A POSITIVE DENSITY ANALOGUE OF THE LIEB-THIRRING INEQUALITY

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Abstract. The Lieb-Thirring inequalities give a bound on the negative eigenvalues of a Schrödinger operator in terms of an $L^p$ norm of the potential. These are dual to bounds on the $H^1$-norms of a system of orthonormal functions. Here we extend these bounds to analogous inequalities for perturbations of the Fermi sea of non-interacting particles, i.e., for perturbations of the continuous spectrum of the Laplacian by local potentials.

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1. Introduction

The Pauli exclusion principle for fermions in quantum mechanics has no classical analogue. One of its primary effects is the increase in kinetic energy that accompanies an increase in the density of such particles. Intuitively, this increase should be quantifiable in a manner similar to that predicted by the semi-classical approximation to quantum mechanics, and it is the aim of this paper to show that this can, indeed, be achieved in the case of density perturbations of an ideal Fermi gas.

We begin with some definitions [25, Chapters 3 and 4]. The state of a finite system of \( N \) fermions of \( q \) spin states each (\( q = 2 \) for electrons) is described by a density matrix \( \Gamma \), which may or not be pure. Associated with a state is a one-body density matrix \( \gamma \) (a reduction of \( \Gamma \)) which is an operator on \( L^2(\mathbb{R}^d, \mathbb{C}^q) \). The essential properties of \( \gamma \) are that \( 0 \leq \gamma \leq 1 \) as an operator and that \( \text{Tr} \gamma = N \). It is a fact proved by Coleman in [5] (see also [25, Thm. 3.2]) that any \( \gamma \) with these properties arises from some state \( \Gamma \), i.e., no other restrictions on \( \gamma \) are required by quantum mechanics.

The electron density is \( \rho_\gamma(x) = \text{Tr}_\mathbb{C}^q \gamma(x, x) \), in which \( \gamma(x, x) \) is a \( q \times q \) matrix. The kinetic energy of the \( N \) particle system depends only on \( \gamma \) and is given by \( \text{Tr}(-\Delta)\gamma \) in units where \( \hbar = 2m = 1 \) and with \( \Delta = \nabla^2 \) denoting the Laplacian.

The semi-classical approximation for the kinetic energy is

\[
\text{Tr}(-\Delta)\gamma \approx K_{\text{sc}}(d) \int_{\mathbb{R}^d} \rho_\gamma(x)^{1+2/d} \, dx,
\]

with the constant

\[
K_{\text{sc}}(d) := \frac{d}{d+2} \left( \frac{d(2\pi)^d}{q|S^{d-1}|} \right)^{\frac{2}{d}}.
\]

The Lieb-Thirring inequality [27] states that there is a constant \( 0 < r_d \leq 1 \) such that

\[
\text{Tr}(-\Delta)\gamma \geq r_d K_{\text{sc}}(d) \int_{\mathbb{R}^d} \rho_\gamma(x)^{1+2/d} \, dx
\]

for any one-body density matrix \( 0 \leq \gamma \leq 1 \). The original value [27] for \( r_3 \) was 0.185 but it has been improved since then to 0.672 [6], and it is current belief that it equals one (for \( d \geq 3 \)). This subject continues to be actively studied (see for instance the recent works [6, 11, 12, 18] and the reviews [17, 11]). Note that the inequality (1.2) does not require \( \text{Tr} \gamma \) to be an integer; it need not even be finite.

We can turn the matter around and, instead of specifying \( \gamma \), think of specifying a density \( \rho(x) \) and asking for the minimum kinetic energy needed to achieve this particle density. The Lieb-Thirring inequality above gives a universal answer to this question in terms of the semi-classical approximation. Here, we are implicitly using the fact that for any given function \( \rho(x) \geq 0 \) with \( \int_{\mathbb{R}^d} \rho(x) \, dx = N \) there is a fermionic \( N \)-particle density matrix whose one-body reduced density matrix \( \gamma \) satisfies \( \rho_\gamma(x) = \rho(x) \), see [20, Thm. 1.2].

It is important for many applications that the right side of the inequality (1.2) is additive in position space. If we partition \( \mathbb{R}^3 \) into disjoint subsets, the right side is just the sum of the corresponding local energies. While this does not hold for the left side, it nearly does. The bound shows that there is some truth to this
approximate additivity. This additivity, or locality, played an important role in a proof of the stability of matter [26].

While the inequality (1.2) was the object of principal interest in [26], the actual proof of (1.2) went via the Legendre transform of (1.2) with respect to $\rho$. This is an inequality about the sum of the negative eigenvalues ($E_j$) of the Schrödinger operator $-\Delta + V$ for an arbitrary potential $V$, namely,

\[ \sum_j |E_j| = \text{Tr}(-\Delta + V)^- \leq \hat{r}_d L_{sc}(d) \int_{\mathbb{R}^d} V(x)^{1+d/2} \, dx, \]

where $X_- := \max\{-X, 0\} \geq 0$ denotes the negative part of a number or a self-adjoint operator $X$,

\[ L_{sc}(d) = \frac{2^{d-1}}{d(d+2)(2\pi)^{d}}, \]

and $\hat{r}_d$ is a universal constant independent of $V$. In fact, the relation between the constants in (1.2) and (1.3) is given by

\[ (p L_{sc}(d))^{p'} (p' K_{sc}(d))^{p} = 1 \quad \text{and} \quad (\hat{r}_d)^{p'} (r_d)^{p} = 1, \]

where $p = 1 + d/2$ and $p' = 1 + 2/d$; see [26, 25]. The duality between $\rho$ and $V$, and between kinetic energy and Schrödinger eigenvalue sums is one of the important inputs in density functional theory [20].

A question that is not only natural but of significance for condensed matter physics is the analogue of (1.2) when we start, not with the vacuum, but with a background of fermions with some prescribed constant density $\rho_0 > 0$. How much kinetic energy does it then cost to make a local perturbation $\delta \rho(x)$? This time $\delta \rho$ can be negative, as long as $\rho_0 + \delta \rho \geq 0$ everywhere. We would expect that the semi-classical expression will guide us here as well and, indeed, it does so, as we will show in this paper.

The principal difficulty that has to be overcome is that inequality (1.2) was obtained in [27] by first proving (1.3), a route that does not seem to be helpful now. The picture was changed by a paper of Rumin [35] in which inequality (1.2) was obtained directly, without estimates on eigenvalues. (The constant obtained this way is not, however, as good as the 0.672 quoted above.) We are able to utilize some ideas in [35] to help solve our problem.

The first thing is to formulate a mathematically precise statement of what it means to make a local perturbation of an ideal Fermi gas. One could think of putting $N$ electrons in a large box of volume $v$, computing the change in kinetic energy, and then passing to the thermodynamic limit $v \to \infty$ with $\rho_0 = N/v$ fixed. For this, appropriate boundary conditions have to be imposed. To avoid this discussion we pose the problem for an infinite sea with specified chemical potential $\mu > 0$. The chemical potential of the ideal Fermi gas is

\[ \mu = \frac{2 + d}{d} K_{sc}(d) \rho_0^{2/d}. \]

It is often called the Fermi energy and can be interpreted, physically, as the kinetic energy needed to add one more particle to the Fermi sea.

We then look at the operator $-\Delta - \mu$ in $L^2(\mathbb{R}^d, \mathbb{C}^q)$, which, in our context, plays the role of $-\Delta$ in inequality (1.2). The energy observable of a particle is now defined to be $-\Delta - \mu$, which is negative for states in the Fermi sea and positive for states
outside the sea. The energy to create either a particle outside the Fermi sea or a hole inside the sea is positive. The (grand-canonical) energy of the unperturbed Fermi sea is $\text{Tr}(-\Delta - \mu)\Pi^- = -\infty$, where $\Pi^-$ denotes the projection onto the negative spectral subspace of $-\Delta - \mu$. Clearly, $\Pi^-(x,x) = \rho_0/q\mathbb{1}_C \delta$ (we will often not write the identity matrix $\mathbb{1}_C \delta$ for simplicity). Our interest is in the formal difference in energy between the state described by a one-body density matrix $0 \leq \gamma \leq 1$ and the state described by $\Pi^-$, and this is non-negative since the minimum total energy (given $\mu$) is the uniform, filled Fermi sea. Our main result is a lower bound for this difference in terms of the semi-classical expression in all dimensions $d \geq 2$, namely,

\begin{align}
(1.5) \quad & \text{Tr} \left( (-\Delta - \mu)\gamma - (-\Delta - \mu)\Pi^- \right) = \text{Tr}(-\Delta - \mu)(\gamma - \Pi^-) \\
& \geq r_d \ K_{sc}(d) \int_{\mathbb{R}^d} \left( \rho_{\gamma}(x)^{1+\frac{d}{2}} - (\rho_0)^{1+\frac{d}{2}} - \frac{2+d}{d} (\rho_0)^{\frac{d}{2}} (\rho_{\gamma}(x) - \rho_0) \right) dx
\end{align}

for some universal $r_d$ that does not depend on $\gamma$. The trace in this expression might not exist in the usual sense, that is, $(-\Delta - \mu)(\gamma - \Pi^-)$ might not be trace class. This situation will be dealt with more carefully in the sequel.

The Legendre transform of the right side of the inequality (1.5) will give us an inequality for the change in energy of the Fermi sea when a one-body potential $V$ is added to $-\Delta - \mu$. The positive density analogue of (1.3) in dimensions $d \geq 2$ is

\begin{align}
(1.6) \quad & \text{Tr} \left( (-\Delta - \mu + V) - (-\Delta - \mu) \right) + \rho_0 \int_{\mathbb{R}^d} V(x) dx \\
& \leq \tilde{r}_d \ L_{sc}(d) \int_{\mathbb{R}^d} \left( (V(x) - \mu)^{1+\frac{d}{2}} - \mu^{1+\frac{d}{2}} + \frac{2+d}{2} \mu^\frac{d}{2} V(x) \right) dx.
\end{align}

Of course, $(-\Delta - \mu)_- = -(-\Delta - \mu)\Pi^-$. The quantity above is not necessarily trace class but the trace can, nevertheless, be defined; see Definition 2.2 below.

The inequalities (1.5) and (1.6) are only valid in dimensions $d \geq 2$. In dimension $d = 1$, a divergence related to the Peierls instability [29] appears, and a Lieb-Thirring inequality of the form of (1.5) or (1.6) cannot hold for $\mu > 0$. This will be discussed in detail in this paper.

Our main inequalities (1.5) and (1.6) were announced and discussed in [7]. In particular, the constants $r_3 \simeq 0.1279$ and $r_2 \simeq 0.04493$ were given. We do not expect them to be optimal and it is a challenge to improve them. One interesting case in which the sharp constant in (1.5) can be found is that in which $\rho_{\gamma}(x)$ is required to be zero for all $x$ in some bounded domain $\Omega$. In Section 2.3 we prove that if the integral on the right side of (1.5) is taken only over $\Omega$, then $r_d = 1$ in this case, and this is obviously optimal.

Our method to prove the inequalities (1.5) and (1.6) is rather general and it can be used to treat other systems. As examples we will also discuss in this paper Lieb-Thirring inequalities in a box of size $L \gg 1$ with periodic boundary conditions, the case of positive temperature, and systems with a periodic background.

The paper is organized as follows. In the next section we introduce some mathematical tools allowing us to give a rigorous meaning to the Lieb-Thirring inequalities (1.5) and (1.6). Our main task will be to correctly define the traces $\text{Tr}(-\Delta - \mu)(\gamma - \Pi^-)$ and $\text{Tr}((-\Delta - \mu + V) - (-\Delta - \mu)_-)$ in such a way that (1.5)
and (1.6) become dual to each other in the appropriate function spaces. In Section 2.3 we consider the case of a weak potential \( tV \) with \( t \ll 1 \), and we compute the second-order term in \( t \) of the left side of (1.6). This will clarify the fact that there cannot be simple Lieb-Thirring inequalities at positive density in dimension \( d = 1 \).

The proofs of all these results are provided in Sections 3 and 4. In Section 2.4 we consider the case of a density matrix \( \gamma \) which vanishes on a given domain \( \Omega \) and we derive a lower bound on the relative kinetic energy which involves the sharp constant \( K_{\text{sc}}(d) \). In Section 5 we prove Lieb-Thirring inequalities in a box with periodic boundary conditions. This allows us to investigate the thermodynamic limit and to extend our results to positive temperature. Finally, in Section 6 we discuss the extension of our results to general background potentials, with an emphasis on periodic systems.

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2. Statement of the main results

In this section, we provide the necessary tools to give a clear mathematical meaning to the inequalities (1.5) and (1.6) which we have announced in the introduction, and we state our main results.

We fix a positive number \( \mu > 0 \) and denote by

\[
\Pi^- := 1(-\Delta \leq \mu)
\]

the spectral projection of the Laplacian associated with the interval \((-\infty, \mu)\), describing a free Fermi gas in its ground state with chemical potential \( \mu \). As recalled before, the ambient Hilbert space is \( L^2(\mathbb{R}^d, \mathbb{C}^q) \) where \( q \) is the number of spin states per particle (which is 2 for unpolarized electrons but which will be taken arbitrary in this work). The gas, described by the projection \( \Pi^- \), has the constant density

\[
\rho_0 = q(2\pi)^{-d} \int_{|p|^2 < \mu} dp = \frac{q[S^{d-1}]}{d(2\pi)^d} \mu^\frac{d}{2}.
\]

The kinetic energy per unit volume agrees with the semi-classical formula

\[
q(2\pi)^{-d} \int_{|p|^2 < \mu} |p|^2 dp = (d/2) L_{\text{sc}}(d) \mu^{1+\frac{d}{2}} = K_{\text{sc}}(d) (\rho_0)^{1+\frac{d}{2}}
\]

where the semi-classical constants \( K_{\text{sc}}(d) \) and \( L_{\text{sc}}(d) \) are given by (1.1) and (1.4) above.

2.1. Lower bound on the variation of kinetic energy. We consider a fermionic state, with one-body density matrix \( 0 \leq \gamma \leq 1 \) acting on \( L^2(\mathbb{R}^d, \mathbb{C}^q) \), which we think of as a perturbation of the reference state \( \Pi^- \) defined before in (2.1). We are interested in proving a lower bound on the kinetic energy (including \( \mu \)) of \( \gamma \), counted relatively to that of \( \Pi^- \), of the form of (1.5). To make sense of this inequality, we use as main variable \( Q := \gamma - \Pi^- \) which satisfies the constraint

\[
-\Pi^- \leq Q \leq 1 - \Pi^- := \Pi^+.
\]

Our goal is to prove a lower bound on \( \text{Tr}(-\Delta - \mu)Q \). Our first task will be to give a clear meaning to this quantity, in a rather general sense. The constraint (2.4)
can also be written \((Q + \Pi^-)^2 \leq Q + \Pi^-\). Expanding \((Q + \Pi^-)^2\) shows that (2.4) is equivalent to
\[
(2.5) \quad Q^2 \leq Q^{++} - Q^{--}
\]
where we have introduced the notation \(Q^{\tau \tau'} := \Pi' Q \Pi'\) for \(\tau, \tau' \in \{\pm\}\). In particular, we have \(Q^{++} \geq 0\) and \(Q^{--} \leq 0\). Furthermore there is equality in (2.5) if and only if \(\gamma = Q + \Pi^-\) is an orthogonal projection.

For smooth-enough finite rank operators \(Q\), the following computation is justified:
\[
\text{Tr}(-\Delta - \mu)Q = \text{Tr}\left(\Pi^+(-\Delta - \mu)Q \Pi^+ + \Pi^-(-\Delta - \mu)Q \Pi^-\right)
\]
\[
= \text{Tr}| - \Delta - \mu|\left(Q^{++} - Q^{--}\right).
\]
As we have seen, we always have \(Q^{++} - Q^{--} \geq 0\), hence \(\text{Tr}(\Delta - \mu)Q \geq 0\) (changing the density of particles inside or outside of the Fermi sea costs a positive energy once \(\mu\), the energy of the Fermi level, has been subtracted). We now use this fact to give a general meaning to \(\text{Tr}(\Delta - \mu)Q\), in the sense of quadratic forms.

**Definition 2.1** (Relative kinetic energy). Let \(Q\) be a bounded self-adjoint operator such that \(|-\Delta - \mu|^{1/2}Q^{\pm \pm}|-\Delta - \mu|^{1/2}\) are trace-class\(^1\). We define
\[
(2.6) \quad \text{Tr}_0(-\Delta - \mu)Q := \text{Tr}| - \Delta - \mu|^{1/2}\left(Q^{++} - Q^{--}\right)| - \Delta - \mu|^{1/2}.
\]
If \(Q\) is a bounded operator such that \(\pm Q^{\pm \pm} \geq 0\), then we extend the previous expression by letting
\[
\text{Tr}_0(-\Delta - \mu)Q := +\infty
\]
whenever \(|-\Delta - \mu|^{1/2}Q^{++}|-\Delta - \mu|^{1/2}\) or \(|-\Delta - \mu|^{1/2}Q^{--}|-\Delta - \mu|^{1/2}\) is not trace-class.

Of course we have \(\text{Tr}_0(\Delta - \mu)Q = \text{Tr}(\Delta - \mu)Q\) (the usual trace) when \((\Delta - \mu)Q\) is trace-class. The previous definition of the relative kinetic energy is inspired by similar ideas used in the context of the Dirac equation \([9]\) and of electrons in crystals \([1]\). Later on we will be interested in estimating the kinetic energy of operators of the form \(Q_V = 1(-\Delta + V \leq \mu) - \Pi^-\) for a given potential \(V\). In general we do not expect such operators to be trace-class when \(\mu > 0\) (or even compact, see Remark 2.1 below).

