allowed by the second law of thermodynamics in the form which it assumes in the extension of thermodynamics to general relativity.[6]

The third possibility, that the universe might be subject to a regular periodic change in properties, will be discussed in the present paper. This alternative has been discarded in the past since the classical form of the second law of thermodynamics would permit thermodynamic changes to take place in the universe at a finite rate only in the direction of increasing entropy, thus preventing any return of the universe to an earlier condition. It has now been demonstrated, however, that the relativistic form of the second law of thermodynamics would permit important changes to take place at a finite rate entirely reversibly without any increase in entropy at all. Thus in the case of a model of the universe filled solely with radiation, I have shown[4] that a reversible expansion of the universe could take place at a finite rate without increase in entropy, and that this expansion would be accompanied by a number of phenomena, such as the flow of radiation away from any given observer out into space, which would previously have been regarded as necessarily irreversible. And in the case of a model filled with a mixture of perfect gas and radiation, under the subsidiary assumption that the matter maintains

[2] See for example MacMillan, Astrophys. J. **48,** 35 (1919); Science **62,** 63, 96, 121 (1925); Millikan and Cameron, Proc. Nat. Acad. **14,** 637 (1928); Millikan, Nature **126,** 14 (1930).

[3] Boltzmann, Wied. Ann. **60,** 392 (1897).

[4] Tolman, Phys. Rev. **37,** 1639 (1931).

[5] Tolman, Phys. Rev. **38,** 797 (1931).

[6] The principles for extending thermodynamics to general relativity were developed in previous papers:—Tolman, Proc. Nat. Acad. **14,** 268 (1928); ibid. **14,** 701 (1928); Phys. Rev. **35,** 875 (1930); ibid. **35,** 896 (1930); and have been applied to problems of a different character than the present in other papers:—Tolman, Proc. Nat. Acad. **14,** 348 (1928); ibid. **14,** 353 (1928); ibid. **17,** 153 (1931); Phys. Rev. **35,** 904 (1930); Tolman and Ehrenfest, Phys. Rev. **36,** 1791 (1930).

itself in equilibrium with the radiation, I have been able to show[5] that a reversible transformation of matter into radiation could take place, and would be accompanied by a reversible expansion and reversible flow of radiation out into space.

Such examples of reversible changes make it very interesting to investigate the occurrence of models of the universe which would exhibit periodic behaviour, since such would now appear thermodynamically possible. It was pointed out, nevertheless, in the articles cited that neither of the above models would actually furnish a strictly periodic solution of the equations of motion, and this will be confirmed by more general considerations in the present article. Recently, however, a simple model of the universe has been discussed by Einstein[7] which exhibits a possibility for quasi-periodic solutions of a type which must now also be considered.

§2. Outline of Present Article

The discussion in the present article will be limited to models in which the universe can be regarded from a large-scale point of view as filled with a perfect fluid, having properties which change with the time but are independent of position. Furthermore, the time coordinate will be chosen so as to agree with the proper time as used by local observers at rest with respect to the material in the model, since it is evident that equations periodic in form with respect to an arbitrarily chosen time coordinate might have no special physical interest.

The possibilities for change with time in such models consist in an expansion or contraction in the proper volume of the universe. The equations of motion for such changes will first be obtained from the principles of relativistic mechanics, and the analytical requirement for these equations to correspond to a periodic expansion and contraction will be exhibited. Combining this analytical requirement with the thermodynamic requirement for periodic behaviour, namely that the motion should be reversible in the sense of relativistic thermodynamics, it will then be shown that no strictly periodic solution for the equations of motion would be possible for a model of this kind filled with any combination of incoherent matter and radiation. This general result confirms the conclusion already found for the two special models mentioned above, and also has the consequence of ruling out as physically probable the very interesting periodic line element recently proposed by Takéuchi.[8]

At first sight the result would appear to prove conclusively that no model whatever of the kind considered could exhibit a continued sequence of identical expansions and contractions. It will next be shown, however, that a new method of attack can be undertaken, by abandoning the attempt to find strictly periodic solutions of the equations of motion and considering instead quasi-periodic solutions which correspond to an expansion of the universe from zero proper volume to an upper limit and contraction again to the

[7] Einstein, Berl. Ber. (1931) p. 235. See also de Sitter, Bull. Astron. Inst. Netherlands **6**, 141 (1931).

