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Motivic structures in quantum field theory

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ABSTRACT. This is a writeup of the lecture given by the author at the String-Math 2011 conference in Philadelphia. It gives an overview of recent work of the author, in collaboration with Aluffi and with Ceyhan, on some aspects of the occurrence of motivic structures in perturbative quantum field theory.

1. Motives and quantum fields

The theory of motives originated with Grothendieck's idea of a "universal cohomology" theory for algebraic varieties, underlying the different realizations (de Rham, Betti, étale cohomologies, Hodge structures). The first detailed account of Grothendieck's theory of motives was given by Manin in [**36**].

In this sense, categories of motives "interpolate" between categories of varieties and their cohomologies. In particular, since categories of motives are constructed by replacing the stricter notion of morphisms of algebraic varieties with a more general notion of *correspondences*, they tend to be better behaved than the category of varieties itself. In the best possible case, one obtains an *abelian* category. This is indeed the case when one works with the category of *pure motives*, which means motives associated to smooth projective varieties, with the correspondences given by algebraic cycles modulo the *numerical* equivalence relation. In this case, a result of Jannsen [**29**] shows that the category one obtains is abelian.

1.1. Pure motives. More precisely, the objects of the category of pure motives are triples (X, p, m) of a smooth projective variety X, an endomorphism $p \in \mathrm{End}(X)$ with $p^2 = p$, and an integer $m \in \mathbb{Z}$. Morphisms are given by

$$\mathrm{Hom}((X,p,m),(Y,q,n)) = q\mathrm{Corr}^{m-n}_{/\sim,\mathbb{Q}}(X,Y)\,p,$$

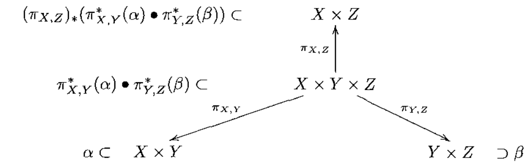
where $\mathrm{Corr}^{m-n}_{/\sim,\mathbb{Q}}(X,Y)$ denotes the $\mathbb{Q}$-vector space of equivalence classes of algebraic cycles in $X \times Y$ of codimension $m-n$. The algebraic cycles are considered modulo the numerical equivalence relation, which means that a cycle Z is numerically trivial ($Z \sim 0$) if its intersection pairing with arbitrary other cycles is zero. The composition of morphisms

$$\mathrm{Corr}(X,Y) \times \mathrm{Corr}(Y,Z) \to \mathrm{Corr}(X,Z)$$

is induced by the intersection product in $X \times Y \times Z$,

$$(\pi_{X,Z})_*(\pi^*_{X,Y}(\alpha) \bullet \pi^*_{Y,Z}(\beta)).$$

Here $\pi_{X,Z}$, $\pi_{X,Y}$, $\pi_{Y,Z}$ are the projection maps from $X \times Y \times Z$ to $X \times Z$, $X \times Y$, and $Y \times Z$, respectively, $\bullet$ denotes the intersection product, and α and β are (linear combinations of) cycles in $X \times Y$ and $Y \times Z$, respectively, as illustrated in the following diagram:



The category $\mathcal{M}_{num,\mathbb{Q}}(k)$ of numerical pure motives obtained in this way is a semi-simple abelian category **[29]**. The Tate objects in the category of pure motives are defined as the objects of the form $\mathbb{Q}(m) = \mathbb{Q}(1)^{\otimes m}$. The Tate motive $\mathbb{Q}(1)$ is defined as the formal inverse of the Lefschetz motive $\mathbb{L}$, which is characterized by the property that the motive of the projective line $\mathbb{P}^1$ is of the form $1 + \mathbb{L}$.

1.2. Mixed motives. In view of the applications of motives to quantum field theory, however, one cannot restrict oneself to the setting of smooth projective varieties. In fact, the varieties that occur in that context are typically *singular* hypersurfaces. The theory of motives of varieties that are not smooth projective is much more complicated than the case of pure motives and it goes under the name of *mixed motives*.

In this case, one obtains only a *triangulated category* $\mathcal{DM}$ of mixed motives, for which there are several different (but equivalent) constructions, by Voevodsky **[42]**, Hanamura **[27]**, and Levine **[32]**. The triangulated structure reflects the long exact sequences in cohomology associated to embeddings of subvarieties $Y \subset X$,

$$\mathfrak{m}(Y) \to \mathfrak{m}(X) \to \mathfrak{m}(X \smallsetminus Y) \to \mathfrak{m}(Y)[1],$$

where $\mathfrak{m}(X)$ denotes the object in the category $\mathcal{DM}$ defined by a variety X, and the homotopy invariance property

$$\mathfrak{m}(X \times \mathbb{A}^1) = \mathfrak{m}(X)(-1)[2],$$

where (-1) denotes the Tate twist, obtained by tensoring with the Tate motive.

Inside the triangulated category $\mathcal{DM}$ of mixed motives one can identify a subcategory $\mathcal{DMT} \subset \mathcal{DM}$, which is generated by the Tate objects $\mathbb{Q}(m)$. This is called the (triangulated) category of mixed Tate motives. In the case of varieties defined over a number field, a crucial vanishing result **[33]** allows the construction of an abelian category of mixed Tate motives, obtained as the heart of a t-structure in this triangulated category. For the reader interested in a gentle introduction to the subject, a brief account of motives for physicists can be found in **[38]**, while a general overview on motives and quantum field theory is given in the author's monograph **[37]**.

1.3. Motives and periods. An aspect of the theory of motives which is of direct relevance to quantum field theory is its relation to periods of varieties. Periods are a special class of numbers that can be obtained by integrating an algebraic differential form over a cycle defined by algebraic equations in an algebraic variety, see **[30]**. Interestingly, the motivic nature of the variety determines what kind of numbers can arise as periods. In particular, it was conjectured (and recently

proved by Francis Brown **[16]**) that the periods of mixed Tate motives over $\mathbb{Z}$ are $\mathbb{Q}[(2\pi i)^{-1}]$-linear combinations of multiple zeta values. The latter are numbers of the form

$$\zeta(n_1, \ldots, n_r) = \sum_{0<k_1<\cdots<k_r} \frac{1}{k_1^{n_1} \cdots k_r^{n_r}}$$

with $n_j \geq 1$ and $n_r \geq 2$.

This very special class of numbers is precisely the original source of the connection between motives and quantum field theory. An extensive investigation by Broadhurst and Kreimer **[15]** revealed the pervasive occurrence of multiple zeta values in computations of residues of Feynman integrals. A way to begin to understand how that may relate to some underlying motivic properties (see **[12]**) is to express the Feynman integral computations in terms of the well known Feynman parametric form.

1.4. Feynman amplitudes. For a massless (Euclidean) perturbative scalar field theory, the Feynman rules prescribe propagators associated to the internal edges e_i of each Feynman graph Γ that are given by quadratic forms q_i in the momentum variables and momentum conservation laws at all vertices. The resulting Feynman amplitude is then expressed as an integral of the form (**[10]**, **[28]**, **[37]**)

$$U(\Gamma) = \int \frac{\delta(\sum_{i=1}^n \epsilon_{v,i} k_i + \sum_{j=1}^N \epsilon_{v,j} p_j)}{q_1 \cdots q_n} d^D k_1 \cdots d^D k_n \tag{1.1}$$

where $n = \#E_{int}(\Gamma)$, $N = \#E_{ext}(\Gamma)$ are the numbers of internal and external edges of the graph, the q_i are quadratic forms in the momentum variables k_i, the p_j are assigned external momenta, and δ is the Dirac delta function imposing the momentum conservation law; D is the spacetime dimension, and

$$\epsilon_{e,v} = \begin{cases} +1 & t(e) = v \\ -1 & s(e) = v \\ 0 & \text{otherwise,} \end{cases}$$

with $s(e)$ and $t(e)$ the source and target vertices of the oriented egde e. In the following, we will use the notation $U(\Gamma)$ for the Feynman amplitude, as in (1.1), or $U(\Gamma, p)$ when we want to explicitly stress the dependence on the datum of the external momenta p.