**Remark 2.1.** When \(\gamma = Q + \Pi^-\) is itself an orthogonal projection, \(\gamma^2 = \gamma\), we have equality in (2.5) and we obtain
\[
(2.7) \quad \text{Tr}_0(-\Delta - \mu)Q = \text{Tr}| - \Delta - \mu|^{1/2}Q^2| - \Delta - \mu|^{1/2} = \left\|\begin{array}{c} Q \\ Q^2 \\ Q \end{array}\right\|_{\mathcal{S}_2}^2
\]
where \(\mathcal{S}_2\) denotes the ideal of Hilbert-Schmidt operators on \(L^2(\mathbb{R}^d, \mathbb{C}^d)\).

We are now ready to state our rigorous version of (1.5).

**Theorem 2.1** (Lieb-Thirring inequality, density version, \(d \geq 2\)). Assume that \(d \geq 2\) and \(\mu \geq 0\). Let \(Q\) be a self-adjoint operator such that \(-\Pi^- \leq Q \leq \Pi^+\) and

\(^1\)In the whole paper we use the notation \(Q^{\pm \pm}\) for the two operators \(Q^{++}\) and \(Q^{--}\), and the notation \(Q^{\pm \mp}\) for \(Q^{+-}\) and \(Q^{-+}\).
such that \(|-\Delta - \mu|^{1/2}Q^{\pm}| - \Delta - \mu|^{1/2}\) are trace-class. Then \(Q\) is locally trace-class and the corresponding density satisfies
\[
\rho_Q \in L^{1+\frac{2}{d}}(\mathbb{R}^d) + L^2(\mathbb{R}^d).
\]
Moreover, there exists a positive constant \(K(d) \leq K_{sc}(d)\) (depending only on \(d \geq 2\)) such that
\[
(2.9)\quad \text{Tr}_0(-\Delta - \mu)Q \geq K(d) \int_{\mathbb{R}^d} \left( (\rho_0 + \rho_Q(x))^{1+\frac{2}{d}} - (\rho_0)^{1+\frac{2}{d}} - \frac{2}{d} (\rho_0)\frac{2}{d} \rho_Q(x) \right) dx
\]
with \(\rho_0\) the constant density of the Fermi gas, given by (2.2).

We recall that a locally trace-class self-adjoint operator \(A\) is such that \(\text{Tr}|\chi A \chi| < \infty\) for every bounded function \(\chi\) of compact support. In this case, the associated density \(\rho_A\) is the unique real-valued function in \(L^1_{\text{loc}}(\mathbb{R}^d)\) satisfying \(\text{Tr}(\chi A \chi) = \int_{\mathbb{R}^d} \chi(x)^T \rho_A(x) dx\).

Note that since \(Q \geq -\Pi^-\) in Theorem 2.1, we have \(\rho_Q(x) \geq -\rho_0\) for all \(x \in \mathbb{R}^d\). The function
\[
(2.10)\quad \delta T_{\mu}^{sc}(\rho) := (\rho_0 + \rho)^{1+\frac{2}{d}} - (\rho_0)^{1+\frac{2}{d}} - \frac{d + 2}{d} (\rho_0)\frac{2}{d} \rho
\]
is non-negative and convex for \(\rho \geq -\rho_0\). Hence, the integrand on the right side of (2.9) is always non-negative. The function \(\delta T_{\mu}^{sc}(\rho)\) behaves like \(\rho^{1+2/d}\) for large \(\rho\), and like \(\rho^2\) for small \(\rho\). Moreover, it satisfies the scaling property
\[
(2.11)\quad \delta T_{\mu}^{sc}(\rho) = \mu^{d/2} \delta T_{1}^{sc}(\rho \mu^{-2/d})
\]
and one has
\[
\lim_{\mu \to 0} \delta T_{\mu}^{sc}(\rho) = \rho^{1+\frac{2}{d}}
\]
uniformly on \(\mathbb{R}^+\). In the limit \(\mu \to 0\) (which is the same as \(\rho_0 \to 0\) by (2.2)), the inequality (2.9) reduces to the usual Lieb-Thirring inequality [27] [26] [25].

(2.12)\quad \forall 0 \leq \gamma \leq 1, \quad \text{Tr}(-\Delta)\gamma \geq K(d) \int_{\mathbb{R}^d} \rho_\gamma(x)^{1+\frac{2}{d}} dx.

The best constant in this inequality is smaller than or equal to \(K_{sc}(d)\), the semiclassical constant defined above in (1.1). Hence, \(K(d) \leq K_{sc}(d)\) must hold. From the scaling property (2.11), we know that the best constant in (2.9) is independent of \(\mu \geq 0\). However, the best constant for \(\mu = 0\) in the Lieb-Thirring estimate (2.12) is not necessarily equal to the best constant for (2.9). The recent estimates [6] for the Lieb-Thirring constant in (2.12) do not a priori give any information on the positive density analogue (2.9).

The proof of Theorem 2.1 is detailed later in Section 3.2. It uses the convexity of \(\delta T_{\mu}^{sc}\), to estimate separately the densities corresponding to the two diagonal terms \(Q^{\pm}\) and the two off-diagonal terms \(Q^{\pm \pm}\). The estimate on the diagonal terms \(Q^{\pm}\) is based on a new method which has recently been introduced by Rumin [35]. This estimate works similarly in dimension \(d = 1\). The off-diagonal terms \(Q^{\pm \pm}\) are studied by a direct and explicit method which does not cover the case \(d = 1\).

There cannot be an inequality like (2.9) in dimension \(d = 1\) for \(\mu > 0\). This surprising fact is due to a special divergence of the off-diagonal terms \(Q^{\pm \pm}\) at
the Fermi points (see Section 2.3 below for details). However, we can prove the following:

**Theorem 2.2** (Lieb-Thirring inequality, density version, \( d = 1 \)). Assume that \( d = 1 \) and \( \mu > 0 \). Let \( Q \) be a self-adjoint operator such that \(-\Pi^- \leq Q \leq \Pi^+\) and such that \(|-\Delta-\mu|^{1/2} Q^{\pm \pm} \leq |-\Delta-\mu|^{1/2}\) are trace-class. Then \( Q \) is locally trace-class and the corresponding densities satisfy

\[
\text{Tr}_0(-\Delta - \mu)Q \geq K(1) \left( \left( \rho_0 + \rho_{Q^{++}} + \rho_{Q^{--}} \right)^3 - (\rho_0)^3 - 3(\rho_0)^2 \left( \rho_{Q^{++}} + \rho_{Q^{--}} \right) \right) dx
\]

\[
+ K'(1) \int_{\mathbb{R}} \frac{\sqrt{\mu} |k|}{(\sqrt{\mu} + |k|) \log \left( \frac{2\sqrt{\mu} + |k|}{2\sqrt{\mu} - |k|} \right)} |\hat{\rho}_{Q^{++}}(k) + \hat{\rho}_{Q^{--}}(k)|^2 dk,
\]

with \( \rho_0 \) the constant density of the Fermi gas, given by (2.2).

Note the logarithmic divergence of the function in the denominator, at \( |k| = 2\sqrt{\mu} \). Hence the last term is not bounded from below by \( \int_{\mathbb{R}} |\rho_{Q^{++}} + \rho_{Q^{--}}|^2 \). In Section 2.3 below, we will see that, up to the value of the prefactors \( K(1) \) and \( K'(1) \), this bound is optimal. In particular, the right side of (2.14) cannot be replaced by a constant times \( \int_{\mathbb{R}} \delta T_{\mu}^{\text{sc}}(\rho_Q) \). In the limit \( \mu \to 0 \), the inequality (2.14) nevertheless reduces to the one-dimensional Lieb-Thirring inequality (2.12).

**Remark 2.2.** Let \( \varphi \in L^2(\mathbb{R}^d, \mathbb{C}^q) \) be any normalized, smooth enough function. Applying (2.9) or (2.13) to \( Q_{\pm} = \pm [\Pi^+ \varphi](\Pi^\pm \varphi) \) and using a simple convexity argument, we obtain the following Sobolev-like inequality:

\[
\int_{\mathbb{R}^d} |p^2 - \mu| |\hat{\varphi}(p)|^2 \geq K(d) \int_{\mathbb{R}^d} \delta T_{\mu}^{\text{sc}}(|\varphi|^2)
\]

for all \( \varphi \) with \( \int_{\mathbb{R}^d} |\varphi|^2 \leq 1 \), and in any dimension \( d \geq 1 \).

### 2.2. Variation of energy in presence of an external potential

In this section we study the dual version of our Lieb-Thirring inequalities (2.9) and (2.14), expressed in terms of an external potential \( V \) (the variable dual to \( \rho \)). We will give a rigorous meaning to (1.6).

Let \( V \) be a real-valued function satisfying

\[
V \in L^2(\mathbb{R}^d) \cap L^{1+\frac{d}{2}}(\mathbb{R}^d) \quad \text{for } d \geq 2
\]

or

\[
V \in L^{3/2}(\mathbb{R}) + L^2(\mathbb{R})
\]

with

\[
\int_{\mathbb{R}} \left( 1 + \frac{\sqrt{\mu} + |k|}{\sqrt{\mu} |k|} \log \left( \frac{2\sqrt{\mu} + |k|}{2\sqrt{\mu} - |k|} \right) \right) |\hat{V}(k)|^2 dk < \infty \quad \text{for } d = 1.
\]
Under our assumption \((2.16)\), the operator \(-\Delta + V\) is self-adjoint on \(H^2(\mathbb{R}^d)\), by the Rellich-Kato Theorem \([33]\). In dimension \(d = 1\), our assumption \((2.17)\) allows to define the Friedrichs self-adjoint realization of \(-\Delta + V\), by the KLNM theorem \([33]\).

We now define
\[
(2.18) \quad Q_V := \Pi_V^+ - \Pi_V^- \quad \text{where} \quad \Pi_V^+ := 1(-\Delta + V \leq \mu),
\]
as well as \(\Pi_V^- := 1 - \Pi_V^+\).

**Remark 2.3.** The real number \(\mu\) could a priori be an eigenvalue of \(-\Delta + V\). Then, Theorems 2.3 and 2.4 below hold exactly the same if \(\Pi_V^+\) is replaced by \(\Pi_V^+ + \delta\), where \(\delta\) is an orthogonal projection whose range is contained in \(\ker(-\Delta + V - \mu)\).

In dimension \(d \geq 3\), it is indeed known \([15]\) that, under our assumption \((2.16)\) on \(V\), the self-adjoint operator \(-\Delta + V\) has no positive eigenvalue, thus \(\mu\) is not is the point spectrum of \(-\Delta + V\). However, \(\mu\) could be an eigenvalue of \(-\Delta + V\) in dimensions \(d = 1\) and \(d = 2\).

Similarly as in Definition 2.1 we can define a relative total energy as follows.

**Definition 2.2** (Relative total energy). *Let \(R\) be a bounded self-adjoint operator such that \(|-\Delta - \mu + V|^{1/2}\Pi_V^+ R \Pi_V^+| - \Delta - \mu + V|^{1/2}\) are trace-class. We define (2.19)*
\[
\text{Tr}_V(-\Delta - \mu + V) R := \text{Tr}|-\Delta - \mu + V|^{1/2} (\Pi_V^+ R \Pi_V^+ - \Pi_V^- R \Pi_V^-)| - \Delta - \mu + V|^{1/2}.
\]
*If \(R\) is a bounded operator such that \(\pm \Pi_V^+ R \Pi_V^+ \geq 0\), then we extend the previous expression by letting

\[
\text{Tr}_V(-\Delta - \mu + V) R := +\infty
\]
*whenever \(|-\Delta - \mu + V|^{1/2}\Pi_V^+ R \Pi_V^+| - \Delta - \mu|^{1/2}\) or \(|-\Delta - \mu|^{1/2}\Pi_V^+ R \Pi_V^-| - \Delta - \mu|^{1/2}\) is not trace-class.*

Since \(Q_V\) is the difference of the two orthogonal projections \(\Pi_V^-\) and \(\Pi^-\), we have at the same time
\[-\Pi^- \leq Q_V \leq \Pi^+ \quad \text{and} \quad -\Pi_V^+ \leq -Q_V \leq \Pi_V^-.
\]
Hence both \(\text{Tr}_0(-\Delta - \mu) Q_V\) and \(\text{Tr}_V(-\Delta - \mu + V) Q_V\) make sense by Definitions 2.1 and 2.2. With our definitions we have
\[
(2.20) \quad \text{Tr}_0(-\Delta - \mu) Q_V = \left\|Q_V| - \Delta - \mu|^{1/2}\right\|^2_{L^2}
\]
and
\[
(2.21) \quad \text{Tr}_V(-\Delta - \mu + V) Q_V = -\left\|Q_V| - \Delta - \mu + V|^{1/2}\right\|^2_{L^2}.
\]
In the theorem below, we show that, under suitable assumptions on \(V\), the two quantities \((2.20)\) and \((2.21)\) are finite and that
\[
(2.22) \quad \text{Tr}_V(-\Delta - \mu + V) Q_V = \text{Tr}_0(-\Delta - \mu) Q_V + \int_{\mathbb{R}^d} V \rho_{Q_V},
\]
as expected. We also derive an estimate on \(\text{Tr}_V(-\Delta - \mu + V) Q_V\) which is the dual version of \((2.9)\) for \(d \geq 2\).
Theorem 2.3 (Lieb-Thirring inequality, potential version, \(d \geq 2\)). Assume that \(\mu \geq 0\) and \(d \geq 2\). Let \(V\) be a real-valued function in \(L^2(\mathbb{R}^d) \cap L^{1+d/2}(\mathbb{R}^d)\).

- Both \(Q_V | - \Delta - \mu|^{1/2} \) and \(Q_V | - \Delta - \mu + V|^{1/2} \) are Hilbert-Schmidt operators, hence (2.20) and (2.21) are finite.

- The relative total energy \(\text{Tr}_V(-\Delta - \mu + V)Q_V\) can be expressed as

\[
\text{(2.23)} \quad \text{Tr}_V(-\Delta - \mu + V)Q_V = \min_{-\Pi \leq Q \leq \Pi^+} \left( \text{Tr}_0(-\Delta - \mu)Q + \int_{\mathbb{R}^d} V(x) \rho_Q(x) \, dx \right).
\]

The minimum in this formula is attained for \(Q = Q_V\). In particular, (2.22) holds true.

- We have the inequality

\[
\text{(2.24)} \quad \text{Tr}_V(-\Delta - \mu + V)Q_V \geq -L(d) \int_{\mathbb{R}^d} \left( (V(x) - \mu)^{1+d/2} - \mu^{1+d/2} + \frac{2 + d}{2} \mu^{d/2} V(x) \right) \, dx
\]

with

\[
L(d) = \frac{2}{d+2} \left( \frac{d}{(d+2)K(d)} \right)^{d/2} \geq L_{\text{sc}}(d)
\]

and where \(K(d)\) is the optimal constant in (2.9).

We recall that the semi-classical constant \(L_{\text{sc}}(d)\) is defined above in (1.4).

Let us comment on our result. We can formally write

\[
\text{(2.25)} \quad \text{Tr}_V(-\Delta - \mu + V)Q_V \quad \approx \quad \text{Tr} \left( (\Delta - \mu + V) - (\Delta - \mu) \right) - \rho_0 \int_{\mathbb{R}^d} V
\]

where \(\rho_0\) is the constant density of the translation-invariant state \(\Pi^-\), recalled in (2.2). The first term of the right side is the formal difference between the total (grand-canonical) energy of the Fermi gas in the presence of the local perturbation \(V\), and its total (grand-canonical) energy in the translation-invariant setting without any potential. The term \(\rho_0 \int_{\mathbb{R}^d} V\), which makes sense under the additional assumption that \(V \in L^1(\mathbb{R}^d)\), is also the first order term obtained by perturbation theory when the first term is formally expanded in powers of \(V\).

The semi-classical approximation of the right side of (2.20) is

\[
L_{\text{sc}}(d) \int_{\mathbb{R}^d} \left( (V(x) - \mu)^{1+d/2} - \mu^{1+d/2} + \frac{2 + d}{2} \mu^{d/2} V(x) \right) \, dx
\]

and, up to the value of the multiplicative constant \(L(d)\), it is precisely the right side of our estimate (2.21). Our result therefore says that the variation of energy obtained by including the potential \(V\) in the system is \(O(1)\) in the thermodynamic limit, and (2.21) provides a precise estimate in terms of the size of \(V\). Since the term \(\rho_0 \int_{\mathbb{R}^d} V\) is obtained via first-order perturbation theory, the semi-classical term on the right side of (2.24) is therefore an estimate on the validity of the first order approximation.
In Section 5.2, we will render the formal equality (2.25) more rigorous, by means of a thermodynamic limit argument. More precisely, we show in Theorem 5.3 that

\[(2.26) \quad \text{Tr}_V(-\Delta - \mu + V)Q_V = \lim_{L \to \infty} \left( -\text{Tr}_{L^2(C_L)}(-\Delta_L - \mu + V\mathbb{1}_{C_L}) - \text{Tr}_{L^2(C_L)}(-\Delta_L - \mu) - \rho_0 \int_{C_L} V \right) \]

where $-\Delta_L$ is the Laplacian on a box $C_L = [-L/2, L/2]^d$ with periodic boundary conditions. This will also justify our definition of the total free energy. A tool to prove (2.26) is to derive a Lieb-Thirring inequality similar to (2.24), for a system living in a box with periodic boundary conditions (Theorem 5.2).