[8] Takéuchi, Proc. Phys. Math. Soc. Japan **13**, 166 (1931).

starting point, the analytical treatment failing however outside the range between these points of zero volume.

A wide range of possibilities for solutions of this type will be found to exist. Furthermore, cases will be found where the analytical requirements for these quasi-periodic solutions can be satisfied by systems which expand and contract reversibly without increase in entropy. In such cases, since the only physical possibility after contraction to zero volume would be another expansion, and since the system arrives at the point of zero volume after contraction with unchanged entropy, it would seem allowable to expect a continued sequence of identical expansions and contractions, even though the mathematical analysis fails to carry us through the exceptional point of zero volume.

As mentioned above, one quasi-periodic solution corresponding to expansion from zero volume to an upper limit and return has already been given by Einstein.[7] It corresponds to a model in which the contents of the universe are taken as incoherent matter exerting zero pressure and unaccompanied by radiation. In this article it will be specially pointed out that the expansion and contraction of the Einstein model would not be accompanied by increase in entropy and hence could presumably be repeated over and over again.

Since the actual universe presumably contains radiation as well as matter, it will also be interesting to consider a model filled solely with black-body radiation which is the opposite extreme to the Einstein model containing nothing but matter. The form of line element for such a model will be exhibited, and it will be shown that this model would also expand and contract reversibly. Attention will then be turned to a more complicated model containing an equilibrium mixture of black-body radiation and perfect monatomic gas, and it will be shown that such a model could also furnish quasi-periodic solutions corresponding to reversible expansions and contractions. Finally in conclusion, a few remarks will be made concerning the possible bearing of such considerations on the problem of the behaviour of the actual universe.

§ 3. The Mechanics of the Nonstatic Universe

The line element for a nonstatic universe filled with a uniform distribution of matter and energy can be derived,[9] by treating the contents of the universe for the purposes of large-scale considerations as a perfect fluid, and written in the form

$$ds^2 = -\frac{e^{g(t)}}{[1 + r^2/4R^2]^2}(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2) + dt^2 \tag{1}$$

where r, θ and ϕ are the spatial coordinates, t is the time coordinate, R is a constant, and the dependence of the line element on the time is given by the exponent $g(t)$.

With this choice of coordinates, particles which are at rest in the coordinate system will not be subject to acceleration but will remain perman-

[9] See Tolman, Proc. Nat. Acad. **16**, 320 (1930).

ently at rest, and it will be noted that the time coordinate t will agree with the proper time as used by local observers at rest with respect to such particles.

The proper pressure p_0 and proper macroscopic density ρ_{00} of the fluid corresponding to this line element are given by the equations[10]

$$8\pi p_0 = -\frac{1}{R^2}e^{-g} - \ddot{g} - \tfrac{3}{4}\dot{g}^2 + \Lambda \tag{2}$$

$$8\pi\rho_{00} = \frac{3}{R^2}e^{-g} + \tfrac{3}{4}\dot{g}^2 - \Lambda \tag{3}$$

where Λ is the cosmological constant. The pressure and density are independent of position, and are dependent as shown on the exponent g and its time derivatives $\dot{g}$ and $\ddot{g}$, and thus on the time itself.

The proper volume associated with the coordinate range $\delta r\ \delta\theta\ \delta\phi$ in accordance with the above line element, is evidently

$$\delta V_0 = \frac{r^2 \sin\theta\, e^{3g/2}}{[1 + r^2/4R^2]^3}\,\delta r\,\delta\theta\,\delta\phi \tag{4}$$

and since the dependence of the line element on the time occurs solely because of the functional dependence of the exponent g on t, it is evident that changes in the state of the universe with time can be characterized as expansions or contractions in proper volume, according as g is increasing or decreasing with the time.