These Feynman amplitudes satisfy some formal properties that make it possible to reduce the combinatorics of graphs involved in the perturbative expansion. In fact, one can consider only connected graphs, since

$$U(\Gamma_1 \amalg \Gamma_2, p) = U(\Gamma_1, p_1) U(\Gamma_2, p_2)$$

and one can further reduce to 1PI graphs, namely those graphs that cannot be disconnected by the removal of a single internal edge. Indeed, for a connected graph described as $\Gamma = \cup_{v \in T} \Gamma_v$, a tree with vertices replaced by 1PI graphs Γ_v, one has

$$U(\Gamma, p) = \prod_{v \in T} U(\Gamma_v, p_v) \frac{\delta((p_v)_e - (p_{v'})_e)}{q_e((p_v)_e)}.$$

These formal properties can be abstracted to define formal "algebro-geometric Feynman rules", which can be obtained, for instance, though invariants of singular varieties based Chern classes, or from classes in the Grothendieck ring of varieties, and which behave, in certain respects, like physical Feynman rules, see **[3]**, **[4]**, **[1]**.

1.5. Parametric form of Feynman integrals. The Feynman parametric form of the amplitude is obtained from (1.1) by passing to Schwinger parameters through the identity

$$q_1^{-k_1}\cdots q_n^{-k_n} = \frac{1}{\Gamma(k_1)\cdots\Gamma(k_n)}\int_{\mathbb{R}^n} e^{-(s_1q_1+\cdots+s_nq_n)}\, s_1^{k_1-1}\cdots s_n^{k_n-1}\, ds_1\cdots ds_n.$$

The so called Feynman trick then expresses

$$\frac{1}{q_1\cdots q_n} = (n-1)!\int \frac{\delta(1-\sum_{i=1}^n t_i)}{(t_1q_1+\cdots+t_nq_n)^n}\, dt_1\cdots dt_n.$$

Let $\{\ell_r\}_{r=1,\ldots,b_1(\Gamma)}$ denote a choice of a basis of $H_1(\Gamma)$, namely a collection of *loops* ℓ_r of edges in the graph Γ that generate the first homology. With a change of variables $k_i = u_i + \sum_{r=1}^{\ell} \eta_{ir} x_r$, where

$$\eta_{ir} = \begin{cases} \pm 1 & \text{edge } \pm e_i \in \text{ loop } \ell_r \\ 0 & \text{otherwise} \end{cases}$$

one then obtains the Feynman amplitude in the form

$$U(\Gamma) = \frac{\Gamma(n-D\ell/2)}{(4\pi)^{\ell D/2}}\int_{\sigma_n} \frac{\omega_n}{\Psi_\Gamma(t)^{D/2} V_\Gamma(t,p)^{n-D\ell/2}}$$

where the integration is over the simplex $\sigma_n = \{t\in\mathbb{R}_+^n \,|\, \sum_i t_i = 1\}$, with ω_n the volume form. The integrand is given by the *graph polynomials*: the Kirchhoff polynomial. This can be written as a determinant involving the matrix η_{ir}, and also explicitly as a sum over spanning trees Tcof the graph Γ,

$$\Psi_\Gamma(t) = \det M_\Gamma(t) = \sum_T \prod_{e\notin T} t_e \quad \text{with} \quad (M_\Gamma)_{kr}(t) = \sum_{i=0}^n t_i \eta_{ik}\eta_{ir}.$$

One also has

$$V_\Gamma(t,p) = \frac{P_\Gamma(t,p)}{\Psi_\Gamma(t)} \quad \text{with} \quad P_\Gamma(p,t) = \sum_{C\subset\Gamma} s_C \prod_{e\in C} t_e.$$

Here the sum is over cut-sets C (complements of spanning trees plus one edge), and $s_C = (\sum_{v\in V(\Gamma_1)} P_v)^2$ are functions of the external momenta, with $P_v = \sum_{e\in E_{ext}(\Gamma), t(e)=v} p_e$, with the conservation law $\sum_{e\in E_{ext}(\Gamma)} p_e = 0$. One has $\deg\Psi_\Gamma = b_1(\Gamma) = \deg P_\Gamma - 1$. In the stable range where $-n + D\ell/2 \geq 0$, the Feynman amplitude is an integral of a ratio of polynomials of the form

$$U(\Gamma) = \frac{\Gamma(n-D\ell/2)}{(4\pi)^{\ell D/2}}\int_{\sigma_n} \frac{P_\Gamma(t,p)^{-n+D\ell/2}\omega_n}{\Psi_\Gamma(t)^{-n+D(\ell+1)/2}},$$

while in the simpler log divergent case with $n = D\ell/2$ one has

$$\int_{\sigma_n} \frac{\omega_n}{\Psi_\Gamma(t)^{D/2}}.$$

1.6. Divergences and periods. In this form, one sees the ultraviolet divergence in the form of a pole of a Gamma function. Up to that divergent factor, the "residue" is given by the integral

$$\int_{\sigma_n} \frac{P_\Gamma(t,p)^{-n+D\ell/2}\omega_n}{\Psi_\Gamma(t)^{-n+D(\ell+1)/2}}. \tag{1.2}$$

Unfortunately, this integral itself is still in general divergent. These infrared divergences come from the intersections of the domain of integration with the locus defined by the vanishing of the polynomial in the denominator. Thus, these integrals need to be further regularized and renormalized, before one can properly interpret them as periods. This issue is discussed at length in the work of Bloch, Esnault, Kreimer **[12]**, and Bloch and Kreimer **[13]**, **[14]**, where algebro-geometric techniques based on blowups, monodromies, and limiting mixed Hodge structures are used to address the renormalization problem for these divergences.

For simplicity of exposition, I am going to ignore here these important divergence issues and refer the readers to **[12]**, **[13]**, **[14]** for an appropriate discussion. If the integrals (1.2) were convergent, they would indeed be periods. In fact, one can define the graph hypersurfaces as

$$\hat{X}_\Gamma = \{t\in\mathbb{A}^n \,|\, \Psi_\Gamma(t) = 0\},$$

in the affine case. Since the polynomial Ψ_Γ is homogenous with deg $= b_1(\Gamma)$, one can also consider the projective hypersurface

$$X_\Gamma = \{t\in\mathbb{P}^{n-1} \,|\, \Psi_\Gamma(t) = 0\}.$$

In terms of explicit calculations of classes in the Grothendieck ring, it is often convenient to go back and forth between the affine formulation in terms of $\hat{X}_\Gamma$ and the projective one in terms of X_Γ.

The domain of integration in (1.2) is the simplex σ_n, which is a chain with non-trivial boundary. Thus, the integral can be regarded as a period for a *relative* cohomology, namely

$$H^{n-1}(\mathbb{P}^{n-1}\smallsetminus X_\Gamma, \Sigma_n\smallsetminus(\Sigma_n\cap X_\Gamma)),$$

where $\Sigma_n = \{\prod_i t_i = 0\}$ is a divisor that contains the boundary $\partial\sigma_n$ of the domain of integration. (We are assuming we work in the stable range $-n+D\ell/2\geq 0$, where the other graph polynomial $P_\Gamma(t,p)$ does not appear in the denominator.) When one takes the divergences into account, as in **[12]**, **[13]**, this relative cohomology is in fact replaced by a similar relative cohomology involving a toric variety obtained as an iterated blowup of $\mathbb{P}^{n-1}$. More precisely, the divergences happen where $X_\Gamma\cap\sigma_n\neq\emptyset$. This locus is clearly contained inside the divisor $\Sigma_n\supset\partial\sigma_n$ of coordinate hyperplanes. In fact, since the graph polynomial has positive coefficients, it has no real solutions in the interior of the positive quadrant, so the only place where the hypersurface can meet the simplex is along the boundary $\partial\sigma_n$. One proceeds as in **[13]** to an iterated series of blowups of $\mathbb{P}^{n-1}$ along the coordinate linear spaces defined by edges of 1PI subgraphs. The result is a toric variety $P(\Gamma)$. The iterated blowups that define $P(\Gamma)$ have the effect of separating the strict transform of X_Γ from the non-negative real points. Deforming the integration chain leads to a monodromy problem and the subtraction of divergences is then achieved in terms of Poincaré residues and limiting mixed Hodge structures (see **[13]**).