The estimate (2.24) follows from the density estimate (2.9) and the variational principle (2.23), by noting that

\[
\text{Tr}_0(-\Delta - \mu)Q + \int_{\mathbb{R}^d} V \rho_Q \geq K(d) \int_{\mathbb{R}^d} \delta T^{\text{sc}}(\rho_Q) + \int_{\mathbb{R}^d} V \rho_Q
\]

for all $-\Pi^- \leq Q \leq \Pi^+$, by Theorem 2.1. Optimizing the right side with respect to $\rho_Q$ (keeping in mind that $\rho_Q$ is pointwise bounded from below by $-\rho_0$), yields (2.24).

Similarly, if we assume that (2.24) is known, we can derive (2.9) by choosing $V = \frac{\partial (\delta T^{\text{sc}})}{\partial \rho}(\rho_Q)$.

Hence (2.24) and (2.9) are dual to each other.

In dimension $d = 1$, using the weaker lower bound (2.14) on $\text{Tr}_0(-\Delta - \mu)Q$, we can prove the following result.

**Theorem 2.4** (Lieb-Thirring inequality, potential version, $d = 1$). Assume that $\mu > 0$ and $d = 1$. Let $V \in L^{3/2}(\mathbb{R}) + L^2(\mathbb{R})$ be a real-valued function such that

\[(2.27) \quad \int_{\mathbb{R}} \sqrt{\mu + |k|} \log \left( \frac{2\sqrt{\mu} + |k|}{2\sqrt{\mu} - |k|} \right) |\hat{V}(k)|^2 dk < \infty.\]

Then all the conclusions of Theorem 2.3 remain true, except that (2.24) must be replaced by

\[(2.28) \quad \text{Tr}_V(-\Delta - \mu + V)Q_V \geq -L(1) \int_{\mathbb{R}} \left( (V(x) - \mu)^{\frac{1}{2}} - \frac{3}{2} \mu^{\frac{1}{2}} + \frac{3}{2} \mu^{\frac{1}{2}} V(x) \right) dx

\[\quad - L'(1) \int_{\mathbb{R}} \sqrt{\mu + |k|} \log \left( \frac{2\sqrt{\mu} + |k|}{2\sqrt{\mu} - |k|} \right) |\hat{V}(k)|^2 dk\]

with

\[L(1) = \frac{2}{3} \left( \frac{1}{3K(1)} \right)^{1/2} \geq L_{\text{sc}}(1) \quad \text{and} \quad L'(1) = \frac{1}{4K'(1)}.\]

We will see in Section 2.3 below that it is not possible to take $L'(1) = 0$.

When $\mu \to 0$ the inequalities (2.24) and (2.28) reduce again to the usual Lieb-Thirring inequality [26, 27] which is the dual version of (2.12):

\[(2.29) \quad 0 \leq \text{Tr}(-\Delta + V) - L(d) \int_{\mathbb{R}^d} V(x)^{1 + \frac{d}{2}} dx.\]
Remark 2.4. In general \( Q_V \) is not a compact operator. Indeed, it was shown by Pushnitski \cite{Pushnitski2011} (see also \cite{Shubin1997}) that the essential spectrum of \( Q_V \) is
\[
\sigma_{\text{ess}}(Q_V) = \left[ -\frac{|S(\mu) - 1|}{2}, \frac{|S(\mu) - 1|}{2} \right]
\]
where \( S(\mu) \) is the scattering matrix associated to the pair \((-\Delta, -\Delta + V)\). Hence \( Q_V \) is not compact, unless \( S(\mu) = 1 \). Similarly, one does not expect, in general, that \((-\Delta - \mu)Q_V\) and \((-\Delta - \mu + V)Q_V\) are trace-class, rendering Definitions \ref{def:1} and \ref{def:2} necessary.

Remark 2.5 (Relation with the spectral shift function). The spectral shift function \( \zeta_V(\lambda) \) formally satisfies \cite{Faddeev1964}
\[
\int_{-\infty}^{\infty} \zeta_V(\lambda) \, d\lambda = -\text{Tr}_V(-\Delta - \mu + V)Q_V - \rho_0 \int_{\mathbb{R}^d} V.
\]
If \( V \) is in \( L^1(\mathbb{R}^d) \) and satisfies the assumptions \ref{assumption1} or \ref{assumption2}, it is possible to define \( \zeta_V \) as the (distributional) derivative of the right side with respect to \( \mu \).

2.3. Second-order perturbation theory and the 1D case. In this section we compute the variation of energy when a potential \( tV \) is inserted in the system, to second-order in \( t \). In particular we will show that in the one-dimensional case \( d = 1 \), the constant \( L'(1) \) in the lower bound \ref{lower_bound} cannot be taken equal to 0.

The following result, whose proof is sketched in Section \ref{sketch}, is well known in the physics literature \cite{Stefanov2002}.

Theorem 2.5 (Second-order perturbation theory). Assume \( d \geq 1 \). Let \( V \) be a real-valued function in \( L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \) and \( \mu > 0 \). Then, using the notation of the previous section,
\[
\lim_{t \to 0} \frac{\text{Tr}_V(-\Delta - \mu + tV)Q_V}{t^2} = -\mu \frac{d-1}{2} \int_{\mathbb{R}^d} \Psi_d\left(\sqrt{\mu}\right) \left| \hat{V}(k) \right|^2 \, dk,
\]
where
\[
\Psi_d(|k|) = (2\pi)^{-d} \int_{|p-k|^2 \geq 1} \frac{dp}{|p-k|^2 - |p|^2},
\]
(2.31)
\[
= \begin{cases} 
\frac{1}{4\pi|k|} \log \left( \frac{2 + |k|}{|2 - |k||} \right) & \text{if } d = 1, \\
\frac{|S^{d-2}|}{2^d (2\pi)^d} \int_0^1 \log \left( \frac{2\sqrt{1-r^2} + |k|}{2\sqrt{1-r^2} - |k|} \right) r^{d-2} \, dr & \text{if } d \geq 2.
\end{cases}
\]
(2.32)
In particular, when \( d = 1 \) the constant \( L'(1) \) appearing in \ref{lower_bound} must satisfy \( L'(1) \geq q/(12\pi) \).

Our proof is valid under much weaker assumptions on the potential \( V \), but we have not tried to optimize this. The divergence at \( |k| = 2\sqrt{\mu} \) of \( \Psi_1(\cdot/\sqrt{\mu}) \) in dimension \( d = 1 \) is well-known, and it is sometimes called the Peierls instability \cite{Stefanov2002} Sec. 4.3. When the interactions among the particles are turned on, the system becomes unstable because of the large number of possible electron-hole excitations between the two points \( \pm 2\sqrt{\mu} \). A macroscopic deformation of the system can sometimes lead to the opening of a gap at the Fermi points \cite{Stefanov2002, Lieb1985, Lieb1986, Lieb1987, Lieb1990, Lieb1991, Lieb1992}. In higher dimensions, the second-order response function \( \Psi_d \) is bounded (this also
follows from our bound (2.24), but it is seen to have an infinite derivative at
\(|k| = 2\sqrt{\mu}\), a fact sometimes referred to as Migdal-Kohn anomaly \[28, 16\].

We note that the semi-classical approximation to the left side of (2.31) satisfies
\[
(2.33) \quad \lim_{t \to 0} \frac{-(tV - \mu)^{1+\frac{d}{2}} - \mu^{1+\frac{d}{2}} + \frac{2 + d}{2} \mu^{\frac{d}{2}} tV}{t^2} = -\frac{d(d+2)}{8} \mu^{\frac{d}{2}-1} V^2.
\]
This proves that for \(d = 1\), it is not possible to take \(L'(1) = 0\) in (2.28), since
the response function diverges at the Fermi points \(k = \pm 2\sqrt{\mu}\) whereas the semi-
classical second-order term stays finite. A closer inspection of the constants reveals
that \(L'(1) \geq q/(12\pi)\) must hold, as stated in Theorem 2.5.

It is possible to calculate \(\Psi_2\) and \(\Psi_3\) exactly:
\[
(2.34) \quad \Psi_2(|k|) = \frac{1}{8\pi} - \frac{1}{8\pi} \sqrt{\left(1 - \frac{4}{|k|^2}\right)}
\]
\[
(2.35) \quad \Psi_3(|k|) = \frac{1}{16\pi^2} \left(1 + \frac{1}{|k|} \left(1 - \frac{|k|^2}{4}\right) \log \left(\frac{2 + |k|}{|2 - |k||}\right)\right).
\]
Furthermore, we have the following recursion relation
\[
(2.36) \quad \Psi_d(|k|) = \frac{|S|^{d-3}}{(2\pi)^{d-2}} \int_0^1 \frac{r^{d-3}}{\sqrt{1 - r^2}} \Psi_2 \left(\frac{|k|}{\sqrt{1 - r^2}}\right) \, dr, \quad \text{for } d \geq 3,
\]
which implies that \(\Psi_d\) is strictly decreasing for all \(d \geq 3\) (whereas for \(d = 2\), \(\Psi_2\) is
constant on \([0, 2]\) and strictly decreasing on \([2, \infty)\)). We deduce that
\[
\|\Psi_d\|_{L^\infty(\mathbb{R}^+)} = \Psi_d(0) = \frac{|S|^{d-2}}{2(2\pi)^{d}} \int_0^1 \frac{r^{d-2}}{\sqrt{1 - r^2}} \, dr = \frac{d(d+2)}{8q} L_{sc}(d), \quad \text{for } d \geq 2.
\]
Observe that in dimensions \(d \geq 2\), perturbation theory predicts the same value for
the constant \(L(d)\) as semi-classics does. This is not so surprising since the largest
constant is obtained if \(V\) is supported close to 0, hence \(V\) is very spread out in
\(x\)-space, which puts us in the semi-classical regime.

**Remark 2.6.** As is detailed in Section 4.3 below, the second-order perturbation
of the energy arises from the first-order term in the expansion of \(Q_{1V}\). This term
is purely off-diagonal (the corresponding \((Q_{1V})_{\pm\pm}\) vanish to first order in \(t\)). This
emphasizes the fact that the absence of a Lieb-Thirring inequality in 1D is due to a
possible divergence of the off-diagonal densities \(\rho_{Q_{\pm\pm}}\) in Fourier space at \(|k| = 2\sqrt{\mu}\).

The corresponding first-order density is proportional to \(\Psi_{d+1} \star V\). For potentials \(V\)
whose Fourier transform does not vanish at the Fermi surface, this density decays
slowly in \(x\)-space, due to the lack of regularity of \(\Psi_d\) at \(|k| = 2\sqrt{\mu}\).

### 2.4 A sharp inequality.

We state and prove in this section a lower bound on the relative kinetic energy needed to banish all the particles in a domain, \(\Omega\), from the Fermi gas. This inequality involves the sharp constant \(K_{sc}(d)\) and it is the positive density analogue of a result due to Li and Yau \[19\].

**Theorem 2.6 (A sharp estimate for the energy shift).** Assume \(d \geq 1\). Let \(\Omega\) be
an open subset of \(\mathbb{R}^d\) of finite measure. Let \(\mu \geq 0\) and denote, as before, \(\Pi^- :=

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1(−Δ ≤ μ). For any fermionic density matrix such that

\[ 0 ≤ γ ≤ 1_{\mathbb{R}^d \setminus \Omega}, \quad \text{i.e.,} \quad 0 ≤ \langle f, γf \rangle_{L^2(\mathbb{R}^d)} ≤ \int_{\mathbb{R}^d \setminus \Omega} |f(x)|^2 dx, \quad \forall f ∈ L^2(\mathbb{R}^d), \]

and such that \(|−Δ − μ|^{1/2}Q^{±}| − Δ − μ|^{1/2}\) are trace-class, with \(Q := γ − Π^−\), we have

\[ \text{Tr}_0(−Δ − μ)Q ≥ \frac{2}{d} K_{sc}(d) ρ_0^{1 + \frac{d}{2}} |\Omega|. \]

The constant in this inequality is best possible. In dimension \(d ≥ 2\), applying Theorem 2.1 and using that \(ρ_γ(x) = 0\) on \(Ω\), we get

\[ \text{Tr}_0(−Δ − μ)Q ≥ K(d) ρ_0^{1 + d/2} |\Omega| \]

+ \(K(d) \int_{\mathbb{R}^d \setminus \Omega} \left( ρ_γ^{1 + d/2} − ρ_0^{1 + d/2} − \frac{2 + d}{d} ρ_0^{d/2} (ρ_γ − ρ_0) \right) dx. \]

Here \(K(d)\) is not optimal but, on the other hand, the bound also quantifies the fact that \(ρ_γ\) cannot be equal to \(ρ_0\) close to the boundary because of the Dirichlet conditions.

Proof of Theorem 2.6. We have

\[ Q^{−−} = Π^−(γ − Π^−)Π^− ≤ Π^− 1_{\mathbb{R}^d \setminus \Omega} Π^− − Π^− = −Π^− 1_Ω Π^−. \]

Using that \(Q^{++} ≥ 0\), we get

\[ \text{Tr}_0(−Δ − μ)Q = \text{Tr} |−Δ − μ|^{1/2}(Q^{++} − Q^{−−}) − Δ − μ|^{1/2} \]

\[ ≥ \text{Tr}(−Δ − μ)^{1/2} 1_Ω (−Δ − μ)^{1/2} \]

\[ = (2π)^{−d} |\Omega| \int_{\mathbb{R}^d} (|p|^2 − μ)_{−} dp. \]

Recalling the definition of \(K_{sc}(d)\), we obtain the claim. \(\square\)

3. Kinetic energy inequalities: Proof of Theorems 2.1 and 2.2

3.1. Preliminaries. In this section we state and prove some preliminary results that will be useful in the proof of our main theorems.

Throughout the paper we denote by \(\mathcal{K} = \mathcal{S}_∞\) (resp. \(\mathcal{B}\)) the algebra of compact (resp. bounded) operators on \(L^2(\mathbb{R}^d, \mathbb{C}^N)\). The usual norm of bounded operators is simply denoted by \(|\cdot|\). We also denote by \(\mathcal{S}_p\) (for \(1 ≤ p < ∞\)) the ideal of compact operators \(A\) on \(L^2(\mathbb{R}^d, \mathbb{C}^N)\) such that \(\text{Tr} |A|^p < ∞\), endowed with its norm \(|A|_{\mathcal{S}_p} = (\text{Tr} |A|^p)^{1/p}\).

In order to simplify the statements below, we introduce the following Banach space

\[ \mathcal{X} := \{ Q = Q^* ∈ \mathcal{B} : Q| − Δ − μ|^{1/2} ∈ \mathcal{S}_2, \]

\[ |−Δ − μ|^{1/2} Q^{±±} | − Δ − μ|^{1/2} ∈ \mathcal{S}_1 \}, \]

\[ \mathcal{S}_1 \]
endowed with its natural norm
\begin{equation}
\|Q\|_X := \|Q\| + \|Q| - \Delta - \mu|^{1/2}\|_{\mathcal{E}_2} + \|\Delta - \mu|^{1/2}Q^{++}\| - \Delta - \mu|^{1/2}\|_{\mathcal{E}_1} + \|\Delta - \mu|^{1/2}Q^{--}\| - \Delta - \mu|^{1/2}\|_{\mathcal{E}_1}.
\end{equation}

For the sake of simplicity, we do not emphasize the dependence in \(\mu\) in our notation. The space \(\mathcal{X}\) has a natural weak topology which is the intersection of the ones associated with the spaces appearing in the definition (3.2) of \(\mathcal{X}\). Here \(Q_n \to Q\) in \(\mathcal{X}\) means \(Q_n \to Q\) weakly-\(\ast\) in \(\mathcal{B}\), \(Q_n\) is weakly \(\ast\)-convergent in \(\mathcal{E}_2\) and \(|-\Delta - \mu|^{1/2} Q_n^{\pm}\| - \Delta - \mu|^{1/2}\) weakly in \(\mathcal{E}_1\). The unit ball of \(\mathcal{X}\) is weakly compact for this topology, by the Banach-Alaoglu theorem. The following convex subset of \(\mathcal{X}\) will play an important role:
\begin{equation}
\mathcal{K} := \{Q \in \mathcal{X} : -\Pi^- \leq Q \leq \Pi^+\}.
\end{equation}

Our first result deals with the continuity of the map \(Q \in \mathcal{X} \mapsto \rho_Q\) in \(L^1_{\text{loc}}(\mathbb{R}^d)\).

**Lemma 3.1** (Operators in \(\mathcal{X}\) are locally trace-class). We assume that \(\mu \geq 0\) and \(d \geq 1\). Let \(Q\) be a self-adjoint bounded operator in \(\mathcal{X}\). Then, for every bounded function \(\eta\) of compact support, there exists a constant \(C_\eta\) such that
\begin{equation}
\eta Q \eta \|_{\mathcal{E}_1} \leq C_\eta \|Q\|_X.
\end{equation}
Hence \(Q\) is locally trace class and \(\rho_Q\) is well-defined in \(L^1_{\text{loc}}(\mathbb{R}^d)\).

Furthermore, the map \(Q \in \mathcal{X} \mapsto \eta Q \eta \in \mathcal{S}_1\) is weakly continuous: If we have a sequence \(\{Q_n\}\) such that \(Q_n \to Q\) weakly in \(\mathcal{X}\), then \(\eta Q_n \eta \to \eta Q \eta\) strongly in \(\mathcal{S}_1\). In particular, \(\rho_{Q_n} \to \rho_Q\) strongly in \(L^1_{\text{loc}}(\mathbb{R}^d)\).