In accordance with the expressions for pressure and density given by (2) and (3), this change in volume of the universe is governed by the simple equation[11]

$$\frac{d}{dt}(\rho_{00}\, e^{3g/2}) + p_0 \frac{d}{dt}(e^{3g/2}) = 0. \tag{5}$$

Or noting Eq. (4), we can write

$$\frac{d}{dt}(\rho_{00}\,\delta V_0) + p_0 \frac{d}{dt}(\delta V_0) = 0 \tag{6}$$

which shows that the change in the energy content associated with any small element of fluid is equal to the negative of the work performed on the surroundings.

§ 4. The Analytical Requirements for a Strictly Periodic Solution

Since changes in the line element with time occur because of the dependence of g on t, it is evident that the conditions for a strictly periodic behaviour of the model can be met only if g passes periodically through its minimum and maximum values. Using the subscripts 1 and 2 to denote the values assumed by different quantities when g has its minimum and maximum values, we can write as the analytical condition for the minimum

[10] See preceding reference, Eqs. (34).

[11] See Tolman, Proc. Nat. Acad. **16,** 409 (1930); Eq. (4).

$$g = g_1 \qquad \dot{g}_1 = 0 \qquad \ddot{g}_1 \geqq 0 \tag{7}$$

and as the condition for the maximum

$$g = g_2 \qquad \dot{g}_2 = 0 \qquad \ddot{g}_2 \leqq 0 \tag{8}$$

and by intercombination these expressions give as a necessary requirement for a strictly periodic solution

$$g_1 < g_2 \qquad \dot{g}_1 = \dot{g}_2 = 0 \qquad \ddot{g}_1 \geqq \ddot{g}_2 \tag{9}$$

§ 5. The Thermodynamic Requirements for a Strictly Periodic Solution

As the thermodynamic requirement for a strictly periodic solution, it is evident that the changes taking place in our model of the universe must be thermodynamically reversible, since otherwise it would not be possible for the model to return to an earlier state. As previously shown[12] in accordance with the principles of relativistic thermodynamics, the condition for reversibility will be met in a system of the kind considered if we have the equation

$$\frac{d}{dt}(\phi_0 \ \delta V_0) = 0 \tag{10}$$

holding at each point in the fluid, where ϕ_0 is the proper density of entropy as measured by a local observer at rest with respect to the fluid, and δV_0 is the proper volume of a small element of the fluid.

Since the proper volume in accordance with the principles of general relativity is dependent on the gravitational potentials $g_{\mu\nu}$, this condition for reversibility permits a mutual change in entropy density and gravitational field which was not contemplated in the classical thermodynamics. The equation states that the total entropy for each small element of fluid shall be constant as measured by a local observer, and this can evidently be met if the change in proper volume for the element occurs without flow of heat into the element, with exact balance between the internal and external pressures acting at the boundary of the element, and without irreversible processes taking place inside the element. In the case of the systems we are discussing the condition for no flow of heat into the element is evidently met in general on account of Eq. (6), and the exact balance between internal and external pressures is assured by the uniformity of pressure throughout the whole model as shown by Eq. (2); so that the avoidance of irreversible changes within the element is the only part of the requirement that must be specifically watched. If the requirement for reversibility is met, it will be noted that the changes taking place in the properties of the fluid filling our model of the universe would be those which would result from a reversible adiabatic change in volume of that kind of fluid.

[12] See reference 4, Eq. (18).

§ 6. Impossibility of a Strictly Periodic Solution

It can now be shown that the two separate sets of requirements for a strictly periodic solution could not both be satisfied by any fluid for filling the model of the universe which has reasonable physical properties.