1.7. The polynomial countability question. Thus, from this point of view, the question of why one finds multiple zeta values when evaluating residues of Feynman integrals can be reformulated as a question on whether a relative cohomology group of a hypersurface complement is a realization of a mixed Tate motive. Very early into the development of the subject, Kontsevich conjectured that the graph hypersurfaces X_Γ themselves may be always mixed Tate. More precisely, the conjecture was formulated in terms of a closely related property, *polynomial countability.* A variety X that is defined over $\mathbb{Z}$ admits reductions X_p modulo various primes p. These give varieties defined over finite fields $\mathbb{F}_p$ and one can count the number of their algebraic points $X_p(\mathbb{F}_q)$ for field extensions $\mathbb{F}_q$ with $q = p^m$. The variety X is polynomially countable if the counting function $\#X_p(\mathbb{F}_q)$ is a polynomial in q with $\mathbb{Z}$ coefficients. (One can also consider a weaker conditions where this holds for all but finitely many primes.) This polynomial countability property is related to the mixed Tate nature of the motive of X through the relation between the counting of points over finite fields and the classes in the Grothendieck ring of varieties that we discuss more precisely below. The conjecture was, at first, verified for all graphs with up to twelve edges in [**40**], but was later disproved by a remarkable general result of Belkale and Brosnan [**8**], which showed that, on the contrary, the graph hypersurfaces can be "arbitrarily complicated" as motives, in a sense that we'll discuss more precisely below. Recently, explicit counterexamples were found by Doryn [**22**] and by Brown and Schnetz [**19**]. The counterexample identified in [**22**] fails the polynomially countable condition, but may still satisfy a weaker version for all but finitely many primes, while the example of [**19**] exhibits a more substantial failure of polynomial countability.

1.8. Virtual motives as universal Euler characteristics. While the category of mixed motives has a very complicated construction, and it is generally hard to work directly with the mixed motives themselves, there is a very good invariant of motives that can be more easily computed and which is very useful in order to detect properties of motives. This is given by the class in the Grothendieck ring of varieties $K_0(\mathcal{V})$.

The generators $[X]$ of $K_0(\mathcal{V})$ are the isomorphism classes of smooth quasi-projective varieties and the relations are the inclusion-exclusion (or scissor congruence) relation

$$[X] = [X \smallsetminus Y] + [Y],$$

for a closed embedding of a smooth closed subvariety $Y \subset X$ and the product relation that gives the ring structure,

$$[X] \cdot [Y] = [X \times Y].$$

Classes in the Grothendieck ring $K_0(\mathcal{V})$ are also called *virtual motives.*

The Grothendieck ring has an alternative presentation, as shown in [**9**], where the generators are isomorphism classes of *smooth projective* varieties and the relations are given by the blowup formula $[Bl_Y(X)] = [X] - [Y] + [E]$, with E the exceptional divisor of the blowup, together with the trivial relation $[\emptyset] = 0$ and the ring structure given again by the product.

In the Grothendieck ring of varieties the (virtual) Tate motives are the elements of the subring $\mathbb{Z}[\mathbb{L}]$, where $\mathbb{L} = [\mathbb{A}^1]$ is the class of the affine line, the Lefschetz motive.

Notice that, if X is a variety that is neither smooth nor projective, but which has a stratification by strata that are smooth (quasi)projective, then one has a corresponding well defined class $[X]$ in $K_0(\mathcal{V})$, obtained in terms of the classes of the strata. Thus, if X is a variety whose motive $\mathfrak{m}(X)$ in the category of mixed Tate motives is mixed Tate, one knows that the corresponding class $[X]$ will also be mixed Tate as a virtual motive, that is, an element of the subring $\mathbb{Z}[\mathbb{L}]$. This is essentially because a mixed Tate motive has a filtration whose graded pieces are pure Tate motives. Subject to some conjectures (such as the Tate conjecture, see the discussion of this issue in [**7**], for instance), a converse statement would also hold, namely if the class in the Grothendieck ring is a virtual Tate motive, the the variety is a (mixed) Tate motive. The conjectural statement needed here is essentially the fact that the numbers N_p of points over finite fields suffice to determine the motive. Since the counting of points over finite fields is an additive invariant that factors through the Grothendieck ring, the class $[X]$ would then also suffice.

The class $[X]$ of a variety in the Grothendieck ring $K_0(\mathcal{V})$ is regarded as a "universal Euler characteristic" for the variety, [**9**]. The reason lies in the following fact. An *additive invariant* of varieties is a function on algebraic varieties with values in a commutative ring $\mathcal{R}$ which satisfies $\chi(X) = \chi(Y)$, for isomorphic varieties $X \cong Y$, and

$$\chi(X) = \chi(Y) + \chi(X \smallsetminus Y),$$

for a closed embedding of a subvariety $Y \subset X$, and

$$\chi(X \times Y) = \chi(X)\chi(Y).$$

Any additive invariant of varieties factors through the Grothendieck ring, as a ring homomorphism

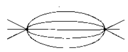
$$\chi : K_0(\mathcal{V}) \to \mathcal{R}.$$

One readily recognizes that the properties of additive invariants are exactly those that are satisfied by the Euler characteristic: inclusion-exclusion and multiplicativity on products. Indeed, examples of additive invariants are: the topological Euler characteristic or the virtual Hodge polynomial in the case of complex varieties; the couting of algebraic points in the case of varieties over finite fields; the Gillet–Soulé motivic Euler characteristic $\chi_{mot} : K_0(\mathcal{V})[\mathbb{L}^{-1}] \to K_0(\mathcal{M})$, given by $\chi_{mot}(X) = [(X, id, 0)]$ when X is smooth projective and by a complex $\chi_{mot}(X) = W^{\cdot}(X)$ in more general case, as defined in [**25**].

2. Virtual motives of graph hypersurfaces

In this section we recall some results on the virtual motives of the graph hypersurfaces. We focus on the recent work of Aluffi and the author ([**2**]–[**5**]).

2.1. Graph hypersurfaces and the Grothendieck ring. Suppose X is a variety defined over $\mathbb{Z}$. The conditions that a variety X is a mixed Tate motive, that its class $[X]$ in the Grothendieck ring $K_0(\mathcal{V})$ is a virtual mixed Tate motive (it lies in the $\mathbb{Z}[\mathbb{L}]$ subring), and that X is polynomially countable are closely related. If X is a mixed Tate motive then its virtual motive is mixed Tate and the counting of algebraic points for the reductions modulo the various primes is a polynomial function for all but finitely many primes (to account for primes of bad reduction). In fact, if one assumes certain conjectures (Tate conjecture), then knowledge of the



counting function $\#X_p(\mathbb{F}_q)$ determines the motives, so that the converse would also be true.

The result of Belkale and Brosnan [**8**] mentioned above, which disproved the Kontsevich conjecture on polynomial countability of the graph hypersurfaces can then be stated in the following way: the graph hypersurfaces X_Γ generate the Grothendieck ring $K_0(\mathcal{V})$, *localized* at $\mathbb{L}^n - \mathbb{L}$, for all $n > 1$.

This means that, in an appropriate sense, the graph hypersurfaces can be arbitrarily complicated as motives: any class in the (localized) Grothendieck ring can be obtained from graph hypersurfaces, including those that are not mixed Tate. Thus, as the graphs become more and more combinatorially complicated, one expects to see more and more non-mixed-Tate examples appear among them, even though the results of [**40**] show that the classes remain mixed Tate for all graphs with up to twelve edges. The occurrence of the first explicit counterexamples of [**22**] and of [**19**] shortly past the previously explored range confirms this understanding. The result of Belkale and Brosnan [**8**] is a very deep and elaborate result, which depends on a universality theorem for matroids.

It is worth pointing out that there is an interesting dichotomy in the Grothendieck ring between what happens when one *inverts* $\mathbb{L}$ and when one instead *sets* $\mathbb{L}$ *to zero*. As shown by Larsen and Lunts [**31**] in the context of motivic integration, setting $\mathbb{L}$ to zero in the Grothendieck ring $K_0(\mathcal{V}_\mathbb{C})$ gives the ring of *stable birational equivalence classes* of varieties

$$\mathbb{Z}[SB] = K_0(\mathcal{V})|_{\mathbb{L}=0}.$$

It was shown by Aluffi and the author in [**5**], by a very simple and direct argument, that the graph hypersurfaces span the subring $\mathbb{Z}$ inside $\mathbb{Z}[SB]$. Thus, while the graph hypersurfaces are "as general as possible" when one inverts $\mathbb{L}^n - \mathbb{L}$, they are just equivalent to points when one sets $\mathbb{L} = 0$. This property depends upon a deletion-contraction type formula, that we discuss in more detail below.