**Proof.** We consider the spectral projection \(\Pi_1 := \mathbb{1}( -\Delta \leq \max(1, 2\mu) )\), which localizes in a ball containing strictly the Fermi surface, and we denote by \(\Pi_2 = 1 - \Pi_1\) its complement. Then we write \(Q = \sum_{k,\ell = 1, 2} \Pi_k Q \Pi_\ell\) and estimate each term separately. We start with \(\Pi_2 Q \Pi_2\) which we treat as follows
\[\eta \Pi_2 Q \Pi_2 \eta = \eta \frac{\Pi_2}{-\Delta - \mu} \eta \frac{\Pi_2}{-\Delta - \mu} \frac{\Pi_2}{\Delta - \mu} \eta \frac{\Pi_2}{\Delta - \mu} \eta\]
where we have used that \(\Pi_2 = \Pi_2 \Pi^+\). Since \(\eta\) and \(\Pi_2\) are bounded, it is clear that the previous operator is trace-class. Furthermore, we know that if \(T_n \to T\) weakly-\(\ast\) in \(\mathcal{S}_1\) and \(K\) is compact, then \(KT_n K \to KT K\) strongly in \(\mathcal{S}_1\). Hence the weak continuity follows from the fact that \(\eta \Pi_2| -\Delta - \mu|^{-1/2}\) is compact. For \(\Pi_1 Q \Pi_2\), we write similarly
\[\eta \Pi_1 Q \Pi_2 \eta = \eta \Pi_1 \frac{\Pi_2}{-\Delta - \mu} \eta \frac{\Pi_2}{-\Delta - \mu} \frac{\Pi_2}{\Delta - \mu} \eta \frac{\Pi_2}{\Delta - \mu} \eta\]
and use that \(\eta \Pi_1 \in \mathcal{E}_2, Q| -\Delta - \mu|^{-1/2} \in \mathcal{E}_2\) and \(\Pi_2| -\Delta - \mu|^{-1/2}\eta \in \mathcal{K}\). The argument is then similar as before. Finally, for \(\Pi_1 Q \Pi_1\), we simply use that \(\eta \Pi_1 \in \mathcal{E}_2\) and that \(Q\) is bounded. The rest follows.

**Remark 3.1.** The previous proof does not use the fact that \(|-\Delta - \mu|^{1/2}Q^{--}| -\Delta - \mu|^{1/2}\) is trace-class.

The following says that finite rank operators are dense in \(\mathcal{X}\) in the appropriate sense.
Lemma 3.2 (Density of finite rank operators). For every $Q \in X$, there exists a sequence $Q_n \in X$ of finite rank operators, such that $(-\Delta)Q_n \in \mathcal{B}$ and

- $Q_n \to Q$ strongly (that is, $Q_nf \to Qf$ strongly in $L^2(\mathbb{R}^d, \mathbb{C}^n)$ for every fixed $f \in L^2(\mathbb{R}^d, \mathbb{C}^n)$);
- $\lim_{n \to \infty} \left\| (Q_n - Q) - \Delta - \mu \right\|^{1/2}_{\mathcal{S}_2} = 0$;
- $\lim_{n \to \infty} \left\| -\Delta - \mu \right\|^{1/2}(Q_n - Q)_{\pm \pm} - \Delta - \mu \right\|^{1/2}_{\mathcal{S}_1} = 0$;
- $\rho_{Q_n} \to \rho_Q$ strongly in $L^1_{\text{loc}}(\mathbb{R}^d)$.

Furthermore, if $Q$ belongs to the convex set $\mathcal{K}$ defined in (3.3), then $Q_n$ can be chosen in $\mathcal{K}$ for all $n$.

Note that operators $Q \in X$ are not all compact, hence in general $|Q_n - Q| \not\to 0$.

Proof. We start by approximating $Q$ by a sequence of Hilbert-Schmidt operators $Q_n$, with $(Q_n)_{\pm \pm} \in \mathcal{S}_1$. Let us define the orthogonal projection $P_n := 1(1/n \leq |\Delta - \mu| \leq n)$, which localizes in momentum space away from the Fermi surface and from infinity. We now define $Q_n := P_nQP_n$. It is easy to verify that $Q_n$ is a Hilbert-Schmidt operator by choice of $P_n$ and, similarly that $(Q_n)_{\pm \pm}$ are trace-class. We have $P_n \to 1$ strongly in $L^2(\mathbb{R}^d)$. Since $Q$ is bounded, we obtain that $Q_n \to Q$ strongly. Also, it is well-known that when $A \in \mathcal{S}_p$ for some $1 \leq p \leq \infty$, then $P_nAP_n \to A$ strongly in $\mathcal{S}_p$. In particular, we have that $P_nQ|\Delta - \mu|^{1/2}P_n = Q_n|\Delta - \mu|^{1/2} \to Q|\Delta - \mu|^{1/2}$ strongly in $\mathcal{S}_2$, using that $P_n$ commutes with $|\Delta - \mu|^{1/2}$. The convergence of the trace-class terms is similar, and the strong convergence of $\rho_{Q_n}$ in $L^1_{\text{loc}}(\mathbb{R}^d)$ follows from Lemma 3.1. Finally, we note that, since $P_n$ commutes with $\Pi^-$, $Q_n$ belongs to $\mathcal{K}$ for all $n$, whenever $Q$ is itself in $\mathcal{K}$.

For a proof that $Q_n$ can itself be approximated by smooth finite rank operators in $\mathcal{K}$, see [10, Theorem 6].

3.2. Proof of Theorem 2.1. kinetic Lieb-Thirring inequality for $d \geq 2$. In this section we prove Theorem 2.1 for $\mu > 0$ (the case $\mu = 0$ is well-known [20, 24, 25]). Replacing $Q$ by $U_nQU_n^*$ where $(U_n \mu f)(x) := \mu^{d/4}f(\sqrt{\mu}x)$, it is easy to verify that (2.9) follows from the case $\mu = 1$, which we will assume throughout the proof. Also we assume for simplicity that the number of spin states is $q = 1$ but the proof for the general case is identical. Finally, since the semi-classical energy difference $\delta T^c$ (defined in (2.10)) is non-negative, the right side of our Lieb-Thirring inequality (2.9) is lower semi-continuous with respect to $\rho_Q$. This shows that, by Lemma 3.2, we can prove (2.9) assuming that $Q$ is a smooth-enough finite rank operator, and deduce the general case by density.

Recall our notation $Q^{-} = \Pi^-Q\Pi^-$, $Q^{++} = \Pi^+Q\Pi^+$ and so on. We will estimate the density arising from each term separately. The constraint $-\Pi^- \leq Q \leq \Pi^+$ is equivalent to $Q^2 \leq Q^{++} - Q^{-}$.

Step 1. Estimate on $Q^{\pm \pm}$. In order to bound the density arising from the diagonal terms, we will use the following generalization of the Lieb-Thirring inequality.

Lemma 3.3 (Lieb-Thirring inequality with positive Fermi level). Assume $d \geq 1$. Let $0 \leq \gamma \leq 1$ be a self-adjoint operator on $L^2(\mathbb{R}^d)$ such that $|\Delta + 1|^{1/2}\gamma|\Delta + 1|^{1/2}$
is trace-class. Then $\gamma$ is locally trace-class and its density satisfies
\begin{equation}
(3.5) \quad \text{Tr} |\Delta + 1|^{1/2} |\Delta + 1|^{1/2} \geq \hat{K}(d) \int_{\mathbb{R}^d} \delta T^w_1(\rho_\gamma(x)) \, dx
\end{equation}
where
\begin{equation*}
\delta T^w_1(\rho) = (\rho_0 + \rho)^{1+\frac{d}{2}} - (\rho_0)^{1+\frac{d}{2}} - \frac{2 + d}{d} (\rho_0)^{\frac{d}{2}} \rho
\end{equation*}
with $\rho_0 = |S^{d-1}|(2\pi)^d/d$, and where $\hat{K}(d)$ is a positive constant depending only on $d$.

The proof of Lemma 3.3 follows ideas of Rumin [35]. Note that Lemma 3.3 is also valid in dimension $d = 1$.

Proof. We follow a recent method of Rumin [35]. We introduce the spectral projection $P_e := 1(\{\Delta + 1 \geq e\})$ in such a way that we have the layer cake representation
\begin{equation*}
|\Delta + 1| = \int_0^\infty P_e \, de.
\end{equation*}
Let now $0 \leq \gamma \leq 1$ be a smooth-enough finite rank operator. We have
\begin{equation}
(3.6) \quad \text{Tr} |\Delta + 1| \gamma = \int_0^\infty de \, \text{Tr}(P_e \gamma P_e) = \int_0^\infty de \int_{\mathbb{R}^d} \rho_e(x) \, dx
\end{equation}
where $\rho_e$ is the density of the finite-rank operator $P_e \gamma P_e$. We now consider a bounded set $A \subseteq \mathbb{R}^d$ and estimate
\begin{equation*}
\int_A \rho_e(x) \, dx = \text{Tr}(1_A P_e \gamma P_e) = \left\| 1_A P_e \gamma^{1/2} \right\|_{\mathcal{E}_2}^2
\geq \left( \left\| 1_A \gamma^{1/2} \right\|_{\mathcal{E}_2} - \left\| 1_A P_e^\perp \gamma^{1/2} \right\|_{\mathcal{E}_2} \right)^2
= \left( \int_A \rho \right)^{1/2} - \left\| 1_A P_e^\perp \gamma^{1/2} \right\|_{\mathcal{E}_2}^2.
\end{equation*}
(3.7)
Note that, since $\Vert \gamma \Vert \leq 1$,
\begin{equation*}
\left\| 1_A P_e^\perp \gamma^{1/2} \right\|_{\mathcal{E}_2}^2 = \text{Tr}(1_A P_e^\perp \gamma P_e^\perp 1_A) \leq \left\| 1_A P_e^\perp \right\|_{\mathcal{E}_2}^2 \Vert \gamma \Vert \leq |A| \, f(e)
\end{equation*}
where
\begin{equation*}
f(e) := (2\pi)^{-d} \left\{ \left\{ |p^2 - 1| \leq e \right\} \right\} = \frac{|S^{d-1}|}{d(2\pi)^d} \left( (1 + e)^{d/2} - (1 - e)^{d/2} \right).
\end{equation*}
Taking $A$ to be a ball of radius $\varepsilon \to 0$ centered at $x$, we obtain from (3.7) the pointwise estimate
\begin{equation*}
\rho_e(x) \geq \left( \sqrt{\rho(x)} - \sqrt{f(e)} \right)_+^2.
\end{equation*}
We may now insert this in (3.6) and obtain
\begin{equation*}
\text{Tr} |\Delta + 1| \gamma \geq \int_{\mathbb{R}^d} dx \int_0^\infty de \left( \sqrt{\rho(x)} - \sqrt{f(e)} \right)_+^2 = \int_{\mathbb{R}^d} R_d(\rho(x)) \, dx
\end{equation*}
with
\begin{equation}
R_d(\rho) := \int_0^\infty \left( \sqrt{\rho} - \sqrt{f(e)} \right)_+^2 de.
\end{equation}
At zero we have
\[ R_d(\rho) \sim_{\rho \to 0} \frac{(2\pi)^d}{6|S^{d-1}|} \rho^2. \]

At infinity, one can compute that
\[ R_d(\rho) \sim_{\rho \to \infty} \frac{d}{d+4} K_{sc}(d) \rho^{1+2/d}. \]

Hence there is a constant \( \tilde{K}(d) \) such that (3.5) holds.

We have written the proof for a smooth enough finite-rank operator. The general case follows from an approximation argument based on Lemma 3.2. This completes the proof of Lemma 3.3. \( \square \)

Since \(|\Delta + 1|^{1/2}Q^{\pm \pm}|\Delta + 1|^{1/2}\) is trace-class by assumption and \(0 \leq Q^{++} \leq \Pi^+ \leq 1, -1 \leq -\Pi \leq Q^- \leq 0\), we immediately obtain from Lemma 3.3 that
\[ \pm \text{Tr} |\Delta + 1|^{1/2}Q^{\pm \pm}|\Delta + 1|^{1/2} \geq \tilde{K}(d) \int_{\mathbb{R}^d} \delta T_{sc}(\rho_{Q^{\pm \pm}}(x)) \, dx. \]

It therefore remains to estimate the density arising from the off-diagonal terms \( Q^{+ -} \) and \( Q^{- +} \).

**Step 2.** Estimate on \( Q^{\pm \pm} \). It is enough to consider \( Q^{+ +} = \Pi^- Q^+ \), since \( \rho_{Q^{++}} + \rho_{Q^{+-}} = 2R \rho_{Q^{++}} \). In order to estimate the density \( \rho_{Q^{++}} \) in the whole space \( \mathbb{R}^d \), we argue by duality and write
\[ \int_{\mathbb{R}^d} V \rho_{Q^{++}} = \text{Tr}(V \Pi^- Q^+) = \text{Tr} \left( \frac{\Pi^+}{|\Delta + 1|^{1/4}} V \frac{\Pi^-}{|\Delta + 1|^{1/4}} |\Delta + 1|^{1/4} Q |\Delta + 1|^{1/4} \right). \]

This calculation is valid if \( V \) is bounded and compactly supported, since \( Q \) is a smooth-enough finite-rank operator. Using Schwarz’s inequality and that \( Q^2 \leq Q^{++} - Q^- \), we have
\[ \| |\Delta + 1|^{1/4} Q |\Delta + 1|^{1/4} \|_{\mathcal{E}_2} \leq \| |\Delta + 1|^{1/2} Q \|_{\mathcal{E}_2} = (\text{Tr} |\Delta + 1| Q^2)^{1/2} \leq (\text{Tr}(|\Delta - 1| Q)^{1/2}. \]

Returning to (3.10), we obtain
\[ \left| \int_{\mathbb{R}^d} V \rho_{Q^{++}} \right| \leq \left\| \frac{\Pi^+}{|\Delta + 1|^{1/4}} V \frac{\Pi^-}{|\Delta + 1|^{1/4}} \right\|_{\mathcal{E}_2} (\text{Tr}(|\Delta - 1| Q)^{1/2}. \]

We now compute
\[ \left\| \frac{\Pi^+}{|\Delta + 1|^{1/4}} V \frac{\Pi^-}{|\Delta + 1|^{1/4}} \right\|_{\mathcal{E}_2}^2 = (2\pi)^{-d} \int_{|p|^2 \leq 1} \int_{|q|^2 \geq 1} \frac{|\tilde{V}(p-q)|^2}{(1 - |p|^2)^{1/2} (|q|^2 - 1)^{1/2}} \, dp \, dq \]
\[ = (2\pi)^{-d} \int_{\mathbb{R}^d} |\tilde{V}(k)|^2 \Phi_d(|k|) \, dk \]
where
\[ \Phi_d(|k|) := \int_{|p| \leq 1} \int_{|p-k| \geq 1} \frac{dp}{(1 - |p|^2)^{1/2} (|p-k|^2 - 1)^{1/2}}. \]

We will use the following fundamental result.
Lemma 3.4. For \( d \geq 2 \), the function \( \Phi_d \) is bounded on \( \mathbb{R}^d \). The function \( \Phi_1 \) is not bounded in a neighborhood of \( k = 2 \).

For clarity the proof of Lemma 3.4 is postponed until the end of the proof of Theorem 2.1. We deduce that

\[
\| \Pi^+ |\Delta + 1|^{1/4} V \Pi^- |\Delta + 1|^{1/4} \|_{\mathfrak{S}_2} \leq (2\pi)^{-d/2} \| \Phi_d \|_{L^\infty(\mathbb{R}^+)}^{1/2} \| V \|_{L^2(\mathbb{R}^d)},
\]

which leads to the estimate

\[
(3.13) \quad \int_{\mathbb{R}^d} |\rho_{Q-}|^2 \leq (2\pi)^{-d} \| \Phi_d \|_{L^\infty(\mathbb{R}^+)} \Tr_0(-\Delta - 1)Q.
\]

We can extend \( \delta T_1^{sc} \) for \( \rho \leq -\rho_0 \) linearly as follows

\[
\delta T_1^{sc}(\rho) = (\rho_0 + \rho)^{\frac{1}{2}} \Delta^\frac{3}{2} - (\rho_0)^{\frac{1}{2}} \frac{2 + d}{d} (\rho_0)^{\frac{2}{d}} \rho.
\]

The function is now convex on the whole line \( \mathbb{R} \). Note that for \( d \geq 2 \), we have \( |\rho|^2 \geq c \delta T_1^{sc}(\rho) \) for all \( \rho \), hence we have also shown that

\[
(3.14) \quad c \int_{\mathbb{R}^d} \delta T_1^{sc}(\rho_{Q-}) \leq \Tr_0(-\Delta - 1)Q
\]

for a small enough constant \( c > 0 \).

**Remark 3.2.** Modifying the previous proof by using \( \Pi^+ |\Delta + 1|^{-\alpha} V \Pi^- |\Delta + 1|^{-\alpha} \) with an appropriate power \( \alpha \) and \( \mathfrak{S}_p \) norms, one can show that

\[
(3.15) \quad \int_{\mathbb{R}^d} |\rho_{Q-}|^p + \int_{\mathbb{R}^d} |\rho_{Q-}|^p \leq C(d, p) \Tr_0(-\Delta - 1)Q
\]

holds for all \( 2 \leq p < \infty \) and all \( d \geq 2 \).