On the one hand in accordance with the general expressions for density and pressure given by (2) and (3), and the necessary analytical requirement for a strictly periodic solution given by (9), we can evidently write

$$8\pi(\rho_1 - \rho_2) = \frac{3}{R^2}(e^{-g_1} - e^{-g_2}) \tag{11}$$

$$8\pi(p_1 - p_2) = -\frac{1}{R^2}(e^{-g_1} - e^{-g_2}) - (\ddot{g}_1 - \ddot{g}_2) \tag{12}$$

$$g_1 < g_2 \quad (13a) \qquad \text{and} \quad \ddot{g}_1 \geqq \ddot{g}_2 \quad (13b)$$

where $(\rho_1 - \rho_2)$ is the difference in energy density when the universe has its minimum and maximum volumes, and $(p_1 - p_2)$ is the difference in pressure.

On the other hand in accordance with the thermodynamic requirement for a periodic solution given in § 5, the changes in density and pressure when the universe passes from its minimum to its maximum volume must be those which would result from a reversible adiabatic expansion of the fluid, which would evidently give us

$$\rho_1 > \rho_2 \quad (14a) \qquad \text{and} \quad p_1 \geqq p_2 \quad (14b)$$

for any kind of fluid with which we might wish to regard the universe as filled.

All of the above relations, however, cannot be simultaneously satisfied. If the density is to decrease on expansion, then in accordance with (11) and (13a), it is evident that R must be real and R^2 positive. And with R^2 positive it will then be seen from (12), (13a) and (13b) that the pressure would increase on expansion, in contradiction to (14b).

It is evident that the relations (14a) and (14b) on which this reasoning is based would hold for any combination of incoherent matter and radiation which we might take as the fluid for filling the universe. By a special assignment of assumed properties, it might of course be possible to obtain strictly periodic solutions, but it does not appear probable that they would have actual physical interest. Thus a fluid which would correspond to the assumption made by Takéuchi,[13]

$$g = \sin kt \tag{15}$$

would appear to have very arbitrary properties.

§ 7. Analytical Requirements for a Quasi-Periodic Solution

In view of the above result we may now turn our attention to the possibility of quasi-periodic solutions, in which the proper volume of the universe increases from zero to an upper limit and returns, the mathematical analysis, however, failing to carry us through the exceptional point of zero volume. A

[13] See reference 8.

wide variety of possibilities presents itself for such solutions, and we may confine our attention for the present to those in which we take the constant R as real, and the cosmological constant Λ equal to zero as in Einstein's original simple form of the equations of relativistic mechanics.[14]

We can then obtain an expansion from zero proper volume $g_1 = -\infty$ to an upper limit $g_2 > g_1$ and return again to $g_3 = -\infty$, with reasonable values for density and pressure, if we place the following restrictions on g and its derivatives,

$$g_1 = -\infty \qquad \dot{g}_1 > 0 \qquad \ddot{g}_1 = -\infty \tag{16}$$

$$g_2 > g_1 \qquad \dot{g}_2 = 0 \qquad \ddot{g}_2 \leqq -\frac{1}{R^2} e^{-g} \tag{17}$$

$$g_3 = -\infty \qquad \dot{g}_3 < 0 \qquad \ddot{g}_3 = -\infty \tag{18}$$

In accordance with these restrictions, the model will evidently pass through a maximum volume at $g = g_2$. Furthermore, making use of the general expressions for proper density and pressure given by (2) and (3), the model will exhibit at the limits of the expansion, with $\Lambda = 0$, the densities

$$8\pi\rho_1 = \infty \qquad \text{and} \quad 8\pi\rho_2 \geqq 0 \tag{19}$$

and the pressures

$$8\pi p_1 = -\infty + \infty \qquad \text{and} \quad 8\pi p_2 \geqq 0 \tag{20}$$

and these expressions evidently place no undue restrictions on the physical properties of the fluid with which we fill the model.