It is also interesting to point out that the localization of $K_0(\mathcal{V})$ at all the elements $\mathbb{L}^n - \mathbb{L}$ is isomorphic to the Grothendieck ring of special Artin stacks, as proved by Toën in [**41**], though it is not yet clear what this identification may be saying in this quantum field theory context.

2.2. Computing in the Grothendieck ring. While computing virtual motives is generally easier than working in the category of mixed motives, it still happens rarely that one can give a completely explicit calculation of the class $[X_\Gamma]$ of a graph hypersurface X_Γ. We recall here a sufficiently simple example from [**2**], where one can carry out a calculation completely explicitly.

The *banana graphs* are graphs with two vertices and n parallel edges between then, as in the figure.

The graph polynomial of the n-th banana graph Γ_n is

$$\Psi_{\Gamma_n}(t) = t_1 \cdots t_n \left(\frac{1}{t_1} + \cdots + \frac{1}{t_n}\right)$$

and the class in the Grothendieck ring turns out to be explicitly given by the formula

$$[X_{\Gamma_n}] = \frac{\mathbb{L}^n - 1}{\mathbb{L} - 1} - \frac{(\mathbb{L}-1)^n - (-1)^n}{\mathbb{L}} - n\,(\mathbb{L}-1)^{n-2}, \tag{2.1}$$

with $\mathbb{L} = [\mathbb{A}^1]$ the Lefschetz motive, as above.

The reason why, in this case, one can carry out the computation explicitly is that one can compare the hypersurface of a planar graph with that of the dual graph via the Cremona transformation.

The Cremona transformation is the map

$$\mathcal{C} : (t_1 : \cdots : t_n) \mapsto \left(\frac{1}{t_1} : \cdots : \frac{1}{t_n}\right)$$

defined outside the singularities locus $\mathcal{S}_n$ of the divisor of coordinate hyperplanes $\Sigma_n = \{\prod_i t_i = 0\}$. This is the locus defined by the ideal

$$I_{\mathcal{S}_n} = (t_1 \cdots t_{n-1}, t_1 \cdots t_{n-2} t_n, \cdots, t_1 t_3 \cdots t_n).$$

The relation between the graph polynomial of a planar graph and that of the dual graph is then given through the Cremona transformation as

$$\Psi_\Gamma(t_1, \ldots, t_n) = \left(\prod_e t_e\right) \Psi_{\Gamma^\vee}(t_1^{-1}, \ldots, t_n^{-1})$$

so that we have

$$\mathcal{C}(X_\Gamma \cap (\mathbb{P}^{n-1} \smallsetminus \Sigma_n)) = X_{\Gamma^\vee} \cap (\mathbb{P}^{n-1} \smallsetminus \Sigma_n).$$

This gives an isomorphism of X_Γ and $X_{\Gamma^\vee}$ outside of Σ_n, see [**2**] for more details.

For a banana graph Γ_n the dual graph $\Gamma_n^\vee$ is just a polygon with n sides, for which the graph hypersurface $X_{\Gamma_n^\vee} = \mathcal{L}$ is just a hyperplane in $\mathbb{P}^{n-1}$. One then computes separately the contribution to the virtual motive coming from the intersection with the locus Σ_n,

$$[X_{\Gamma_n}] = [X_{\Gamma_n} \cap \Sigma_n] + [X_{\Gamma_n} \smallsetminus \Sigma_n]$$

and from the complement. One finds, in terms of the class $\mathbb{T} = [\mathbb{G}_m] = [\mathbb{A}^1] - [\mathbb{A}^0]$ of the multiplicative group,

$$[\mathcal{L} \smallsetminus \Sigma_n] = [\mathcal{L}] - [\mathcal{L} \cap \Sigma_n] = \frac{\mathbb{T}^{n-1} - (-1)^{n-1}}{\mathbb{T} + 1}$$

$$X_{\Gamma_n} \cap \Sigma_n = \mathcal{S}_n \quad \text{with} \quad [\mathcal{S}_n] = [\Sigma_n] - n\mathbb{T}^{n-2},$$

so that $[X_{\Gamma_n}] = [\mathcal{S}_n] + [\mathcal{L} \smallsetminus \Sigma_n]$ gives the formula (2.1).

2.3. Sums over graphs. When one works with graphs that are not necessarily planar, one can still look at the image of the graph hypersurfaces under the Cremona transformation. The dual hypersurface obtained in that way is then not necessarily a graph hypersurface, but it can still give useful information on the original graph hypersurface. This approach was used by Bloch in [**11**] to prove a very interesting result about virtual motives of certain sums of graphs.

If one considers only graphs that have no looping edges or parallel edges, then such graphs can always be realized as subgraphs of the complete graph on the same number of vertices. By analyzing the graph hypersurface of the complete graph, and its relation to those of subgraphs, through the Cremona transformation method, Bloch showed in [**11**] that the sum

$$S_N = \sum_{\#V(\Gamma)=N} [X_\Gamma] \frac{N!}{\#\mathrm{Aut}(\Gamma)} \tag{2.2}$$

is always in the subring $\mathbb{Z}[\mathbb{L}]$ of virtual Tate motives, even though the individual terms $[X_\Gamma]$ will, in general, not be in $\mathbb{Z}[\mathbb{L}]$.

This result suggests that, although individual graphs may give non-mixed-Tate contributions, the sum over graphs in Feynman amplitudes may still be mixed Tate. There are two subtle problems in this, though: the first is that the typical sum over graphs in Feynman amplitudes is a sum over graphs with a fixed number of loops, not of vertices, and it is not clear whether a result like (2.2) may be expected in that case. The second problem is with the fact that periods do not factor through the Grothendieck ring, so the (renormalized) Feynman amplitudes $U(\Gamma)$ also will not decompose well according to a decomposition of the variety X_Γ as a class $[X_\Gamma]$ in the Grothendieck ring. Thus, even if a result analogous to (2.2) could be obtained for the more physical sum over loops, the result for classes in the Grothendieck ring would not directly imply a result of the same type for the Feynman amplitudes.

2.4. Deletion–contraction relations. The graph polynomial satisfies a deletion–contraction relation. Namely, for a graph Γ with $n \geq 2$ edges, with $\deg \Psi_\Gamma = \ell > 0$, one has

$$\Psi_\Gamma = t_e \Psi_{\Gamma\smallsetminus e} + \Psi_{\Gamma/e}$$

where the polynomials of the deletion and the contraction are, respectively, given by

$$\Psi_{\Gamma\smallsetminus e} = \frac{\partial \Psi_\Gamma}{\partial t_n} \quad \text{and} \quad \Psi_{\Gamma/e} = \Psi_\Gamma|_{t_n=0}.$$

It is a well known result about the graph polynomials (Proposition 5.2 of [**39**]) that the irreducible components of a graph hypersurface are themselves graph hypersurfaces of subgraphs that cannot be disconnected by removing a single vertex. Thus, one can assume that the graph hypersurface is irreducible, or work separately on each irreducible component. One can then apply the following general fact, proved in [**4**]. For $X = \{\psi = 0\} \subset \mathbb{P}^{n-1}$, and $Y = \{F = 0\} \subset \mathbb{P}^{n-2}$ with a polynomial ψ satisfying a decomposition

$$\psi(t_1, \ldots, t_n) = t_n F(t_1, \ldots, t_{n-1}) + G(t_1, \ldots, t_{n-1}),$$

one obtains that the projection from $(0 : \cdots : 0 : 1)$ *gives an isomorphism*

$$X \smallsetminus (X \cap \overline{Y}) \xrightarrow{\sim} \mathbb{P}^{n-2} \smallsetminus Y,$$

where $\overline{Y}$ is the cone of Y in $\mathbb{P}^{n-1}$.