**Conclusion.** Putting (3.9) and (3.14) together, we deduce by convexity of \( \delta T_1^{sc} \) that

\[
\Tr_0(-\Delta - 1)Q \geq c \int_{\mathbb{R}^d} \delta T_1^{sc}(\rho_{Q+}) + \delta T_1^{sc}(\rho_{Q-}) + \delta T_1^{sc}(\rho_{Q+}) + \delta T_1^{sc}(\rho_{Q-})
\]

\[
\geq 4c \int_{\mathbb{R}^d} \delta T_1^{sc}\left(\frac{\rho_{Q+} + \rho_{Q-} + \rho_{Q-} + \rho_{Q+}}{4}\right)
\]

\[
\geq K(d) \int_{\mathbb{R}^d} \delta T_1^{sc}(\rho_{Q})
\]

for a small enough constant \( K(d) > 0 \). This completes the proof of Theorem 2.1. \( \Box \)

**Remark 3.3.** Our method yields explicit values for the constant \( K(d) \) appearing in the statement of Theorem 2.1, see [7].

It remains to prove Lemma 3.4.

**Proof of Lemma 3.4.** To study \( \Phi_d(k) \) for \( d \geq 2 \), we make the decomposition \( p = (s, p_\perp) \) with \( s = p \cdot k \) and find

\[
\Phi_d(k) = \int_{s^2 + |p_\perp|^2 \leq 1} ds dp_\perp \int_{s-k)^2 + |p_\perp|^2 \geq 1} \frac{ds dp_\perp}{(1 - s^2 - |p_{\perp}|^2)^{1/2} ((s-k)^2 + |p_{\perp}|^2 - 1)^{1/2}}
\]

\[
= |S^{d-2}| \int_{s^2 + r^2 \leq 1} ds dr \int_{s-k)^2 + r^2 \geq 1} \frac{ds}{(1 - s^2 - r^2)^{1/2} ((s-k)^2 + r^2 - 1)^{1/2}}.
\]

(3.16)
For \( k \geq 2 \) and \( |s| \leq 1 \), it is clear that \((s - k)^2 + r^2 \geq 1\). The integration domain is therefore independent of \( k \) when \( k \geq 2 \). It is then easy to verify that \( \Phi_d \) is decreasing and continuous on \( (2, \infty) \). Hence we only have to prove that it is bounded in a neighborhood of \([0, 2]\). Next we note that

\[
\Phi_d(k) = |S^d - 2| \int_0^1 \frac{r^{d-2}}{\sqrt{1 - r^2}} \Phi_1 \left( \frac{k}{\sqrt{1 - r^2}} \right) \, dr
\]

where we recall that

\[
\Phi_1(x) = \int_{v^2 \leq 1} \frac{dv}{\sqrt{1 - v^2} (v - x)^2 - 1} = \int_{v^2 = 1} \frac{dv}{\sqrt{1 - v^2} (v - x)^2 - 1}.
\]

It is an exercise to verify that \( \Phi_1 \) is a continuous function on \( \mathbb{R}^+ \setminus \{2\} \) (in particular it has a finite limit at \( x = 0 \), and that

\[
\Phi_1(x) \sim \frac{1}{2} \log |x - 2|, \quad \Phi_1(x) \sim \frac{\pi}{x}.
\]

Using that, for instance,

\[
\Phi_1(x) \leq \frac{C}{|x - 2|^2},
\]

and letting \( u = \sqrt{1 - r^2} \), we obtain

\[
\Phi_d(k) \leq C \int_0^1 \frac{du}{\sqrt{1 - u^2} |k - 2u|^2} \leq C \left( \int_0^1 \frac{du}{(1 - u)^{3/2}} \right)^{\frac{2}{3}} \left( \int_0^1 \frac{du}{|k - 2u|^2} \right)^{\frac{1}{2}}
\]

by Hölder’s inequality. The right side is bounded with respect to \( k \), hence \( \Phi_d \) is uniformly bounded for \( d \geq 2 \). By a similar proof one can verify that \( \Phi_d \) is also a continuous function on \( \mathbb{R}^+ \). This completes the proof of Lemma 3.4. \( \square \)

**Remark 3.4.** It is possible to calculate the exact maximum value of \( \Phi_d \), which might be interesting for physical applications \([2]\). Starting from (3.16) and letting \( t = r^2 \), we obtain

\[
\Phi_3(k) = \pi \int_{s^2 + t \leq 1} ds \, dt \frac{1}{(1 - s^2 - t)^{1/2} ((s - k)^2 + t - 1)^{1/2}}
\]

\[
= \pi \int_{\min(1, k/2)}^{\min(1, -1 + k)} ds \int_0^{1 - s^2} dt \frac{1}{(1 - s^2 - t)^{1/2} ((s - k)^2 + t - 1)}
\]

\[
+ \pi \int_{-1}^{\min(1, -1 + k)} ds \int_0^{1 - s^2} dt \frac{1}{(1 - s^2 - t)^{1/2} ((s - k)^2 + t - 1)}.
\]

In order to compute these integrals we use the fact that

\[
\int \frac{dt}{\sqrt{(a - t)(t - b)}} = -2 \arcsin \sqrt{\frac{a - t}{a - b}}
\]

whenever \( a > b \). We find for \( 0 \leq k \leq 2 \), with \( s = -1 + ku \),

\[
\Phi_3(k) = \pi^2 + 2 \pi k \left( \int_0^1 \arcsin \sqrt{\frac{2 - ku}{2 + k(1 - 2u)}} \, du - \frac{\pi}{4} \right).
\]
We have
\[
\frac{d}{dk} \left( \frac{2 - ku}{2 + k(1 - 2u)} \right) = \frac{2(u - 1)}{(2 + k(1 - 2u))^2} \leq 0
\]

hence the function
\[
f(k) := \int_0^1 \arcsin \sqrt{\frac{2 - ku}{2 + k(1 - 2u)}} \, du - \frac{\pi}{4}
\]

appearing in the parenthesis in (3.19) is decreasing with respect to \(k\), by monotonicity of \(t \mapsto \arcsin(\sqrt{t})\). Its value at \(k = 0\) is
\[
f(0) = \int_0^1 \arcsin \sqrt{u} \, du - \frac{\pi}{4} = 0.
\]

Therefore, \(f(k) \leq 0\) for \(0 < k \leq 2\). Now \(\Phi'_3(k) = 2\pi (f(k) + k f'(k)) \leq 0\), hence \(\Phi_3\) is decreasing on \([0, 2]\). Since we know already that \(\Phi_3\) also decreases on \([2, \infty)\), we conclude that
\[
\max_{\mathbb{R}^+} \Phi_3 = \Phi_3(0) = \pi^2.
\]

Similarly as in (3.17), we can express \(\Phi_d\) in terms of \(\Phi_3\) for \(d \geq 4\) by assuming, for instance, \(\mathbf{k} = k(1, 0, ..., 0)\) and writing \(p = (q, p_\perp)\) with \(q \in \mathbb{R}^3\) and \(p_\perp \in \mathbb{R}^{d-3}\). We obtain the recursion relation
\[
(3.20) \quad \Phi_d(k) = |S^{d-4}| \int_0^1 \sqrt{1 - r^2} \, \Phi_3 \left( \frac{k}{\sqrt{1 - r^2}} \right) r^{d-4} \, dr, \quad \text{for } d \geq 4.
\]

As we have shown above that \(\Phi_3\) is strictly decreasing, this proves that \(\Phi_d\) is also strictly decreasing, hence that
\[
\max_{\mathbb{R}^+} \Phi_d = \Phi_d(0) = \pi^2 |S^{d-4}| \int_0^1 \sqrt{1 - r^2} \, r^{d-4} \, dr, \quad \text{for } d \geq 4.
\]

3.3. **Proof of Theorem 2.2**: kinetic Lieb-Thirring inequality for \(d = 1\).

In the one-dimensional case \(d = 1\), the same proof as that of Theorem 2.1 leads to a bound of the form
\[
(3.21) \quad \text{Tr}_0(-\Delta - 1)Q \geq K(1) \int_{\mathbb{R}} \delta T^{\mu_c} \left( \rho_{Q^{++}} + \rho_{Q^{--}} \right)
\]

\[
+ K''(1) \int_{\mathbb{R}} \frac{|\rho_{Q^{+-}}(k) + \rho_{Q^{--}}(k)|^2}{\Phi_1(|k|)} \, dk.
\]

Using the known behavior of \(\Phi_1\) at \(|k| = 2\) and when \(|k| \to \infty\), one can state this bound as in (2.14).

\[\square\]

4. **Potential inequalities: Proof of Theorems 2.3, 2.4 and 2.5**

For the standard Lieb-Thirring inequalities [26, 27] (the case where \(\mu = 0\)), there is a duality between the kinetic energy and the potential versions of the inequality, and this duality is based on a variational principle for sums of eigenvalues. A similar variational principle is also valid inside the continuous spectrum and can be used to deduce Theorems 2.3 and 2.4 from Theorems 2.1 and 2.2.
Theorem 4.1 (Variational principle). Let \( \mu \geq 0 \) and \( V \) be a real-valued function. Assume that \( V \in L^2(\mathbb{R}^d) \cap L^{1+d/2}(\mathbb{R}^d) \) when \( d \geq 2 \), and that \( V \in L^{3/2}(\mathbb{R}) + L^2(\mathbb{R}) \) with
\[
\int_{\mathbb{R}} \frac{\sqrt{\mu} + |k|}{\sqrt{\mu} |k|} \log \left( \frac{2\sqrt{\mu} + |k|}{2\sqrt{\mu} - |k|} \right) |\hat{V}(k)|^2 \, dk < \infty
\]
when \( d = 1 \). Then both \( Q_V | -\Delta - \mu |^{1/2} \) and \( Q_V | -\Delta - \mu + V |^{1/2} \) are Hilbert-Schmidt operators and
\[
\text{Tr}_V(-\Delta - \mu + V)Q_V = \inf_{Q \in \mathcal{K}} \left( \text{Tr}_0(-\Delta - \mu)Q + \int_{\mathbb{R}^d} V(x) \mu_Q(x) \, dx \right),
\]
where \( \mathcal{K} \) was defined in (4.3). The infimum in (4.2) is attained for \( Q = Q_V \).

To motivate this theorem, we explain its analogue for self-adjoint finite-dimensional matrices \( A \) and \( B \). The starting point is the well-known formula for the sum of eigenvalues [21 Thm. 12.1]
\[
- \text{Tr}(A + B)_- = \inf_{0 \leq \gamma \leq 1} \text{Tr}(A + B)\gamma.
\]
Introducing the spectral projection \( \Pi^- = \mathbb{1}(A \leq 0) \) onto the negative spectral subspace of \( A \) and changing variables, \( \gamma = Q + \Pi^- \), we obtain
\[
- \text{Tr}(A + B)_- = \inf_{-\Pi^- \leq Q \leq 1 - \Pi^-} \text{Tr}(A + B)Q + \text{Tr}(A + B)\Pi^-,
\]
that is, with the notation \( \Pi_B^- = \mathbb{1}(A + B \leq 0) \),
\[
\text{Tr}(A + B)(\Pi_B^- - \Pi^-) = \inf_{-\Pi^- \leq Q \leq 1 - \Pi^-} \text{Tr}(A + B)Q.
\]
The right-side is obviously the analogue of the corresponding term in (4.2), with \( A = -\Delta - \mu \) and \( B = V \). The left-side is negative, which can be seen by taking \( Q = 0 \) on the right, or by noticing that
\[
\text{Tr}(A + B)(\Pi_B^- - \Pi^-) = - \text{Tr}|A + B|(\Pi_B^- - \Pi^-)^2.
\]
This, clearly, is the analogue of \( \text{Tr}_V(-\Delta + V - \mu)Q_V \), see Definition (4.2).

4.1. Proof of the Lieb-Thirring inequalities in a potential \( V \). Here we explain how to prove the Lieb-Thirring inequalities (4.21) and (4.28), assuming Theorem (4.1). As in the proof of Theorem (4.1), we assume \( \mu = 1 \), the general case being obtained by a simple scaling argument. By Theorems (4.1) and (4.1) we have for \( d \geq 2 \)
\[
0 \geq \text{Tr}_V(-\Delta + V - 1)Q_V \geq \inf_{\rho \geq -\rho_0} \left( K(d) \int_{\mathbb{R}^d} \delta \mathcal{T}_{\rho_0}(\rho) + \int_{\mathbb{R}^d} V \rho \right)
\]
\[
= - L(d) \int_{\mathbb{R}^d} \left( (V(x) - 1)^{1+d/2} - 1 + \frac{2 + d}{2} V(x) \right) \, dx.
\]
The second equality follows from a simple optimization argument. When \( d = 1 \), we argue similarly. We decompose \( \rho = \rho_{Q++} + \rho_{Q--} \) and \( \rho' = \rho_{Q--} + \rho_{Q++} \) and
use (3.21) to obtain
\[ 0 \geq \text{Tr}_V(-\Delta + V - 1)Q_V \]
\[ \geq \inf_{\rho \geq -\rho_0} \left( K(1) \int_{\mathbb{R}} \delta T^xv(\rho) + \int_{\mathbb{R}} V \rho \right) + \inf_{\rho'} \left( K'(1) \int_{\mathbb{R}} \frac{\tilde{V}(k)^2}{F_1(|k|)} dk + \int_{\mathbb{R}} V \rho' \right) \]
\[ = -L(1) \int_{\mathbb{R}} \left( (V(x) - 1)^2 - 1 + \frac{3}{2} V(x) \right) dx - L'(1) \int_{\mathbb{R}} F_1(|k|) \tilde{V}(k)^2 dk \]
with
\[ F_1(|k|) = \frac{1 + |k|}{|k|} \log \left( \frac{2 + |k|}{2 - |k|} \right) \]
and
\[ L(1) = \frac{2}{3} \left( \frac{1}{3K(1)} \right)^{1/2}, \quad L'(1) = \frac{1}{4K'(1)} \]
This concludes the proof of (2.24) and (2.28).

\[ \square \]

4.2. Proof of Theorem 4.1, the variational principle. As before we assume \( \mu = 1 \). We denote by \( I(V) \) the infimum appearing in (4.2):
\[ I(V) := \inf_{Q \in \mathcal{K}} \left( \text{Tr}_0(-\Delta - 1)Q + \int_{\mathbb{R}^d} V(x) \rho_Q(x) dx \right). \]
Note that by Lemma 3.2 we can restrict the infimum to finite-rank states \( Q \in \mathcal{K} \).

We split the proof of the theorem into two parts. First we show that
\[ 0 \geq \text{Tr}_V(-\Delta - 1 + V)Q_V = -\left\| Q_V - \Delta - 1 + V \right\|^{1/2}_{\mathcal{S}_2} \geq I(V) \]
This will show that \( Q_V - \Delta - 1 + V \) is Hilbert-Schmidt. We will also find that \( Q_V - \Delta - 1 \) is Hilbert-Schmidt. To prove (4.3), we approximate \( Q_V \) by a well-chosen sequence \( Q_v \) of smooth operators in \( \mathcal{K} \) satisfying the constraint \(-\Pi^- \leq Q_v \leq \Pi^+\).

In a second step we prove the converse inequality
\[ \text{Tr}_V(-\Delta - 1 + V)Q_V \leq I(V), \]
using the information that \( Q_V - \Delta - 1 + V \) is a Hilbert-Schmidt operator in \( \mathcal{K} \), as stated in Lemma 3.2.

Step 1. Proof of the lower bound (4.2). We introduce the following function
\[ h(x) := -|x| \mathbb{1}(|x| \leq 1) + (2 - |x|) \mathbb{1}(1 \leq |x| \leq 2) \]
and replace \(-\Delta\) by \( K_{\varepsilon} := -\Delta + \varepsilon h(-i\nabla) \) for a small \( \varepsilon > 0 \). The gain is that \(-\Delta + \varepsilon h(-i\nabla)\) now has a gap \( (1 - \varepsilon, 1 + \varepsilon) \) in its spectrum. Note also that we have \( \Pi^- = \mathbb{1}(K_{\varepsilon} \leq 1) \) for all \( \varepsilon > 0 \), hence the free Fermi sea is not changed.

Let us introduce the corresponding regularized operator
\[ Q_v := \mathbb{1}(-\Delta + \varepsilon h(-i\nabla) + V \leq 1) - \Pi^- \]
Note that \(-\Delta + \varepsilon h(-i\nabla) + V \to -\Delta + V\) in the norm resolvent sense. When \( d \geq 3 \), our assumption that \( V \in L^2(\mathbb{R}^d) \cap L^{1+d/2}(\mathbb{R}^d) \) implies \( V \in L^{(d+1)/2}(\mathbb{R}^d) \) hence it follows from a result of Koch and Tataru [15] that \(-\Delta + V\) has no positive eigenvalue. This in turn implies that \( Q_v \to Q_V \) strongly by, e.g., [32, Thm. VIII.24].

In dimensions \( d = 1 \) and \( d = 2 \), it was shown by von Neumann-Wigner [39] and Ionescu-Jerison [13] that there exist potentials \( V \) satisfying our assumptions for
which \( \ker(-\Delta + V - 1) \neq \{0\} \). For this reason, when \( d = 1, 2 \), we assume first that \( V \) has a compact support and is bounded (then \( -\Delta + V \) has no positive eigenvalue and \( Q_V^\varepsilon \to Q_V \) strongly), and only remove this assumption at the very end of the proof.

What we have gained is that the operator \( Q_V^\varepsilon \) is now Hilbert-Schmidt, whereas \( Q_V \) is not even compact in general (Remark 2.4).