It is evident that solutions which agree with the conditions imposed by (16), (17) and (18) could not be regarded analytically as strictly periodic, since the value of $\ddot{g}$ is necessarily minus infinity at the lower limit of zero volume in order to prevent the pressure from becoming negative, and this is incompatible with the analytical conditions for a minimum. Nevertheless, it is evident physically that contraction to zero volume could only be followed by another expansion, and in addition, as noted by Einstein[7] in a similar connection, the idealization on which our considerations have been based can be regarded as failing in the neighborhood of zero volume. Hence from a physical point of view it seems reasonable to consider that solutions of the kind which we are considering could correspond to a series of successive expansions and contractions.

[14] The introduction of the cosmological term $\Lambda g_{\mu\nu}$ into the field equations of relativity was necessary for Einstein's static model of the universe. The term is not inevitably necessary, however, for nonstatic models, and Einstein (reference 7) now welcomes the opportunity to set it equal to zero. There are several arguments in favor of giving Λ the definite value zero. The equations of the theory are thereby simplified; the conclusions drawn from them are rendered less indeterminate; and it no longer becomes necessary to inquire into the significance and magnitude of what would otherwise be a new constant of nature. On the other hand, since the introduction of the Λ-term provides the most general possible expression of the second order which could be used for the energy momentum tensor, a definite assignment of the value $\Lambda = 0$ must be regarded as somewhat arbitrary and not necessarily correct.

§ 8. Thermodynamic Requirements for a Series of Identical Expansions and Contractions

Since the foregoing indicates the possibility of a series of expansions and contractions, we may next inquire into the further possibility that the successive expansions and contractions could have the character of identical repetitions. For this to be possible, it is evident that each expansion and contraction would have to be reversible from the point of view of relativistic thermodynamics, since any increase in entropy would prevent the return to an earlier condition.

If, however, the condition for thermodynamic reversibility

$$\frac{d}{dt}(\phi_0\,\delta V_0) = 0 \tag{21}$$

already discussed in § 5 should be satisfied, a series of identical expansions and contractions would seem possible.

We may now discuss some quasi-periodic solutions fulfilling these requirements.

§ 9. Einstein's Quasi-Periodic Solution

As already mentioned one quasi-periodic model has already been suggested by Einstein. For simplicity denoting the "radius" of the universe by P where

$$P = Re^{g/2} \tag{22}$$

the general behaviour of the Einstein model can be most readily appreciated by solving the differential equation for P which he gives. We obtain

$$\sqrt{\frac{P}{P_0}} = \sin\left(\frac{t}{P_0} + \sqrt{\frac{P}{P_0}}\sqrt{1-\frac{P}{P_0}}\right) \tag{23}$$

where P_0 is a constant, and note that the radius of the universe would expand from the value zero at $t=0$, to the upper limit $P=P_0$ at $t=\pi P_0/2$, and then return once more to zero at $t=\pi P_0$, the solution failing, however, to make P an analytical minimum at the lower limit.

The values of $\dot{g}$ and $\ddot{g}$ which correspond to (22) and (23) can be written in the form

$$\frac{\dot{g}^2}{4} = \frac{P_0}{P^3} - \frac{1}{P^2} \tag{24}$$

and

$$\ddot{g} = -3\frac{P_0}{P^3} + \frac{2}{P^2} \tag{25}$$

and substituting in the general expressions for pressure and density given by (2) and (3), with $\Lambda=0$ we obtain,

$$8\pi p_0 = 0 \tag{26}$$

$$8\pi\rho_{00} = \frac{3P_0}{P^3} = \frac{3P_0}{R^3e^{3g/2}}\,. \tag{27}$$

The model thus corresponds to a universe filled with incoherent particles of matter, exerting no pressure, and unaccompanied by radiation.