This leads directly to a form of deletion–contraction relation for the virtual motives of the affine graph hypersurfaces $\widehat{X}_\Gamma \subset \mathbb{A}^n$, as proved by Aluffi and the author in [**4**]:

$$[\mathbb{A}^n \smallsetminus \widehat{X}_\Gamma] = \mathbb{L} \cdot [\mathbb{A}^{n-1} \smallsetminus (\widehat{X}_{\Gamma\smallsetminus e} \cap \widehat{X}_{\Gamma/e})] - [\mathbb{A}^{n-1} \smallsetminus \widehat{X}_{\Gamma\smallsetminus e}]$$

if e not a bridge (an edge whose removal would increase the number of connected components of the graph) or a looping edge (an edge with source and target at the same vertex);

$$[\mathbb{A}^n \smallsetminus \widehat{X}_\Gamma] = \mathbb{L} \cdot [\mathbb{A}^{n-1} \smallsetminus \widehat{X}_{\Gamma\smallsetminus e}] = \mathbb{L} \cdot [\mathbb{A}^{n-1} \smallsetminus \widehat{X}_{\Gamma/e}]$$

if e bridge;

$$[\mathbb{A}^n \smallsetminus \widehat{X}_\Gamma] = (\mathbb{L} - 1) \cdot [\mathbb{A}^{n-1} \smallsetminus \widehat{X}_{\Gamma\smallsetminus e}] = (\mathbb{L} - 1) \cdot [\mathbb{A}^{n-1} \smallsetminus \widehat{X}_{\Gamma/e}]$$

if e looping edge.

One certainly does not expect an actual deletion–contraction relation for the virtual motives, expressing the class $[\mathbb{A}^n \smallsetminus \widehat{X}_\Gamma]$ solely as a function of the classes $[\mathbb{A}^{n-1} \smallsetminus \widehat{X}_{\Gamma\smallsetminus e}]$ and $[\mathbb{A}^{n-1} \smallsetminus \widehat{X}_{\Gamma/e}]$ of the deletion and the contraction. In fact, if one has such a relation, it would imply that the virtual motives always remain mixed Tate, since we know they are mixed Tate for small graphs. Indeed, the deletion–contraction formula described above is more subtle, and it involves the more complicated term

$$[\mathbb{A}^{n-1} \smallsetminus (\widehat{X}_{\Gamma\smallsetminus e} \cap \widehat{X}_{\Gamma/e})]$$

with the intersection of the hypersurfaces of the deletion and the contraction.

It is precisely this intersection $\widehat{X}_{\Gamma\smallsetminus e} \cap \widehat{X}_{\Gamma/e}$ that is difficult to control motivically. In fact, two varieties can be mixed Tate but intersect along a non-mixed-Tate locus. So, if one tries to use this deletion–contraction formula to compute the class of $\widehat{X}_\Gamma$, for increasingly complicated graphs, in terms of the less complicated $\Gamma \smallsetminus e$ and Γ/e, one will at some point run into non-mixed Tate examples where, although both $\widehat{X}_{\Gamma\smallsetminus e}$ and $\widehat{X}_{\Gamma/e}$ are mixed Tate, the intersection $\widehat{X}_{\Gamma\smallsetminus e} \cap \widehat{X}_{\Gamma/e}$ is not.

2.5. Recursive operations on graphs. Although the presence of the term

$$[\mathbb{A}^{n-1} \smallsetminus (\widehat{X}_{\Gamma\smallsetminus e} \cap \widehat{X}_{\Gamma/e})]$$

in the deletion-contraction formula can make it difficult to explicitly compute the class of $\widehat{X}_\Gamma$ using this method, there are some operations on graphs for which one can carry out the computation explicitly and reduce the formula above to a real deletion-contraction formula. This happens because of a useful cancellation in the intersection term, [**4**].

Here we use the notation $\mathbb{U}(\Gamma) = [\mathbb{A}^n \smallsetminus \widehat{X}_\Gamma]$, which is suggested by the abstract "Feynman rules" type properties of this virtual motive (see [**3**]).

The first such operation is the replacement of an edge of the graph by a number of parallel edges between the same pair of vertices, namely Γ_{me} is obtained from Γ by replacing an edge e by m parallel edges, with $\Gamma_{0e} = \Gamma \smallsetminus e$ and $\Gamma_e = \Gamma$. Then one obtains as in [**4**] an explicit generating function

$$\begin{aligned}\sum_{m\geq 0} \mathbb{U}(\Gamma_{me}) \frac{s^m}{m!} = \; & \frac{e^{\mathbb{T}s} - e^{-s}}{\mathbb{T}+1} \mathbb{U}(\Gamma) \\ & + \frac{e^{\mathbb{T}s} + \mathbb{T}e^{-s}}{\mathbb{T}+1} \mathbb{U}(\Gamma \smallsetminus e) \\ & + \left(s\, e^{\mathbb{T}s} - \frac{e^{\mathbb{T}s} - e^{-s}}{\mathbb{T}+1} \right) \mathbb{U}(\Gamma/e),\end{aligned}$$

in the case where e not bridge nor looping edge, with $\mathbb{T} = [\mathbb{G}_m] \in K_0(\mathcal{V})$. The case of bridges or looping edges gives rise to a similar formula.

This formula is obtained in [**4**] by first considering the case of the doubling of an edge, for which the inclusion-exclusion formula for the virtual motive gives

$$\mathbb{U}(\Gamma_{2e}) = \mathbb{L} \cdot [\mathbb{A}^n \smallsetminus (\hat{X}_\Gamma \cap \hat{X}_{\Gamma_o})] - \mathbb{U}(\Gamma)$$

$$[\hat{X}_\Gamma \cap \hat{X}_{\Gamma_o}] = [\hat{X}_{\Gamma/e}] + (\mathbb{L} - 1) \cdot [\hat{X}_{\Gamma\smallsetminus e} \cap \hat{X}_{\Gamma/e}],$$

where Γ_o denotes the graph obtained by attaching a looping edge to Γ/e. As one can see, in this case a cancellation occurs that eliminates the term with the intersection of the hypersurfaces of the deletion and the contraction and one obtains the recursion formula

$$\mathbb{U}(\Gamma_{2e}) = (\mathbb{L} - 2) \cdot \mathbb{U}(\Gamma) + (\mathbb{L} - 1) \cdot \mathbb{U}(\Gamma \smallsetminus e) + \mathbb{L} \cdot \mathbb{U}(\Gamma/e),$$

which only uses the data $\mathbb{U}(\Gamma)$, $\mathbb{U}(\Gamma \smallsetminus e)$, and $\mathbb{U}(\Gamma/e)$.

A similar recursive formula exists for the operation of splitting an edge by inserting valence two vertices. An example of an application of these recursive formulae is the computation of the virtual motives for "lemon graphs" and chains of polygons, [4]. If Λ_m is the lemon graph m wedges and Γ_m^Λ is the graph obtained by replacing an edge e of Γ with a "lemon wedge" Λ_m, one finds that the virtual motives are given by the generating function:

$$\sum_{m\geq 0} \mathbb{U}(\Gamma_m^\Lambda)s^m = \frac{(1-(\mathbb{T}+1)s)\,\mathbb{U}(\Gamma) + (\mathbb{T}+1)\mathbb{T}s\,\mathbb{U}(\Gamma\smallsetminus e) + (\mathbb{T}+1)^2 s\,\mathbb{U}(\Gamma/e)}{1-\mathbb{T}(\mathbb{T}+1)s - \mathbb{T}(\mathbb{T}+1)^2s^2},$$

in the case where e is not bridge nor looping edge, and by a similar formula otherwise. The recursive relation is in this case of the form

$$\mathbb{U}(\Lambda_{m+1}) = \mathbb{T}(\mathbb{T}+1)\mathbb{U}(\Lambda_m) + \mathbb{T}(\mathbb{T}+1)^2\mathbb{U}(\Lambda_{m-1})$$

and the terms $a_m = \mathbb{U}(\Lambda_m)$ form a *divisibility sequence*, namely $\mathbb{U}(\Lambda_{m-1})$ divides $\mathbb{U}(\Lambda_{n-1})$ if m divides n.

3. Other approaches

We discuss in this section some other closely related approaches to the question of the motivic nature of the residues of Feynman integrals.