**Lemma 4.1.** Under our assumptions on \( Q \)
\[
\text{Tr}_{\varepsilon}(Q_V^\varepsilon - Q_V) \geq K(d) \int_{\mathbb{R}^d} \delta T_1^\varepsilon(\rho_{Q_V^\varepsilon}).
\]

In dimension \( d = 1 \) we at least know that \( \rho_{Q_V^\varepsilon} \in L^1_{\text{loc}}(\mathbb{R}) \) by Lemma 3.1.

**Lemma 4.2.** We have the following equality:
\[
\text{Tr}_{\varepsilon}(Q_V^\varepsilon - Q_V) + \varepsilon \text{Tr} h(-i\nabla)\left((Q_V^\varepsilon)^{++} - (Q_V^\varepsilon)^{-}\right) + \int_{\mathbb{R}^d} V \rho_{Q_V^\varepsilon} = - \text{Tr}(K_{\varepsilon} + V - 1)^{1/2}(Q_V^\varepsilon)^2 K_{\varepsilon} + V - 1)^{1/2} \leq 0.
\]

**Proof.** It is possible to approximate \( Q_V^\varepsilon \) by a sequence \( \{R_n\} \) of smooth finite rank operators such that \( -\Pi^- \leq R_n \leq \Pi^+ \), \((\Delta + i)R_n \to (-\Delta + i)Q_V \) strongly in \( \mathcal{S}_2 \) and \((\Delta + i)(R_n)^{\pm \pm} \to (-\Delta + i)(Q_V)^{\pm \pm} \) strongly in \( \mathcal{S}_1 \). See, e.g., [10] Prop. 2 & App. B. We then write
\[
\text{Tr}(\Delta - 1 + \varepsilon h(-i\nabla) + V)R_n = \text{Tr}(\Delta - 1)R_n + \varepsilon \text{Tr} h(-i\nabla)R_n + \int_{\mathbb{R}^d} V \rho_{R_n} = \text{Tr}[K_{\varepsilon} - 1 + V]\left(\Pi^{+\varepsilon}R_n\Pi_{\varepsilon}^{+\varepsilon} - \Pi^{-\varepsilon}R_n\Pi_{\varepsilon}^{-\varepsilon}\right)
\]
where \( \Pi^{\varepsilon} = 1(K_{\varepsilon} + V \leq 1) \) and pass to the limit \( n \to \infty \).

Since \( Q_V^\varepsilon \in \mathcal{X} \), we deduce from (4.6) that
\[
\text{Tr}[K_{\varepsilon} + V - 1)^{1/2}(Q_V^\varepsilon)^2 K_{\varepsilon} + V - 1)^{1/2} \geq I(V)
\]
for all \( \varepsilon \geq 0 \). In particular \( Q_V^\varepsilon |K_{\varepsilon} + V - 1)^{1/2} \) is uniformly bounded in the Hilbert-Schmidt class \( \mathcal{S}_2 \). Note that the weak limit of \( Q_V^\varepsilon |K_{\varepsilon} + V - 1)^{1/2} \) in \( \mathcal{S}_2 \) can only be \( Q_V |-\Delta + V - 1)^{1/2} \), since \( Q_V \to Q_V \) strongly and \( |K_{\varepsilon} + V - 1)^{1/2}y \to |-\Delta + V - 1)^{1/2}y \)
for every \( y \in H^2(\mathbb{R}^d) \). This latter statement can be seen by writing\footnote{In dimension \( d = 1 \), the domain of \(-\Delta + V \) (hence of \( K_{\varepsilon} + V \)) contains \( H^2(\mathbb{R}) \) by choice of the Friedrichs extension via the KLMN Theorem.}
\[
|K_{\varepsilon} + V - 1)^{1/2}y = \frac{|K_{\varepsilon} + V - 1)^{1/2}}{K_{\varepsilon} + V + i}(-\Delta + V + i)y + \varepsilon \frac{|K_{\varepsilon} + V - 1)^{1/2}}{K_{\varepsilon} + V + i}h(-i\nabla)y \to \varepsilon \to 0
\]
\[
\frac{|-\Delta + V - 1)^{1/2}}{-\Delta + V + i}(-\Delta + V + i)y = |-\Delta + V - 1)^{1/2}y
\]
and using that \( f(x) = |x-1|^{1/2} (x+i)^{-1} \) is a continuous function tending to zero at infinity, thus \( |f(K_\varepsilon + V - 1) - f(-\Delta - 1 + V)| \rightarrow 0 \) by \cite{32} Thm. VIII.20]. Hence we have

\[
|K_\varepsilon + V - 1|^{1/2} Q_V^\varepsilon \rightharpoonup | - \Delta + V - 1|^{1/2} Q_V \quad \text{weakly in } \mathcal{S}_2
\]

and, passing to the weak limit in (4.7), we obtain the claimed inequality (4.3).

From (4.6), we also have the following bound

\[
\|Q_V^\varepsilon \|_{\mathcal{S}_2}^2 + \int V \rho Q_V^\varepsilon = \text{Tr} \vartheta(-\Delta - 1) Q_V^\varepsilon + \int V \rho Q_V^\varepsilon \leq 0
\]

for all \( \varepsilon > 0 \). We deduce for instance that

\[
\|Q_V^\varepsilon \|_{\mathcal{S}_2}^2 \leq - \left( \text{Tr} \vartheta(-\Delta - 1) Q_V^\varepsilon + 2 \int V \rho Q_V^\varepsilon \right) \leq -I(2V).
\]

This uniform bound proves that \( Q_V \| \Delta + 1 \|^{1/2} \in \mathcal{S}_2 \) and that

\[
Q_V \| \Delta + 1 \|^{1/2} \rightarrow Q_V \| \Delta + 1 \|^{1/2} \quad \text{weakly in } \mathcal{S}_2.
\]

In dimensions \( d = 1, 2 \), we have only written the proof for \( V \) a bounded function of compact support. If \( V \) is an arbitrary function satisfying our assumptions (2.16) and (2.17), we apply the result to \( V_R(x) := V(x) \mathbb{1}(|x| \leq R) \mathbb{1}(|V(x)| \leq R) \) and, from (4.3) and (4.9), we obtain uniform estimates of the form

\[
\text{Tr} | - \Delta + V_R - 1|^{1/2}(Q_V \| R)^2 | - \Delta + V_R - 1|^{1/2} \leq -I(V_R)
\]

and

\[
\text{Tr} | \Delta + 1|^{1/2}(Q_V \| R^+ - Q_V \| R^-)| \Delta + 1|^{1/2} \leq -I(2V_R).
\]

Extracting subsequences we now have at best that \( Q_V \| R \rightarrow Q_V + \delta \) weakly as \( R \rightarrow \infty \), where \( 0 \leq \delta \leq \mathbb{1}(-\Delta + V = 1) \). Passing to weak limits as before, we therefore obtain that

\[
\text{Tr} | - \Delta + V - 1|^{1/2}(Q_V)^2 | - \Delta + V - 1|^{1/2} \leq -I(V)
\]

and

\[
\text{Tr} | \Delta + 1|^{1/2}(Q_V^+ - Q_V^-)| \Delta + 1|^{1/2} \leq \text{Tr} | \Delta + 1|^{1/2}(Q_V^+ - Q_V^- + \delta)| \Delta + 1|^{1/2} \leq -I(2V).
\]

It remains to provide the

\[
\text{Proof of Lemma 4.7}. \quad \text{Our claim (4.5) follows from Cauchy’s formula and the resolvent expansion:}
\]

\[
Q_V^\varepsilon = -\frac{1}{2\pi} \sum_{k=1}^J (-1)^k \oint_C \frac{1}{K_\varepsilon - z} \left( V - \frac{1}{K_\varepsilon - z} \right)^k dz
\]

\[
+ \frac{(-1)^J}{2\pi} \oint_C \frac{1}{K_\varepsilon - z} \left( V - \frac{1}{K_\varepsilon - z} \right)^{J+1} (K_\varepsilon - z) \frac{1}{K_\varepsilon + V - z} dz.
\]

Under our assumptions the function \( V \) is \( K_\varepsilon \)-compact, hence \( K_\varepsilon + V \) has the gap \((1 - \varepsilon, 1 + \varepsilon)\) in its essential spectrum and it is bounded from below. In (4.10), we choose for \( C \) a smooth curve enclosing the spectra of \( K_\varepsilon \) and \( K_\varepsilon + V \) below 1, without intersecting them. We will explain below how to choose \( J \).
In order to show that \((1 - \Delta)Q^\varepsilon_V\) is a Hilbert-Schmidt operator for all \(\varepsilon > 0\), we estimate each term in (4.10). Our bounds will depend on \(\varepsilon\). We start by noticing that there is a uniform bound of the form

\[(4.11) \quad \forall z \in \mathcal{C}, \quad \left\| \frac{1 - \Delta}{K_\varepsilon - z} \right\| + \left\| \frac{1}{K_\varepsilon + V - z} \right\| \leq C.\]

The constant \(C\) diverges when \(\varepsilon \to 0\) but we do not emphasize this in our notation. To estimate the last term of (4.10), we use (for \(d \geq 2\)) that

\[(4.12) \quad \left\| V \frac{1}{K_\varepsilon - z} \right\|_{\mathfrak{S}_{1+d/2}} \leq C \|V\|_{L^{1+d/2}(\mathbb{R}^d)}\]

by the Kato-Seiler-Simon inequality [36, Thm 4.1],

\[(4.13) \quad \forall p \geq 2, \quad \|f(-i\nabla)g(x)\|_{\mathfrak{S}_p} \leq \frac{1}{(2\pi)^{d/2}} \|g\|_{L^p(\mathbb{R}^d)} \|f\|_{L^p(\mathbb{R}^d)}.\]

The constant \(C\) in (4.12) also depends on \(\varepsilon\). Choosing \(J \geq 1/2 + d/4\) in (4.11), we obtain by Hölder’s inequality and (4.11)

\[
\left\| (1 - \Delta) \mathcal{F} \left[ \frac{1}{K_\varepsilon - z} \right] \mathcal{F} \left[ V \frac{1}{K_\varepsilon - z} \right]^{J+1} (K_\varepsilon - z) \frac{1}{K_\varepsilon + V - z} dz \right\|_{\mathfrak{S}_{2J+2}} \leq C \|V\|_{L^{1+d/2}(\mathbb{R}^d)}^{J+1}.
\]

We now treat the term corresponding to \(k = 1\) in the first sum of (4.10) and start by noticing that

\[
\mathcal{F} \left[ \frac{\Pi^-}{K_\varepsilon - z} V \frac{\Pi^-}{K_\varepsilon - z} \right] dz = \mathcal{F} \left[ \frac{\Pi^+}{K_\varepsilon - z} V \frac{\Pi^+}{K_\varepsilon - z} \right] dz = 0.
\]

For the other terms, we simply write, for instance,

\[
\left\| (1 - \Delta) \mathcal{F} \left[ \frac{\Pi^-}{K_\varepsilon - z} V \frac{\Pi^+}{K_\varepsilon - z} \right] dz \right\|_{\mathfrak{S}_{2J+2}} \leq C \left\| \Pi^- V \right\|_{\mathfrak{S}_2} \leq C \|V\|_{L^2(\mathbb{R}^d)}
\]

since \(\Pi^- = 1(|\nabla| \leq 1)\) has a compact support in Fourier space.

The argument is the same for the other terms of the first sum in (4.10): We write

\[
\frac{1}{K_\varepsilon - z} = \frac{\Pi^-}{K_\varepsilon - z} + \frac{\Pi^+}{K_\varepsilon - z}
\]

and note first that the term which has only \(\Pi^+\) vanishes after integrating over \(z \in \mathcal{C}\), by the residue formula (the same holds for the term which has only \(\Pi^-\)). The other terms contain at least one \(\Pi^-\) and can be estimated similarly as before.

We deduce, as was claimed in (4.5), that \((1 - \Delta)Q^\varepsilon_V \in \mathfrak{S}_2\) for every \(\varepsilon > 0\). Since \((Q^\varepsilon_V)^2 = (Q^\varepsilon_V)^{+\varepsilon} + (Q^\varepsilon_V)^{-\varepsilon}\), this implies that \((1 - \Delta)(Q^\varepsilon_V)^{\pm\varepsilon}(1 - \Delta) \in \mathfrak{S}_1\). Finally, \((K_\varepsilon + V)(1 - \Delta)^{-1} \) being bounded, we have that \((K_\varepsilon + V)Q^\varepsilon_V \in \mathfrak{S}_2\).

We have written the proof for \(d \geq 2\). The case \(d = 1\) is similar and left to the reader (see also the proof of Theorem 2.5 below). \(\square\)
Step 2. Prove of the upper bound (4.4). To finish the proof, it remains to show the inequality (4.4), that is \( \text{Tr}_V(-\Delta - 1 + V)Q_V \leq I(V) \).

We pick a smooth finite rank operator \( Q \) such that \(-\Pi^- \leq Q \leq \Pi^+\) and \( Q(-\Delta) \) is bounded, and note that

\[
\text{Tr}_0(-\Delta - 1)Q + \int_{\mathbb{R}^d} V \rho_Q \\
= \text{Tr}(-\Delta - 1 + V)Q \\
= \text{Tr}| -\Delta - 1 + V|^{1/2}(\Pi_V^+Q\Pi_V^+ - \Pi_V^-Q\Pi_V^-)| -\Delta - 1 + V|^{1/2}.
\]

We now use that

\[
| -\Delta - 1 + V|^{1/2}\Pi_V^+Q\Pi_V^+| -\Delta - 1 + V|^{1/2} \in \mathcal{S}_1
\]
as we have shown in Step 1. Writing \( Q = (Q - Q_V) + Q_V \) we obtain

\[
\text{Tr}_0(-\Delta - 1)Q + \int_{\mathbb{R}^d} V \rho_Q = \text{Tr}_V(-\Delta - 1 + V)(Q - Q_V) + \text{Tr}_V(-\Delta - 1 + V)Q_V \\
\geq \text{Tr}_V(-\Delta - 1 + V)Q_V.
\]

In the second line we have used that

\[
\text{Tr}_V(-\Delta - 1 + V)(Q - Q_V) \geq 0
\]
since \(-\Pi_V^- \leq Q - Q_V \leq \Pi_V^+\). By the density of finite rank operators in \( \mathcal{X} \) (see Lemma 3.2), we deduce that

\[
\text{Tr}_V(-\Delta + V - 1)Q_V \leq I(V),
\]
which finishes the proof of Theorem 4.1. \( \square \)

Remark 4.1. Our proof also yields the limit

\[
(4.14) \quad \lim_{\varepsilon \to 0} \left\| K_\varepsilon + V - 1 \right\|^{1/2}Q_V^2 - \left| -\Delta + V - 1 \right|^{1/2}Q_V \right\|_{\mathcal{S}_2} = 0.
\]

Indeed, from (4.17), we know that

\[
\limsup_{\varepsilon \to 0} \left\| K_\varepsilon + V - 1 \right\|^{1/2}Q_V^2 \leq -I(V) = \left\| -\Delta + V - 1 \right\|^{1/2}Q_V \right\|_{\mathcal{S}_2}
\]

where the last equality follows from Theorems 2.3 and 2.4. Since we also have proved that \( |K_\varepsilon + V - 1|^{1/2}Q_V^2 \rightharpoonup -\Delta + V - 1 \] weakly in \( \mathcal{S}_2 \), the statement (4.14) follows.

4.3. Proof of Theorem 2.5. Second-order perturbation theory. In this section we sketch the proof of Theorem 2.5. We detail first the one-dimensional case \( d = 1 \) and mention the necessary modifications in higher dimensions afterwards.

We could embark upon expanding \( Q_{tV} \) in powers of \( t \) by directly using the resolvent formula. Since we want to avoid a tedious justification of this expansion, we instead work with the approximate state

\[
Q_{tV}^\epsilon := 1(K_\varepsilon + tV \leq 1) - \Pi^-
\]
which we have already introduced in the proof of Theorems 2.3 and 2.4. We will prove bounds in \( t \) which are uniform in \( \varepsilon \), and pass to the limit \( \varepsilon \to 0 \) in the end, using (4.14). The same method of proof can be used to justify an expansion of \( Q_{tV} \) to any order in \( t \).
We come back to the resolvent expansion (4.10) for $Q^r_{IV}$ which we have already mentioned in the proof of Lemma 4.1 above. In dimension $d = 1$, we write

\begin{equation}
Q^r_{IV} = t Q^r_1 + t^2 Q^r_2 + t^3 Q^r_3(t)
\end{equation}

where

$$Q^r_1 = \frac{1}{2i\pi} \oint_C \frac{1}{K_\varepsilon - z} V \frac{1}{K_\varepsilon - z} \, dz, \quad Q^r_2 = -\frac{1}{2i\pi} \oint_C \left( \frac{1}{K_\varepsilon - z} V \right)^2 \frac{1}{K_\varepsilon - z} \, dz$$

and

$$Q^r_3(t) = \frac{1}{2i\pi} \oint_C \left( \frac{1}{K_\varepsilon - z} V \right)^3 \frac{1}{K_\varepsilon + tV - z} \, dz.$$ 

In the above formulas, we choose for $C$ a curve in the complex plane enclosing the interval $[-R, 1] \subset \mathbb{R}$, where $-R < \inf \sigma(K_\varepsilon + tV)$ for all $0 < \varepsilon < 1$ and all $|t| < 1$. To simplify certain estimates below, we also assume that $|\Im z| \leq 1/2$ for all $z \in C$ (in such a way that $\log |\Im z|^{-1} \geq 0$). For convenience we will make the assumption that $1 \notin \sigma(K_\varepsilon + tV)$ for all $t$ small enough. If 1 is an eigenvalue of $K_\varepsilon + tV$, one has to let the curve $C$ depend on $\varepsilon$, and modify it a bit in a neighborhood of $z = 1$. It can then be verified that our estimates below still hold true. These details are left to the reader for brevity.