From the point of view of the present paper, our interest lies in the entropy of the model. In accordance with the expressions for proper volume and density given by (4) and (27), it is evident that we shall have in the case of this model

$$\frac{d}{dt}(\rho_{00}\,\delta V_0) = 0 \tag{28}$$

owing to the fact that the only quantity g, which depends on the time, cancels from the product. For particles of matter exerting no pressure, it is evident, however, that the proper entropy $\phi_0\,\delta V_0$ will be proportional to the proper mass $\rho_{00}\,\delta V_0$ so that we must also have

$$\frac{d}{dt}(\phi_0\,\delta V_0) = 0 \tag{29}$$

which is the thermodynamic condition for reversibility already discussed in § 5. We may hence conclude that the model expands and contracts reversibly, which is perhaps an obvious conclusion in the case of this simple system. Since the behaviour of the model is thermodynamically reversible, there would be no thermodynamic hindrance which would prevent a continuous sequence of identical expansions and contractions.

§ 10. Quasi-Periodic Solution for a Universe Filled with Radiation

The real universe undoubtedly contains radiation as well as matter. The actual amount of this radiation is unknown. Nevertheless, it may be noted that as low a temperature as 12° absolute would correspond to a density of radiation of 1.7×10^{-31} gm/cm^3 which is approximately the same as Hubble's estimate[15] of the averaged-out density of visible matter in the universe. In any case it will be interesting to consider the extreme example of a model containing nothing but black-body radiation, even if we regard this merely as an intellectual exercise.

As mentioned above[16] I have already considered this model, without making any assumptions as to the value of the cosmological constant Λ, and come to the conclusion that no strictly periodic solution of the equations of motion could be found. The possibility of quasi-periodic solutions presents itself nevertheless, and setting $\Lambda=0$, it can readily be shown that the line element for such a model then has the form

$$ds^2 = -\frac{At - t^2/R^2}{[1 + r^2/4R^2]^2}(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2) + dt^2 \tag{30}$$

where A is a positive constant.

[15] Hubble, Astrophys. J. **64,** 369 (1926).
[16] See reference 4.

In accordance with this form of the line element, the universe would start from zero proper volume at $t=0$, expand to a maximum at $t=AR^2/2$ and contract to zero again at $t=AR^2$, and the behaviour would be symmetrical on the two sides of the maximum as can easily be seen by transforming so that the zero of time comes at that point. Outside the range $t=0$ to $t=AR^2$, the treatment fails since the analytical expression would then correspond to negative volumes which are physically impossible.

The values of g and its differential coefficients corresponding to this line element are

$$e^g = At - t^2/R^2 \tag{31}$$

$$\dot{g} = \frac{1}{t} - \frac{1}{AR^2 - t} \tag{32}$$

and

$$\ddot{g} = -\frac{1}{t^2} - \frac{1}{(AR^2 - t)^2}. \tag{33}$$

Substituting these values into the general expressions for proper pressure and density given by (2) and (3), with Λ set equal to zero, we obtain

$$8\pi p_0 = \frac{1}{4}\frac{A^2}{(At - t^2/R^2)^2} \tag{34}$$

and

$$8\pi\rho_{00} = \frac{3}{4}\frac{A^2}{(At - t^2/R^2)^2} \tag{35}$$

We note at once that the energy density is always three times the pressure, so that the line element does correspond to a universe filled solely with black-body radiation, It is also interesting to observe that the energy density would be infinite at the times $t=0$ and $t=AR^2$ when the volume of the model is zero, and would drop to $3/R^2$ at $t=AR^2/2$ when the volume is at its maximum.