3.1. Determinant hypersurfaces and Schubert cells. Another way of looking at the problem of the nature of the periods and motives involved in the Feynman integral computations is to use the description of the Kirchhoff polynomial Ψ_Γ as a determinant (the matrix-tree theorem) to map the relevant period computation from the complement $\mathbb{A}^n \smallsetminus \hat{X}_\Gamma$, whose motive is hard to compute and generally not mixed Tate, to the complement of a determinant hypersurface, which is always mixed Tate. The cost of doing this is to trade the simpler divisor Σ_n that contains the boundary $\delta\sigma_n$ of the domain of integration, for a more complicated locus whose motivic nature may be complicated to compute explicitly. This is the point of view proposed by Aluffi and the author in [6], which we recall briefly here.

More precisely, one has a map

$$\Upsilon : \mathbb{A}^n \to \mathbb{A}^{\ell^2}, \quad \Upsilon(t)_{kr} = \sum_i t_i \eta_{ik}\eta_{ir},$$

which identifies the affine graph hypersurface with the preimage

$$\hat{X}_\Gamma = \Upsilon^{-1}(\hat{\mathcal{D}}_\ell)$$

of the determinant hypersurface $\hat{\mathcal{D}}_\ell = \{\det(x_{ij}) = 0\}$.

The virtual motive of the determinant hypersurface complement is much simpler than that of the graph hypersurfaces. It is a mixed Tate motive with class in the Grothendieck ring given by the explicit formula

$$[\mathbb{A}^{\ell^2} \smallsetminus \hat{\mathcal{D}}_\ell] = \mathbb{L}^{\binom{\ell}{2}} \prod_{i=1}^{\ell} (\mathbb{L}^i - 1).$$

Thus, when Υ is an embedding, one can compute the Feynman amplitude as

$$U(\Gamma) = \int_{\Upsilon(\sigma_n)} \frac{\mathcal{P}_\Gamma(x,p)^{-n+D\ell/2}\omega_\Gamma(x)}{\det(x)^{-n+(\ell+1)D/2}}.$$

If $\hat{\Sigma}_\Gamma$ is a normal crossings divisor in $\mathbb{A}^{\ell^2}$ with $\Upsilon(\partial\sigma_n) \subset \hat{\Sigma}_\Gamma$, then the main question becomes whether the motive

$$\mathfrak{m}(\mathbb{A}^{\ell^2} \smallsetminus \hat{\mathcal{D}}_\ell, \hat{\Sigma}_\Gamma \smallsetminus (\hat{\Sigma}_\Gamma \cap \hat{\mathcal{D}}_\ell))$$

is a mixed Tate motive. Here the notation $\mathfrak{m}(X,Y)$ is used to denote the object in the category of mixed motives underlying the relative cohomology $H^*(X,Y)$.

It is possible to identify some explicit combinatorial conditions on the graph Γ that ensure that the map

$$\Upsilon : \mathbb{A}^n \smallsetminus \hat{X}_\Gamma \hookrightarrow \mathbb{A}^{\ell^2} \smallsetminus \hat{\mathcal{D}}_\ell$$

is an embedding. One has the following conditions (see [6]): Γ is a closed 2-cell embedded graph $\iota : \Gamma \hookrightarrow S_g$ in a Riemann surface of genus g, with $S_g \smallsetminus \Gamma$ union of open disks (faces). The closure of each is a disk. Two faces have at most one edge in common. Every edge in the boundary of two faces.

Then a sufficient condition that ensures that the map to the determinant hypersurface complement is an embedding is that Γ is a 3-edge-connected graph with closed 2-cell embedding of face width ≥ 3. The face width is the largest $k \in \mathbb{N}$ such that every non-contractible simple closed curve in S_g intersects Γ at least k times. (It is set to be ∞ for planar graphs.) Notice that, in the usual physics terminology, a 2-edge-connected graph is called a 1PI graph, while the 2-vertex-connected condition conjecturally implies face width ≥ 2.

One can then look again at the question of identifying the motive $\mathfrak{m}(X,Y)$.

One first observes that the divisor $\hat{\Sigma}_\Gamma$ is always contained, $\hat{\Sigma}_\Gamma \subset \hat{\Sigma}_{\ell,g}$, in a divisor $\hat{\Sigma}_{\ell,g}$, for $f = \ell - 2g + 1$, where

$$\hat{\Sigma}_{\ell,g} = L_1 \cup \cdots \cup L_{\binom{f}{2}}$$

is a union of linear spaces L_i, so that $\hat{\Sigma}_{\ell,g}$ can be described by equations of the form

$$\begin{cases} x_{ij} & = 0 \quad 1 \leq i < j \leq f-1 \\ x_{i1} + \cdots + x_{i,f-1} & = 0 \quad 1 \leq i \leq f-1 \end{cases}$$

and

$$\mathfrak{m}(\mathbb{A}^{\ell^2} \smallsetminus \hat{\mathcal{D}}_\ell, \hat{\Sigma}_{\ell,g} \smallsetminus (\hat{\Sigma}_{\ell,g} \cap \hat{\mathcal{D}}_\ell)), \tag{3.1}$$

where $\hat{\Sigma}_{\ell,g}$ is a normal crossings divisor containing $\Upsilon_\Gamma(\partial\sigma_n) \subset \hat{\Sigma}_{\ell,g}$, which depends only on $\ell = b_1(\Gamma)$ and on g, the minimum genus of a Riemann surface S_g in which the graph can be embedded.

As observed in [6] a sufficient condition for the motive $\mathfrak{m}(X,Y)$ of the form (3.1) to be mixed Tate is that the *varieties of frames* are mixed Tate motives, where for a collection $V_1, \ldots, V_\ell$ of linear subspaces in a fixed ℓ-dimensional vector space, the variety of frames $\mathbb{F}(V_1,\ldots,V_\ell)$ is defined as

$$\mathbb{F}(V_1,\ldots,V_\ell) := \{(v_1,\ldots,v_\ell) \in \mathbb{A}^{\ell^2} \smallsetminus \hat{\mathcal{D}}_\ell \,|\, v_k \in V_k\}.$$

In the case of two or three subspaces this can be verified directly (see [6]): for two subspaces with $d_{12} = \dim(V_1 \cap V_2)$ one has

$$[\mathbb{F}(V_1,V_2)] = \mathbb{L}^{d_1+d_2} - \mathbb{L}^{d_1} - \mathbb{L}^{d_2} - \mathbb{L}^{d_{12}+1} + \mathbb{L}^{d_{12}} + \mathbb{L}.$$

In the case of three subspaces with $D = \dim(V_1 + V_2 + V_3)$, one has

$$[\mathbb{F}(V_1,V_2,V_3)] = (\mathbb{L}^{d_1} - 1)(\mathbb{L}^{d_2} - 1)(\mathbb{L}^{d_3} - 1)$$

$$-(\mathbb{L}-1)((\mathbb{L}^{d_1}-\mathbb{L})(\mathbb{L}^{d_{23}}-1)+(\mathbb{L}^{d_2}-\mathbb{L})(\mathbb{L}^{d_{13}}-1)+(\mathbb{L}^{d_3}-\mathbb{L})(\mathbb{L}^{d_{12}}-1)$$
$$+(\mathbb{L}-1)^2(\mathbb{L}^{d_1+d_2+d_3-D}-\mathbb{L}^{d_{123}+1})+(\mathbb{L}-1)^3$$

However, one cannot use these cases to start an induction argument.

One can see the difficulty by another formulation given in **[6]**: consider the locus $Flag_{\ell,\{d_i,e_i\}}(\{V_i\})$ of complete flags $0 \subset E_1 \subset E_2 \subset \cdots \subset E_\ell = E$, with $\dim E_i \cap V_i = d_i$ and $\dim E_i \cap V_{i+1} = e_i$. One can ask when these are mixed Tate, for all choices of d_i, e_i.

The variety of frames $\mathbb{F}(V_1,\ldots,V_\ell)$ is a fibration over this $Flag_{\ell,\{d_i,e_i\}}(\{V_i\})$ and the virtual motive satsifies

$$[\mathbb{F}(V_1,\ldots,V_\ell)] = [Flag_{\ell,\{d_i,e_i\}}(\{V_i\})](\mathbb{L}^{d_1}-1)(\mathbb{L}^{d_2}-\mathbb{L}^{e_1})(\mathbb{L}^{d_3}-\mathbb{L}^{e_2})\cdots(\mathbb{L}^{d_r}-\mathbb{L}^{e_{r-1}})$$

where $Flag_{\ell,\{d_i,e_i\}}(\{V_i\})$ is an intersection of unions of Schubert cells in flag varieties. The motivic nature of such loci is known to be a very delicate question related to the Kazhdan–Lusztig conjecture. So one sees where the difficulties resurface under this point of view.