Note that $Q^r_1$ is purely off-diagonal, i.e. $(Q^r_1)^{\pm \pm} = 0$. Using that $|K_\varepsilon - z| > 0$ for all $z \in C$ by definition of $K_\varepsilon$, one can prove (similarly as in the proof of Lemma 4.1), that $Q^r_1$, $Q^r_2$ and $Q^r_3(t)$ are trace-class, and that

\begin{equation}
\text{Tr}_V(K_\varepsilon - 1 + tV) Q^r_{IV} = t^2 \left\{ \text{Tr}(Q^r_1 V) + \text{Tr} Q^r_2(K_\varepsilon - 1) \right\} \\
+ t^3 \left\{ \text{Tr}(Q^r_3 V) + \text{Tr} Q^r_3(t)(K_\varepsilon - 1 + tV) \right\}.
\end{equation}

Each of the terms on the right side makes sense and can be bounded uniformly in $t$ and $\varepsilon$, as we now explain. First, we have

$$|Q^r_2 V|_{\mathcal{E}_1} \leq (2\pi)^{-1} \oint_C \left\| \left( \frac{1}{K_\varepsilon - z} V \right)^3 \right\|_{\mathcal{E}_1} |dz|,$$

and, similarly,

$$|Q^r_3(t)(K_\varepsilon - 1 + tV)|_{\mathcal{E}_1} \leq C \oint_C \left\| \left( \frac{1}{K_\varepsilon - z} V \right)^3 \right\|_{\mathcal{E}_1} |dz|,$$

since $(K_\varepsilon + tV - 1)(K_\varepsilon + tV - z)^{-1}$ is uniformly bounded for $z \in C$, by choice of the curve $C$ in the complex plane. We now use that

$$\left\| \frac{-\Delta - 1}{K_\varepsilon - z} \right\| \leq C$$

for a constant $C$ independent of $z \in C$ and $\varepsilon$, to deduce that

$$\left\| \left( \frac{1}{K_\varepsilon - z} V \right)^3 \right\|_{\mathcal{E}_1} \leq C |\Im z|^{-1/2} \left\| \frac{1}{-\Delta - z} \right\|^{1/2} \left\| \frac{1}{-\Delta - z} \right\|^{1/2} \left\| \frac{1}{-\Delta - z} \right\|^{1/2}.$$


We have the bound

\[
\left\| \frac{1}{|\Delta - z|^{1/2}} V - \frac{1}{|\Delta - z|^{1/2}} \right\|_{L^1} \leq \left\| \frac{1}{|\Delta - z|^{1/2}} \sqrt{|V|} \right\|^2_{L^2} \leq (2\pi)^{-1} \|V\|_{L^1} \int_{\mathbb{R}} \frac{dp}{\sqrt{(p^2 - R\Delta z)^2 + (\Im z)^2}} \\
\leq C \|V\|_{L^1} \log |3z|^{-1},
\]

(4.17)

and, in a similar fashion,

\[
\left\| \frac{1}{|\Delta - z|^{1/2}} V \right\|_{L^2} \leq C \|V\|_{L^2} \log |3z|^{-1}.
\]

Using these two bounds we deduce that

\[
\left\| \left( \frac{1}{K_\varepsilon - z} V \right)^3 \right\|_{L^1} \leq C \|V\|_{L^1}^2 \|V\|_{L^2} \log |3z|^{-1/2}(\log |3z|^{-1})^3.
\]

Integrating over \( z \in \mathbb{C} \), this eventually shows that

\[
\|Q^2 \|_{L^1} + \|Q^3(t)(K_\varepsilon - 1 + tV)\|_{L^1} \leq C \|V\|_{L^1}^2 \|V\|_{L^2} \|V\|_{L^2}^3,
\]

hence that

\[
(4.18) \quad \left| \text{Tr}_{TV}(K_\varepsilon - 1 + tV)Q^2(t) - t^2 \left\{ \text{Tr}(Q^1(t)V) + \text{Tr} Q^2(K_\varepsilon - 1) \right\} \right| \\
\leq C t^3 \|V\|_{L^1}^2 \|V\|_{L^2}^3
\]

with a constant \( C \) independent of \( \varepsilon \) and \( t \).

Using the residue formula we find

\[
\text{Tr}(Q^1(t)V) = -2(2\pi)^{-1} \int \int |p|^2 \leq 1 \int |q|^2 \geq 1 \frac{(|\tilde{V}(p - q)|^2)}{|q|^2 - |p|^2 + \varepsilon(h(q) - h(p))} dp dq
\]

and

\[
\text{Tr} Q^2(K_\varepsilon - 1) = (2\pi)^{-1} \int \int |p|^2 \leq 1 \int |q|^2 \geq 1 \frac{(|\tilde{V}(p - q)|^2)}{|q|^2 - |p|^2 + \varepsilon(h(q) - h(p))} dp dq.
\]

The result in the case \( d = 1 \) now follows from taking first the limit \( \varepsilon \to 0 \) in (4.18), using (4.14), and then \( t \to 0 \).

When \( d \geq 2 \), the proof is similar but a bit more tedious. We start again with the resolvent expansion (4.11), to an order \( J \) such that the last term becomes trace-class.
(when multiplied by $K_\varepsilon + tV - 1$). This means we write

\[
\text{Tr}_{tV} Q_\varepsilon^{z}(K_\varepsilon + tV - 1) = -t^2(2\pi)^{-d} \int \frac{|\hat{V}(p-q)|^2}{|q|^2 - |p|^2 + \varepsilon(h(q) - h(p))} \, dp \, dq \\
\quad + \sum_{j=3}^{J} (-t)^j \oint_C dz \left( \frac{1}{K_\varepsilon - z} \right)^j \frac{K_\varepsilon - 1}{K_\varepsilon - z} \\
- \sum_{j=3}^{J+1} (-t)^j \oint_C dz \left( \frac{1}{K_\varepsilon - z} \right)^j \\
\quad \left. + (-t)^{J+1} \right) \oint_C dz \left( \frac{1}{K_\varepsilon - z} \right)^{J+1} \frac{K_\varepsilon + tV - 1}{K_\varepsilon + tV - z}.
\]

(4.19)

We fix a $J \geq 1 + d/2$ and deduce, similarly as before, that

\[
\left\| \left( \frac{1}{K_\varepsilon - z} V \right)^{J+1} \right\|_{\mathcal{E}_1} \\
\leq C |\Im z|^{-1/2} \left\| \frac{1}{-\Delta - z} V \frac{1}{-\Delta - z} V \frac{1}{-\Delta - z} V \right\|_{\mathcal{E}_{1+d/2}} \\
\leq C |\Im z|^{-1/2} \left( \|V\|_{L^1(\mathbb{R}^d)} \log |\Im z|^{-1} + \|V\|_{L^{1+d/2}(\mathbb{R}^d)} \right)^J \\
\times \left( \|V\|_{L^2(\mathbb{R}^d)} \log |\Im z|^{-1} + \|V\|_{L^\infty(\mathbb{R}^d)} \right)
\]

with a constant $C$ that is independent of $\varepsilon$. For the other terms in (4.19), we have to work a bit more. As an illustration, we only consider the term

\[
\text{Tr}_0 \oint_C dz \left( \frac{1}{K_\varepsilon - z} V \right)^3,
\]

the other terms are treated by the same argument. We decompose

\[
\frac{1}{K_\varepsilon - z} = \frac{\Pi^-}{K_\varepsilon - z} + \frac{\Pi^+}{K_\varepsilon - z}
\]

and expand $((K_\varepsilon - z)^{-1}V)^3$ accordingly. The terms which have only $\Pi^+$ or only $\Pi^-$ vanish after the integration over the curve $C$, by the residue formula. For the other terms, $\Pi^- \Pi^+$ (or its adjoint) must appear at least twice in the trace to be estimated. For instance, we look at the term

(4.20)

\[
\text{Tr} \oint_C dz \frac{\Pi^+}{K_\varepsilon - z} V \frac{\Pi^+}{K_\varepsilon - z} V \frac{\Pi^-}{K_\varepsilon - z} V = \text{Tr} \oint_C dz \frac{\Pi^+}{K_\varepsilon - z} V \frac{\Pi^-}{K_\varepsilon - z} V \frac{\Pi^+}{K_\varepsilon - z} V \frac{\Pi^-}{K_\varepsilon - z} V \Pi^+.
\]

By cyclicity of the trace, this term can be estimated by

(4.21)

\[
|L_{20}| \leq \oint_C |dz| \left\| \frac{\Pi^+}{-\Delta - z} V \frac{\Pi^+}{-\Delta - z} V \frac{\Pi^-}{-\Delta - z} V \frac{\Pi^-}{-\Delta - z} V \right\|^2_{\mathcal{E}_2}.
\]
For the second term in the right side of (4.21), we use that
\[ d \text{ary conditions and dimensions} \]
Decomposing \( |−Δ − z|^{1/2} = |−Δ − z|^{1/2} \mathbb{I}(|Δ + 1| ≥ 1) + |−Δ − z|^{1/2} \mathbb{I}(|Δ + 1| ≤ 1) \) and using that \( V ∈ L^1(\mathbb{R}^d) ∩ L^{1+d/2}(\mathbb{R}^d) \), we find
\[
\begin{align*}
\left\| \Pi^+ \frac{1}{|−Δ − z|^{1/2}} V \Pi^+ \right\| &\leq \left\| \frac{1}{|−Δ − z|^{1/2}} √|V| \right\|^2 \\
&\leq \left( \left\| \frac{\mathbb{I}(|Δ + 1| ≤ 1)}{|−Δ − z|^{1/2}} √|V| \right\| + \left\| \frac{\mathbb{I}(|Δ + 1| ≥ 1)}{|−Δ − z|^{1/2}} √|V| \right\| \right)^2 \\
&\leq C \left( \|V\|_{L^1(\mathbb{R}^d)} \log |3z|^{-1} + \|V\|_{L^{1+d/2}(\mathbb{R}^d)} \right).
\end{align*}
\]
For the second term in the right side of (4.21), we use that
\[
\begin{align*}
\left\| \Pi^+ \frac{1}{|−Δ − z|^{1/2}} V \Pi^- \right\| &\leq \left\| \frac{\mathbb{I}(|Δ + 1| ≤ 1)}{|−Δ − z|^{1/2}} V \Pi^- \right\| + \left\| \frac{\mathbb{I}(|Δ + 1| ≥ 1)}{|−Δ − z|^{1/2}} V \Pi^- \right\| .
\end{align*}
\]
The first term on the right side is estimated as before. For the second one, we use that
\[
\begin{align*}
\left\| \frac{\Pi^+ \mathbb{I}(|Δ + 1| ≥ 1)}{|−Δ − z|^{1/2}} V \Pi^- \right\| &\leq C|3z|^{-1/4} \left\| \frac{\Pi^+}{|−Δ − 1|^{1/4}} V \frac{\Pi^-}{|−Δ − 1|^{1/4}} \right\| .
\end{align*}
\]
This term is now exactly the one which we have calculated before in (3.11) and it is finite under our assumptions on \( V \). Summarizing, we have proved that the term (4.20) is bounded uniformly in \( ε \).

The same argument can be applied to all the terms in (4.19), showing that they are bounded uniformly in \( ε \). This concludes our sketch of the proof of Theorem 2.5.

\( \square \)

5. Thermodynamic limit and positive temperature

5.1. Lieb-Thirring inequalities in a box. In this section, we extend our inequalities (2.29) and (2.30) to the case of a system living in a box of size \( L \), with constants independent of \( L \). For simplicity we restrict ourselves to periodic boundary conditions and dimensions \( d ≥ 2 \).

We denote by \( −Δ_L \) the Laplacian on \( C_L = [−L/2, L/2]^d \), with periodic boundary conditions, and, for any chosen \( μ > 0 \), we introduce \( \Pi^-_{L,μ} := \mathbb{I}(−Δ_L ≤ μ) \). Note that since the spectrum of \( −Δ_L \) is discrete in \( \mathbb{R}^+ \), \( \Pi^-_{L,μ} \) has finite rank for every finite \( L > 0 \) and \( μ ≥ 0 \). The following is a generalization of the density inequality (2.29).

**Theorem 5.1** (Lieb-Thirring inequality in a box, density version, \( d ≥ 2 \)). We assume that \( d ≥ 2 \), \( μ ≥ 0 \) and \( L > 0 \). Let \( Q \) be a self-adjoint operator of finite rank such that \( −\Pi^-_{L,μ} ≤ Q ≤ 1 − \Pi^-_{L,μ} \). Then there exists positive constants \( K(d) \) and \( C \)
(depending only on $d \geq 2$) such that

\begin{equation}
\text{Tr}_{L^2(C_L)}(-\Delta_L - \mu)Q \geq \tilde{K}(d) \left\{ \begin{array}{ll}
\int_{C_L} \delta T_\mu^{SC} \left( \left( |\rho_Q(x)| - CL^{-1} \mu \frac{d-1}{2} \right)_+ \right) \, dx & \text{for } \mu > 1/L^2, \\
\int_{C_L} \delta T_0^{SC} \left( \left( |\rho_Q(x)| - CL^{-d} \right)_+ \right) \, dx & \text{for } \mu \leq 1/L^2,
\end{array} \right.
\end{equation}

where we recall that

$$\delta T_\mu^{SC}(\rho) := (\rho_0 + \rho)^{1+\frac{2}{d}} - (\rho_0)^{1+\frac{2}{d}} - \frac{d+2}{d} (\rho_0)^{\frac{2}{d}} \rho$$

with $\rho_0 = \mu^{d/2} q(2\pi)^{-d} |S^{d-1}|/d$.

The function appearing in the integrand of (5.1) vanishes for $\rho \leq CL^{-1} \mu \frac{d-1}{2}$ (in the case $\mu > 1/L^2$) or for $\rho \leq CL^{-d}$ (in the case $\mu \leq 1/L^2$), and it converges to $\delta T_\mu(|\rho_Q|)$ in the limit $L \to \infty$. Note the absolute value which we have used to simplify our statement. Of course, $\delta T_\mu^{SC}(\rho_Q)$ is comparable to $\delta T_\mu^{SC}(\rho_Q)$.

Using Theorem 5.1, we can now deduce the (dual) potential version in the box. Again, note that for $V \in L^{1+2/(d)}(C_L)$, the spectrum of $-\Delta_L + V$ is discrete and bounded from below, hence there is only a finite number of eigenvalues below each chosen Fermi level $\mu$.

**Theorem 5.2 (Lieb-Thirring inequality in a box, potential version, $d \geq 2$).** Assume that $\mu \geq 0$, $d \geq 2$ and $L > 0$. Let $V$ be a real-valued function in $L^{1+2/(d)}(C_L)$. Then we have

\begin{equation}
0 \geq -\text{Tr}(-\Delta_L + V - \mu)_{-} + \text{Tr}(-\Delta_L - \mu)_{-} - \rho_0 \int_{C_L} V \geq -\tilde{L}(d) \int_{C_L} \left( (V(x) - \mu)^{1+\frac{2}{d}} - \mu^{1+\frac{2}{d}} + \frac{2 + d}{2} \mu^{\frac{d}{2}} V(x) + \frac{\mu^{d-1}}{L} |V(x)| \right) \, dx
\end{equation}

when $\mu > 1/L^2$, and

\begin{equation}
0 \geq -\text{Tr}(-\Delta_L + V - \mu)_{-} + \text{Tr}(-\Delta_L - \mu)_{-} - \rho_0 \int_{C_L} V \geq -\tilde{L}(d) \int_{C_L} \left( (V(x))^{1+\frac{2}{d}} + \frac{1}{L^d} |V(x)| \right) \, dx
\end{equation}

when $\mu \leq 1/L^2$. The constant $\tilde{L}(d)$ only depends on $d$.

Since all operators are finite-rank, the proof simply reduces to computing the Legendre transform of $\rho \mapsto (1 + (|\rho| - \varepsilon)_+)^{\mu} - 1 - \alpha(|\rho| - \varepsilon)_+$. We skip the details and only provide the proof of Theorem 5.1.

**Proof of Theorem 5.1** The proof follows the same two steps as that of Theorem 2.1, but it is slightly more tedious.
Step 1. Estimate on $Q^{\pm \pm}$. We start by estimating the diagonal densities $\rho_{Q^{\pm \pm}}$. Following the strategy of the proof of Lemma 3.3 we get, with $\gamma = Q^{\pm \pm}$,

$$\text{Tr}_{L^2(C_L)} | - \Delta_L - \mu | \gamma \geq \int_{C_L} R_{d,\mu,L}(\rho(x)) \, dx$$

where

$$(5.4) \quad R_{d,\mu,L}(\rho) = \int_0^\infty \left( \sqrt{\nu} - \sqrt{f_{d,\mu,L}(e)} \right)^2 \, de$$

and

$$f_{d,\mu,L}(e) = \frac{1}{L^d} \# \left\{ p \in (2\pi\mathbb{Z}/L)^d : |p^2 - \mu| \leq e \right\}$$

$$= \frac{1}{L^d} \left\{ \#\mathbb{Z}^d \cap B \left( \frac{L\sqrt{\nu}}{2\pi} \left(1 + \frac{e}{\mu}\right)^{1/2} \right) - \#\mathbb{Z}^d \cap B \left( \frac{L\sqrt{\nu}}{2\pi} \left(1 - \frac{e}{\mu}\right)^{1/2} \right) \right\}.$$ 

The following gives an estimate on the function $f_{d,\mu,L}$.