As in the case of the preceding model, our main interest for the present lies in the entropy of the system. In accordance with the known properties of black-body radiation, it is evident that we may write

$$\rho_{00} = aT_0^4 \quad \text{and} \quad \phi_0 = \frac{4}{3}aT_0^3 \tag{36}$$

for the proper (macroscopic) energy density ρ_{00} and proper entropy density ϕ_0 in terms of the Stefan-Boltzmann constant a and the proper temperature T_0. And by combination we can write

$$\phi_0 = \frac{4}{3}a^{1/4}\rho_{00}^{3/4}. \tag{37}$$

Substituting the values given by (35) and (31) we can then obtain for the entropy density

$$\phi_0 = \left(\frac{4}{3}a\right)^{1/4}\left(\frac{A^2}{8\pi}\right)^{3/4}e^{-3g/2} \tag{38}$$

and comparing with the general expression for proper volume given by Eq. (4), we at once see that we shall have the relation

$$\frac{d}{dt}(\phi_0 \, \delta V_0) = 0 \tag{39}$$

owing to the fact that the only quantity g, which depends on the time cancels from the product. This, however, is the condition for thermodynamic reversibility already discussed in § 5. Hence we may also conclude in the case of this model that there would be no thermodynamic hindrance which would prevent a continuous sequence of identical expansions and contractions.

§ 11. Quasi-Periodic Solution for a Universe Containing an Equilibrium Mixture of Matter and Radiation

It will also be interesting to consider the possibility for a quasi-periodic solution in the case of the model, which I have previously treated, of a universe containing a perfect monatomic gas in equilibrium with black-body radiation. If we set the cosmological constant Λ equal to zero, the equations of motion previously obtained[17] for this model become

$$8\pi p_0 = -\frac{1}{R^2} e^{-g} - \ddot{g} - \tfrac{3}{4}\dot{g}^2 = 8\pi(N_0 k T_0 + \tfrac{1}{3} a T_0^4) \tag{40}$$

$$8\pi \rho_{00} = \frac{3}{R^2} e^{-g} + \tfrac{3}{4}\dot{g}^2 = 8\pi(N_0 m c^2 + \tfrac{3}{2} N_0 k T_0 + a T_0^4) \tag{41}$$

$$N_0 = b T_0^{3/2} e^{-mc^2/kT_0} \tag{42}$$

where N_0 is the proper concentration of the gas in number of molecules per unit volume, T_0 the proper temperature, m the mass of one atom, and the constants k, a, c and b have their customary significance.

As demonstrated in the previous article these differential equations show no possibility for a strictly periodic solution of physical significance. To investigate the possibility for a quasi-periodic behaviour of the model, we shall not try to integrate the equations owing to their great complexity, but instead shall content ourselves with a demonstration of the existence of quasi-periodic solutions without obtaining their explicit form.

In accordance with Eq. (40) it is evident that the pressure p_0 is necessarily a positive quantity

$$p_0 \geqq 0 \tag{43}$$

since the concentration N_0 and temperature T_0 are both quantities which physically cannot be negative. Hence from (40), taking R as real, it is evident that we must also have

$$\ddot{g} \leqq -\frac{1}{R^2} e^{-g} \tag{44}$$

So that $\ddot{g}$ will always be negative if the volume of the universe is finite (i.e. $g < \infty$).

[17] See reference 5.

It appears evident, however, that this provides the possibility of choosing conditions so that models of the kind under discussion would pass through a point of maximum expansion. To prove this it is only necessary to admit the possibility of setting up a model of the universe with an equilibrium mixture of the gas and radiation contained in a finite volume which at the moment is not changing with the time. This set-up, with $\dot{g}$ put equal to zero and $\ddot{g}$ as seen above necessarily negative, evidently corresponds to the condition at the maximum point of expansion of a model which starts at zero volume with $\ddot{g}$ equal to $-\infty$ passes through a maximum with $\ddot{g}$ still negative and returns again to zero volume with $\ddot{g}$ again equal to $-\infty$. So that the conceivability of the proposed set-up shows the possibility of quasi-periodic solutions for models of the universe containing an equilibrium mixture of perfect monatomic gas and black body radiation.

It is also interesting to note that this method of showing the possibility for quasi-periodic solutions would apply to any model in which the fluid must be regarded as always exerting a positive pressure.

Since my previous work on this model showed that its expansion or contraction would be thermodynamically reversible, provided we make the somewhat arbitrary assumption that the matter in the model always immediately adjusts itself so as to remain in equilibrium with the radiation, it is evident in this case too that there would be no thermodynamic hindrance to a continuous succession of identical expansions and contractions.