3.2. Motives and Feynman amplitudes in configuration spaces. In what we reviewed so far, the point of view is always on the approach to perturbative quantum field theory where the Feynman integrals of individual Feynman graphs are computed in momentum space. One can also take the dual, configuration space point of view and analyze the question of the occurrence of motives in that setting. I will recall here some recent work of Ceyhan and the author **[20]** that explores the motivic setup for Feynman integrals in configuration space.

In this setting, the singularities of the Feynman amplitudes occur along a hypersurface Z_Γ in X^{V_Γ}, for X the spacetime manifold, whose real locus is along the diagonals

$$\Delta_e = \{(x_v)_{v\in V_\Gamma} \mid x_{v_1} = x_{v_2} \text{ for } \partial_\Gamma(e) = \{v_1, v_2\}\}.$$

The configuration space itself, for a given Feynman graph Γ is given by

$$Conf_\Gamma(X) = X^{V_\Gamma} \smallsetminus \bigcup_{e\in E_\Gamma} \Delta_e = X^{V_\Gamma} \smallsetminus \cup_{\gamma\subset\mathcal{G}_\Gamma}\Delta_\gamma,$$

with $\mathcal{G}_\Gamma$ the set of subgraphs that are *induced* (they contain all edges of Γ between the given subset of vertices) and 2-vertex-connected.

To treat the divergences along the diagonals, one can embed the configuration space

$$Conf_\Gamma(X) \hookrightarrow \prod_{\gamma\in\mathcal{G}_\Gamma} \mathrm{Bl}_{\Delta_\gamma} X^{V_\Gamma}$$

inside an iterated blowup along a family of diagonals associated to suitable subgraphs. This gives rise to a particular case of what is more generally known as the De Concini–Procesi "wonderful compactifications" **[21]**, which in turn generalize the Fulton–MacPherson compactification **[24]**. The compactification can be explicitly described in terms of strata

$$\overline{Conf}_\Gamma(X) = Conf_\Gamma(X) \cup \bigcup_{\mathcal{N}\in\ \mathcal{G}-\text{nests}} X^\circ_{\mathcal{N}}$$

of a stratification by $\mathcal{G}$-nests of subgraphs, see the work of Li Li, **[34]**, **[35]**, for more details on the more general construction of wonderful compactifications for arrangements of subvarieties.

For these wonderful compactifications, which have the purpose of blowing up the locus where the singularities of the Feynman amplitudes occur, one can explicitly compute the motive, both in the category of Voevodsky motives (in the case where X is quasi-projective) and as a virtual motive. In the smooth projective case the Chow motive computation follows from the general result of Li Li for the wonderful compactifications **[35]**.

In the Voevodsky category of mixed motives, we find (see **[20]**) an explicit expression for the motive $\mathfrak{m}(\overline{Conf}_\Gamma(X))$.

One uses the notation $M_{\mathcal{N}} := \{(\mu_\gamma)_{\Delta_\gamma\in\mathcal{G}_\Gamma} : 1 \le \mu_\gamma \le r_\gamma - 1,\ \mu_\gamma \in \mathbb{Z}\}$ with $r_\gamma = r_{\gamma,\mathcal{N}} := \dim(\cap_{\gamma'\in\mathcal{N}:\gamma'\subset\gamma}\Delta_{\gamma'}) - \dim\Delta_\gamma$ and $\|\mu\| := \sum_{\Delta_\gamma\in\mathcal{G}_\Gamma}\mu_\gamma$, as in **[35]**. Then the motives of the compactification is given by

$$\mathfrak{m}(\overline{Conf}_\Gamma(X)) = \mathfrak{m}(X^{V_\Gamma}) \oplus \bigoplus_{\mathcal{N}\in\mathcal{G}_\Gamma\text{-nests},\mu\in M_{\mathcal{N}}} \mathfrak{m}(X^{V_{\Gamma/\delta_{\mathcal{N}}(\Gamma)}})(\|\mu\|)[2\|\mu\|]$$

where the notation $\Gamma/\delta_{\mathcal{N}}(\Gamma)$ means the quotient $\Gamma//(\gamma_1\cup\cdots\cup\gamma_r)$, where $\mathcal{N} = \{\gamma_1,\ldots,\gamma_r\}$ is the $\mathcal{G}$-nest and the quotient double bar $\Gamma//\gamma$ means the graph obtained from Γ by shrinking each connected component of the subgraph γ to a (different) vertex.

One can compute in a similar way the virtual motive in the Grothendieck ring of varieties, which is given by (see **[20]**)

$$[\overline{Conf}_\Gamma(X)] = [X]^{|V_\Gamma|} + \sum_{\mathcal{N}\in\mathcal{G}_\Gamma\text{-nests}} [X]^{|V_{\Gamma/\delta_{\mathcal{N}}(\Gamma)}|} \sum_{\mu\in M_{\mathcal{N}}} \mathbb{L}^{\|\mu\|}.$$

In both cases, the key to obtaining these explicit formulae is the presence of blowup formulae: for mixed motives (**[42]**)

$$\mathfrak{m}(\mathrm{Bl}_V(Y)) \cong \mathfrak{m}(Y) \oplus \bigoplus_{k=1}^{\mathrm{codim}_Y(V)-1} \mathfrak{m}(V)(k)[2k],$$

and for the virtual motives the Bittner relations (**[9]**)

$$[\mathrm{Bl}_V(Y)] = [Y] - [V] + [E] = [Y] + [V]([\mathbb{P}^{\mathrm{codim}_Y(V)-1}] - 1),$$

where E is the exceptional divisor. One then obtains the formulae for the wonderful compactifications of the configuration spaces of Feynman graphs using their description as iterated blowups.

In particular, one sees from these explicit formulae that $\overline{Conf}_\Gamma(X)$ is a mixed Tate motives whenever X is. Moreover, to regularize the Feynman integrals one can lift the amplitude computation to the blowup $\overline{Conf}_\Gamma(X)$. This generates some ambiguities related to monodromies along exceptional divisors of the iterated blowups (the scenario is reminiscent of the situation one encounters in momentum space in **[13]**). The residues of Feynman integrals can then be expressed in terms of periods on hypersurface complement in $\overline{Conf}_\Gamma(X)$ and some of these residues can be computed using Poincaré residues, in terms of periods on intersections of divisors of the stratification, see **[20]**.

4. Further comments, questions and perspectives

All the original cases computed by Broadhurst and Kreimer in **[15]** have now been rigorously proved to be periods of mixed Tate motives in recent work of Francis Brown, **[17]**, by mapping the period computation to moduli space $\bar{\mathcal{M}}_{0,n}$ and using results on multiple zeta values as periods of $\bar{\mathcal{M}}_{0,n}$, of Goncharov and Manin [**26**] and of Brown [**18**].

Heuristically, the fact that the graph hypersurfaces remain mixed Tate for a surprisingly large size of the graph (the first counterexample found by Doryn has 14 edges), is related to the fact that these hypersurfaces are very singular, with singularities in low codimension, which allows for the motive to be much "simpler" than that of less singular hypersurfaces of the same degree. It is in any case an interesting question to study the singularities of the graph hypersurfaces and have good invariant that estimate how singular they are. The Chern–Schwartz–MacPherson classes of singular varieties are a very refined invariant that measures how singular a variety is. These Chern classes of singular varieties can be conveniently assembled into an invariant of the graph hypersurface complement that behaves like an algebro-geometric Feynman rule, in the sense of [**3**]. It is shown in the recent work of Aluffi [**1**] that these invariants also satisfy a deletion-contraction relation and recursions formulae, similar to those we recalled above for the virtual motive.