§ 12. Conclusion

The result of the foregoing investigation of the theoretical requirements for the periodic behaviour of a certain class of nonstatic models of the universe has been threefold. In the first place, it has been shown that we cannot expect a strictly periodic solution of the equations of motion for such models, which would correspond to an oscillation in proper volume between a minimum and maximum value, unless indeed we should assign properties to the material filling the universe of a kind which would presumably not be of actual interest. It has been shown in the second place, however, that we might expect a wide variety of quasi-periodic solutions, which would correspond to an expansion of the model from zero proper volme to a maximum and return, —the solutions failing, however, to provide the analytical conditions for a minimum at the lower limit, even though physically contraction to this limit could only be followed by renewed expansion. Finally, it has been shown in accordance with the principles of relativistic thermodynamics that the conditions necessary for such quasi-periodic solutions could be met by systems which expand and contract reversibly without increase in entropy, so that there would be no thermodynamic hindrance to a continued series of identical expansions and contractions, and this conclusion has been illustrated by three examples.

The three models, which were chosen as examples to illustrate the possibility of a continued series of identical expansions and contractions, were,—first, a universe filled with incoherent matter exerting no pressure and unac-

companied by radiation,—second, a universe filled solely with black-body radiation,—and third, a universe filled with a mixture of perfect monatomic gas in equilibrium with black-body radiation. It is of interest, in connection with speculations as to the possibility that the actual universe might undergo a series of identical expansions and contractions, to find that so wide a range of models show such behaviour. This interest is increased, moreover, by the fact that my previous studies[18] of the second and third of these two models have shown that an ordinary observer would find a number of phenomena during the expansion of these models, such as the flow of radiation out into space and the annihilation of matter, which he would interpret as necessarily irreversible, in spite of the fact that all the processes taking place in such universes would really be reversible when analyzed from the legitimate point of view of relativistic thermodynamics.

In conclusion, however, we must not fail to emphasize that all our specific considerations have dealt with highly idealized models and not with the actual universe. These models have, moreover, some quite unsatisfying properties. In the first place, the models are all limited to a restricted class in which the properties of the contents of the universe, looked at from a large-scale point of view, are assumed to be the same throughout the whole universe at a given instant, and in justification of this assumption we merely have the data obtained from astronomical observations on the nebulae out to a limit of only about 10^8 light years. In the second place, moreover, the models do not concern themselves with many questions which may be very important for a satisfactory cosmological picture, such as the tendency of matter to conglomerate in stars and stellar systems,[19] the relative abundance of the different elements,[20] the relative concentration of matter and radiation,[21] and the rate at which matter and radiation adjust themselves to mutual equilibrium.[22] For these reasons we must not overestimate any insight that we obtain into the possible behaviour of the actual universe as a result of calculations on highly idealized models. Nevertheless, the principles of relativistic thermodynamics seem to have made a real contribution to cosmology, in showing that reversible models can at least be conceived which could exhibit a succession of identical expansions and contractions taking place at a finite rate.

[18] See references 4 and 5.

[19] The treatments which I have given to the flow of radiation in a universe, which is expanding reversibly, show that an ordinary observer in such a model would find a general outward flow of radiation from his region, but do not concern themselves with the more special problem of a directed outward flow from individual stars.

[20] I have shown, Jour. Am. Chem. Soc. **44,** 1902 (1922), that the relative abundance of hydrogen and helium appears to be very different from that demanded by thermodynamic equilibrium and similar findings have been obtained for the relative abundance of isotopes by Urey and Bradley, Phys. Rev. **38,** 718 (1931).

[21] The constant b in Eq. (42) would have to be enormous to secure appreciable concentrations of matter.

[22] The model of a universe containing an equilibrium mixture of gas and radiation would behave reversibly only if the mutual equilibrium were maintained. See reference 5.