While the general result of Belkale and Brosnan [**8**] and the recent explicit examples of Doryn [**22**] and Brown and Schnetz [**19**] show that graph hypersurfaces become non-mixed-Tate as the graphs become sufficiently large, the question remains of whether the specific period given by the Feynman amplitude may still be itself a mixed Tate period. In fact, in principle, it may happen that the specific piece of the cohomology that is involved in that period computation remains mixed Tate even if the variety itself is no longer a mixed Tate motive. Some computation of the middle cohomology carried out by Doryn in [**23**] are compatible with this possibility. However, there are good reasons at this point to believe that this guess itself may be too strong and that Feynman amplitudes that are non-mixed-Tate periods will also appear for sufficiently large graphs. A good source of evidence in this direction is in the recent paper [**19**], where Brown and Schnetz show that, for physically significant graphs one can write the virtual motive of the graph hypersurface in the form $[X_\Gamma] \equiv c_2(\Gamma)\mathbb{L}^2 \mod \mathbb{L}^3$, where what they denote $c_2(\Gamma)$ is a new invariant, a class in the Grothendieck ring of varieties, which they expect will be closely related to the "framing of the motive", which would be the smallest piece of the motive carrying the information on the period. Thus, the fact that these $c_2(\Gamma)$ will eventually be non-mixed-Tate might be used to prove that the Feynman amplitudes themselves will cease to be mixed Tate for appropriate graphs.

References

[1] P. Aluffi, *Chern classes of graph hypersurfaces and deletion-contraction*, arXiv:1106.1447, to appear in Moscow Mathematical Journal.

[2] P. Aluffi, M. Marcolli, *Feynman motives of banana graphs*, Communications in Number Theory and Physics, Vol.3 (2009) N.1, 1–57.

[3] P. Aluffi, M. Marcolli, *Algebro-geometric Feynman rules*, International Journal of Geometric Methods in Modern Physics, Vol.8 (2011) N.1, 203–237.

[4] P. Aluffi, M. Marcolli, *Feynman motives and deletion-contraction relations*, in "Topology of Algebraic Varieties and Singularities", Contemporary Mathematics, Vol.538 (2011) 21–64.

[5] P. Aluffi, M. Marcolli *Graph hypersurfaces and a dichotomy in the Grothendieck ring*, Letters in Mathematical Physics, (2011) Vol.95, 223–232.

[6] P. Aluffi, M. Marcolli, *Parametric Feynman integrals and determinant hypersurfaces*, Advances in Theoretical and Mathematical Physics, Vol.14 (2010) 911–963.

[7] Y. André, *An introduction to motivic zeta functions of motives*, in "Motives, quantum field theory, and pseudodifferential operators", pp.3–17, Clay Math. Proc., 12, Amer. Math. Soc., Providence, RI, 2010.

[8] P. Belkale, P. Brosnan, *Matroids, motives, and a conjecture of Kontsevich*, Duke Math. J. 116 (2003), no. 1, 147–188.

[9] F. Bittner, *The universal Euler characteristic for varieties of characteristic zero*, Compos. Math. 140 (2004), no. 4, 1011–1032.

[10] J. Bjorken, S. Drell, Relativistic Quantum Mechanics, McGraw-Hill, 1964, and Relativistic Quantum Fields, McGraw-Hill, 1965.

[11] S. Bloch, *Motives associated to sums of graphs*, in "The geometry of algebraic cycles", 137–143, Clay Math. Proc., 9, Amer. Math. Soc., Providence, RI, 2010.

[12] S. Bloch, H. Esnault, D. Kreimer, *On motives associated to graph polynomials*, Comm. Math. Phys. 267 (2006), no. 1, 181–225.

[13] S. Bloch, D. Kreimer, *Mixed Hodge structures and renormalization in physics*, Commun. Number Theory Phys. 2 (2008), no. 4, 637–718.

[14] S. Bloch, D. Kreimer, *Feynman amplitudes and Landau singularities for one-loop graphs*, Commun. Number Theory Phys. 4 (2010), no. 4, 709–753.

[15] D. Broadhurst, D. Kreimer, *Association of multiple zeta values with positive knots via Feynman diagrams up to 9 loops*, Phys. Lett. B 393 (1997), no. 3-4, 403–412.

[16] F. Brown, *Mixed Tate motives over* $\mathbb{Z}$, Annals of Math. Vol.175 (2012) N.2, 949–976.

[17] F. Brown, *On the periods of some Feynman integrals*, arXiv:0910.0114.

[18] F. Brown, *Multiple zeta values and periods of moduli spaces* $\mathfrak{M}_{0,n}$, Ann. Sci. Éc. Norm. Supér. (4) 42 (2009), no. 3, 371–489.

[19] F. Brown, O. Schnetz, *A K3 in* ϕ^4, arXiv:1006.4064.

[20] Ö. Ceyhan, M. Marcolli, *Feynman integrals and motives of configuration spaces*, arXiv:1012.5485, to appear in Communications in Mathematical Physics.

[21] C. De Concini, C. Procesi, *Wonderful models of subspace arrangements*, Selecta Math. (N.S.) Vol.1 (1995) N.3, 459–494.

[22] D. Doryn, *On one example and one counterexample in counting rational points on graph hypersurfaces*, Lett. Math. Phys. 97 (2011), no. 3, 303–315.

[23] D. Doryn, *Cohomology of graph hypersurfaces associated to certain Feynman graphs*, Commun. Number Theory Phys. 4 (2010), no. 2, 365–415.

[24] W. Fulton, R. MacPherson, *A compactification of configuration spaces*, Ann. of Math. (2) 139 (1994), no. 1, 183–225.

[25] H. Gillet, C. Soulé, *Descent, motives and K-theory*, J. Reine Angew. Math. 478 (1996), 127–176.

[26] A. Goncharov, Yu.I. Manin, *Multiple* ζ*-motives and moduli spaces* $\bar{M}_{0,n}$, Compos. Math. 140 (2004), no. 1, 1–14.

[27] M. Hanamura, *Mixed motives and algebraic cycles. I*, Math. Res. Lett. 2 (1995), no. 6, 811–821.

[28] C. Itzykson, J.B. Zuber, Quantum Field Theory, Dover Publications, 2006.

[29] U. Jannsen, *Motives, numerical equivalence, and semi-simplicity*, Invent. Math. 107 (1992), no. 3, 447–452.

[30] M. Kontsevich, D. Zagier, *Periods*, in "Mathematics unlimited—2001 and beyond", 771–808, Springer, Berlin, 2001.

[31] M. Larsen, V. Lunts, *Motivic measures and stable birational geometry*, Mosc. Math. J. 3 (2003), no. 1, 85–95.

[32] M. Levine, *Mixed motives*, Mathematical Surveys and Monographs, 57. American Mathematical Society, Providence, RI, 1998.

[33] M. Levine, *Tate motives and the vanishing conjectures for algebraic K-theory*, in "Algebraic K-theory and algebraic topology" 167–188, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., 407, Kluwer, 1993.

[34] L. Li, *Wonderful compactification of an arrangement of subvarieties*, Michigan Math. J. 58 (2009), no. 2, 535–563.

[35] L. Li, *Chow Motive of Fulton–MacPherson configuration spaces and wonderful compactifications*, Michigan Math. J. 58 (2009), no. 2, 565–598.

[36] Yu.I. Manin, *Correspondences, motifs and monoidal transformations*, Mat. Sb. (N.S.) 77 (119) 1968 475–507.

[37] M. Marcolli, *Feynman motives*, World Scientific, *2010*.
[38] A. Rej, *Motives: an introductory survey for physicists. With an appendix by Matilde Marcolli*, Contemp. Math., 539, "Combinatorics and physics", 377–415, Amer. Math. Soc., Providence, RI, 2011.
[39] M. Sato, T. Miwa, M. Jimbo, T. Oshima, *Holonomy Structure of Landau Singularities and Feynman Integrals*, Publ. RIMS, Kyoto Univ. 12 Suppl. (1977), 387–438.
[40] J. Stembridge, *Counting points on varieties over finite fields related to a conjecture of Kontsevich*, Ann. Comb. 2 (1998), no. 4, 365–385.
[41] B. Toën, *Anneaux de Grothendieck des n-champs d'Artin*, arXiv:math/0509098.
[42] V. Voevodsky, *Triangulated categories of motives over a field*, in "Cycles, transfer and motivic homology theories", pp. 188–238, Annals of Mathematical Studies, Vol. 143, Princeton, 2000.

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