Lemma 5.1 (Estimates on $f_{d,\mu,L}$). When $\mu > 1/L^2$, we have

$$f_{d,\mu,L}(e) \leq C \left( \frac{\mu^{d-1}}{L} + \mu^{d-1} e \mathbb{1}(e \leq \mu) + e^{d/2} \mathbb{1}(e \geq \mu) \right)$$

whereas when $0 \leq \mu \leq 1/L^2$, we have

$$f_{d,\mu,L}(e) \leq C \left( \frac{1}{L^d} + e^{d/2} \right),$$

for all $e > 0$.

Note that the estimate $\text{(5.6)}$ on $f_{d,\mu,L}$ in the case $\mu \leq 1/L^2$ is a bit weaker than the one $\text{(5.5)}$ for $\mu > 1/L^2$.

Proof of Lemma 5.1. First, we recall the following well-known property

$$\text{(5.7)} \quad \left| \#\mathbb{Z}^d \cap B(R) - \frac{|S^{d-1}|}{d} R^d \right| \leq C \text{ max } (1, R^{d-1}),$$

which says that the number of points of the lattice $\mathbb{Z}^d$ inside a ball of radius $R$, behaves like the volume of the ball $B(R)$ in the limit of large $R$, whereas it is just bounded for small $R$. The error term can even be replaced by $o(R^{d-1})$ but we do not need this here. Note that the bound $\text{(5.7)}$ implies $\#\mathbb{Z}^d \cap B(R) \leq C(1 + R^d)$.

The proof of $\text{(5.6)}$ is now straightforward: Assuming $\mu \leq 1/L^2$, we simply write

$$f_{d,\mu,L}(e) \leq \frac{1}{L^d} \#\mathbb{Z}^d \cap B \left( \frac{L^{d/2}}{2\pi \sqrt{\mu} + e} \right) \leq C \frac{1}{L^d} \left(1 + L^d (\mu^{d/2} + e^{d/2}) \right) \leq \frac{2C}{L^d} + C e^{d/2}.$$ 

In order to prove $\text{(5.5)}$ we need another estimate. Let $M > 0$ and $0 < x \leq x_0$ for some fixed $x_0 > 0$. Using $\text{(5.7)}$ we obtain

$$\#\mathbb{Z}^d \cap B \left( M(1 + x)^{1/2} \right) - \#\mathbb{Z}^d \cap B \left( M(1 - x)^{1/2} \right) \leq \frac{|S^{d-1}|}{d} M^d \left( (1 + x)^{1/2} - (1 - x)^{1/2} \right) + 2C \text{ max } (1, M^{d-1}) (1 + x_0)^{d-1}$$

$$\leq C \frac{|S^{d-1}|}{d} M^d x + 2C \text{ max } (1, M^{d-1}) (1 + x_0)^{d-1}$$

$$\leq C \left( M^d x + \text{ max } (1, M^{d-1}) \right).$$
We have used that \((1 + x)^{1/2} - (1 - x)^{1/2} \leq Cx\) for all \(0 \leq x \leq x_0\), where \(C\) only depends on \(x_0\). We can use (5.8) to prove (5.5), assuming now \(\mu > 1/L^2\). For \(\varepsilon \leq 3\mu/2\), we use (5.8) with \(M = L\sqrt{\mu}/2\pi \geq 1/(2\pi)\) and \(x = \varepsilon/\mu \leq 3/2\). We obtain
\[
f_{d,\mu,L}(e) \leq \frac{C}{L^d} \left( \frac{L^d \mu^{d/2} e}{(2\pi)^d \mu} + L^{d-1} \mu^{(d-1)/2} \right) \leq C \left( \mu^{d/2} e + \frac{\mu^{d-1}}{L} \right).
\]
Finally, for \(\varepsilon \geq 3\mu/2\) we have
\[
f_{d,\mu,L}(e) = \frac{1}{L^d} \# \mathbb{Z}^d \cap B \left( \frac{L\sqrt{\mu}}{2\pi} \left( 1 + \frac{\varepsilon}{\mu} \right)^{1/2} \right)
\leq C \left( \mu^{d/2} + \varepsilon^{d/2} + \frac{1}{L^d} \right) \leq C \left( \frac{\mu^{d/2}}{L} + \varepsilon^{d/2} \right)
\]
where in the last estimate we have used both \(L^{-1} \leq \mu^{1/2}\) and \(\varepsilon \geq 3\mu/2\). This finishes the proof of Lemma 5.1.

Using the bounds (5.5) and (5.6) on \(f_{d,\mu,L}\), we can now deduce an estimate on \(R_{d,\mu,L}\) appearing in (5.4). To simplify our argument, we introduce
\[
g_{d,\mu,L}(e) = \begin{cases} \mu^{d/2} - 1 \varepsilon(e \leq \mu) + e^{d/2} \varepsilon(e \geq \mu) & \text{for } \mu > 1/L^2, \\ e^{d/2} & \text{for } \mu \leq 1/L^2. \end{cases}
\]
such that (5.5) and (5.6) can be rewritten as
\[
f_{d,\mu,L}(e) \leq \varepsilon_{d,\mu,L} + C g_{d,\mu,L}(e)
\]
with
\[
\varepsilon_{d,\mu,L} = C \begin{cases} L^{-1} \mu^{d/2} & \text{for } \mu > 1/L^2, \\ L^{-d} & \text{for } \mu \leq 1/L^2. \end{cases}
\]
We then have in all cases
\[
\bar{R}_{d,\mu,L}(\rho) = \int_0^{\infty} \left( \sqrt{\rho} - \sqrt{f_{d,\mu,L}(e)} \right)^2 d\rho
\geq \int_0^{\infty} \left( \sqrt{\rho} - \sqrt{\varepsilon_{d,\mu,L}} - \sqrt{g_{d,\mu,L}(e)} \right)^2 d\rho = S_{d,\mu,L} \left( \left( \sqrt{\rho} - \sqrt{\varepsilon_{d,\mu,L}} \right)^2 \right),
\]
with
\[
S_{d,\mu,L}(\rho) = \int_0^{\infty} \left( \rho - \sqrt{g_{d,\mu,L}(e)} \right)^2 d\rho
\geq C \begin{cases} \mu^{d/2} \rho^2 1(\rho \leq \mu^{2/d}) + \rho^{1+2/d} 1(\rho \geq \mu^{2/d}) & \text{for } \mu > 1/L^2, \\ \rho^{1+2/d} & \text{for } \mu \leq 1/L^2. \end{cases}
\]
To conclude, it suffices to note that
\[
\left( \sqrt{\rho} - \sqrt{\varepsilon_{d,\mu,L}} \right)^2 \geq \alpha_\theta (\rho - \theta \varepsilon_{d,\mu,L})_+ 
\]
for any \(\theta\) bounded away from 0 and \(\alpha_\theta\) small enough, and that \(S_{d,\mu,L}(\alpha_\theta \rho) \geq \beta_\theta S_{d,\mu,L}(\rho)\).
\begin{equation}
\text{Step 2. Estimate on } Q^{\pm \mp} \text{.} \end{equation}
We again separate the cases \( \mu > 1/L^2 \) and \( \mu \leq 1/L^2 \).

We start with the case \( \mu > 1/L^2 \) and decompose \( Q^{\pm} \) as
\[ Q^{\pm} = \Pi^+ Q \Pi^- = (\Pi^+_0 + \Pi^+_1) Q (\Pi^-_0 + \Pi^-_1) = Q^{0+} + Q^{1+} + Q^{0-} + Q^{1-} \]
where
\[ \Pi^+_0 = \mathbb{1} (\mu \leq p^2 \leq \mu + \sqrt{\mu}/L) \quad \text{and} \quad \Pi^-_0 = \mathbb{1} (\mu - \sqrt{\mu}/L \leq p^2 \leq \mu) \]
(we remove the index on \( \Pi^+_{\mu,L} \) for simplicity). We have, with \( e_k := L^{-d/2} e^{i k \cdot x} \),
\[ \rho^{0-}_{00} = \frac{1}{L^d} \sum_{k,\ell \in (2\pi \mathbb{Z}/L)^d} (e_k, Q e_\ell) e^{i x (k - \ell)}. \]
The matrix \( (e_k, Q e_\ell) \) has a norm \( \leq 1 \), hence we deduce by Schwarz’s inequality that
\begin{equation}
|\rho^{0-}_{00}| \leq \frac{1}{L^d} \sqrt{\#\{\mu \leq \ell^2 \leq \mu + \sqrt{\mu}/L\}} \sqrt{\#\{\mu - \sqrt{\mu}/L \leq \ell^2 \leq \mu\}} \leq C \mu \frac{d-1}{L}. 
\end{equation}
In the last bound we have used \([5,8]\) and the assumption that \( \mu > 1/L^2 \). For \( \rho^{+1}_{00} \), we write, this time,
\[ |\text{Tr}(V Q^{+1}_{10})| = \left| \text{Tr} \left( \Pi^+_0 V \frac{\Pi^+_1}{-\Delta L - \mu} \right) \right| \leq \sqrt{\text{Tr}(-\Delta L - \mu) Q} \Pi^+_0 V \frac{\Pi^+_1}{|\Delta L - \mu|^{1/2}}. \]
We now have
\[ \left\| \Pi^-_0 V \frac{\Pi^+_1}{-\Delta L - \mu} \right\|_{\mathfrak{H}_2}^2 \leq \frac{1}{L^{2d}} \sum_{\mu - \sqrt{\mu}/L \leq \ell^2 \leq \mu} \frac{\widehat{V}(p - q)^2}{|q^2 - \mu|} \leq \frac{1}{L^{2d}} \sum_{\mu - \sqrt{\mu}/L \leq \ell^2 \leq \mu} \frac{\widehat{V}(k)^2}{\mu - \sqrt{\mu}/L \leq \ell^2 \leq \mu} \leq C \mu \frac{d-2}{L} \int_{C_L} |V|^2 \]
where, in the last estimate we have again used that
\[ L^{-d} \#\{\mu - \sqrt{\mu}/L \leq p^2 \leq \mu\} \leq C \frac{\mu^{d-1}}{L}. \]
From these bounds we deduce that
\begin{equation}
\int_{C_L} |\rho^{+1}_{01}|^2 \leq C \mu \frac{d-2}{L} \text{Tr}(-\Delta L - \mu) Q. \end{equation}
The term \( \rho^{+1}_{01} \) is treated similarly. We conclude this paragraph with an estimate on \( \rho^{+1}_{11} \), which we derive by the same method as for \([3.13]\) in the proof of Theorem 2.1.
\begin{equation}
\int_{C_L} |\rho^{+1}_{11}|^2 \leq (2\pi)^{-d} \left\| \Phi_{d,\mu,L} \right\|_{L^\infty((2\pi \mathbb{Z}/L)^d)} \text{Tr}_{L^2(C_L)}(-\Delta L - 1) Q, \end{equation}
with

\[ \Phi_{d,\mu,L}(k) := \frac{1}{L^d} \sum_{p \in (2\pi \mathbb{Z}/L)^d \atop |p|^2 \leq \mu - \sqrt{\pi}/L} \frac{1}{(\mu - |p|^2)^{1/2} (|p - k|^2 - \mu)^{1/2}}. \]

The function \( \Phi_{d,\mu,L} \) is a Riemann approximation of \( \mu^{(d-2)/2} \Phi_{d}(-\sqrt{\mu}) \). In order to prove that \( \Phi_{d,\mu,L} \) is uniformly bounded on \( (2\pi \mathbb{Z}/L)^d \) by \( C \mu^{(d-2)/2} \), independently of \( L \), we compare it with its limit. For every \( p \) in the sum above, we introduce the ball \( B_p \) of radius \( \eta / L \), centered at \( p \). We will fix the value of \( \eta \) later, but as a first constraint we impose that

\[ \sqrt{\mu - \frac{\sqrt{\mu}}{L}} + \frac{\eta}{L} \leq \sqrt{\mu - \frac{\sqrt{\mu}}{2L}} \quad \text{and} \quad \sqrt{\mu + \frac{\sqrt{\mu}}{L}} - \frac{\eta}{L} \geq \sqrt{\mu + \frac{\sqrt{\mu}}{2L}}. \]

for all \( \mu > 1/L^2 \) and \( L \geq 1 \). It is easy to verify that the previous condition is satisfied when, for instance, \( \eta \leq 1/8 \). The constraints \((5.14)\) imply that

\[ \forall p' \in B_p, \quad |p'|^2 \leq \mu - \frac{\sqrt{\mu}}{2L}, \quad |p' - k|^2 \geq \mu + \frac{\sqrt{\mu}}{2L}. \]

Next we compute the gradient

\[ \nabla \left( \frac{1}{(\mu - |p|^2)^{1/2} (|p - k|^2 - \mu)^{1/2}} \right) = \left( \frac{p}{\mu - |p|^2} - \frac{p - k}{|p - k|^2 - \mu} \right) \frac{1}{(\mu - |p|^2)^{1/2} (|p - k|^2 - \mu)^{1/2}}. \]

For \( p' \) satisfying \((5.15)\), we have

\[ \frac{|p'|}{\mu - |p'|^2} \leq \frac{\sqrt{\mu - \frac{\sqrt{\mu}}{2L}}}{2L} = 2L \sqrt{1 - \frac{1}{1 + 2L \sqrt{\mu}}} \leq 2L \]

and

\[ \frac{|p' - k|}{|p' - k|^2 - \mu} \leq \frac{\sqrt{\mu + \frac{\sqrt{\mu}}{2L}}}{2L} = 2L \sqrt{1 + \frac{1}{1 + 2L \sqrt{\mu}}} \leq \sqrt{6} L. \]

We therefore deduce by Taylor’s formula, that for every \( p' \in B_p \)

\[ \left| \frac{1}{(\mu - |p|^2)^{1/2} (|p - k|^2 - \mu)^{1/2}} - \frac{1}{(\mu - |p'|^2)^{1/2} (|p' - k|^2 - \mu)^{1/2}} \right| \leq (2 + \sqrt{6}) (2\eta) \sup_{q \in B_p} \frac{1}{(\mu - |q|^2)^{1/2} (|q - k|^2 - \mu)^{1/2}}. \]

Choosing \( \eta \) small enough, we can therefore make sure that

\[ \sup_{q \in B_p} \frac{1}{(\mu - |q|^2)^{1/2} (|q - k|^2 - \mu)^{1/2}} \leq 2 \frac{1}{(\mu - |p|^2)^{1/2} (|p - k|^2 - \mu)^{1/2}} \]

and then that

\[ \frac{1}{(\mu - |p|^2)^{1/2} (|p - k|^2 - \mu)^{1/2}} \leq 4 \frac{\inf_{q \in B_p} 1}{(\mu - |q|^2)^{1/2} (|q - k|^2 - \mu)^{1/2}}. \]
Using that the balls $B_p$ are disjoint for $\eta$ small enough, we finally obtain
\[
\Phi_{d,\mu,L}(k) \leq \frac{4}{L^d} \sum_{\substack{p \in (2\pi\mathbb{Z}/L)^d \\
|p|^2 \leq \mu - \sqrt{\mu}/L \\
|p-k|^2 \geq \mu + \sqrt{\mu}/L}} \frac{1}{|B_p|} \int_{B_p} \frac{1}{(\mu - |p'|^2)^{1/2}} \frac{1}{(|p' - k|^2 - \mu)^{1/2}} dp'
\]
\[
\leq \frac{4}{|S^{d-1}|\eta^d} \mu^{d-2} \Phi_d(\|k\|/\sqrt{\mu}) \leq C \mu^{d-2},
\]
since $\Phi_d$ is bounded by Lemma 3.3. Summarizing all our estimates, we have proved that
\[
\mu^{1-\frac{1}{d}} \int_{C_L} |\rho^{+-} - \rho^{+-}_{00}|^2 \leq C \text{Tr}_{L^2(C_L)}(-\Delta_L - \mu) Q.
\]
Using now both that $|x - \varepsilon| \geq (|x| - \varepsilon)_+$ and $\rho^{+-}_{00} \leq C \mu^{(d-1)/2}/L$, we deduce that
\[
\mu^{1-\frac{1}{d}} \int_{C_L} (|\rho^{+-}| - C \mu^{(d-1)/2}/L)_+^2 \leq C \text{Tr}_{L^2(C_L)}(-\Delta_L - \mu) Q
\]
with a constant $C$ that does not depend on $L$, for $\mu > 1/L^2$. This completes the proof of (5.1) when $\mu > 1/L^2$.

The case $\mu \leq 1/L^2$ is similar, except that we only decompose
\[
\Pi^+ = 1(\mu \leq p^2 \leq \mu + 1/L^2) + 1(p^2 > \mu + 1/L^2),
\]
and retain $\Pi^-$. We get two terms $Q^{+-}_0$ and $Q^{+-}_1$. We estimate $\rho^{+-}_0$ in $L^\infty$ as in (5.10), getting
\[
|\rho^{+-}_0| \leq \frac{1}{L^d} \sqrt{\#\{\mu \leq k^2 \leq \mu + 1/L^2\}} \sqrt{\#\{l^2 \leq \mu\}} \leq \frac{C}{L^d},
\]

since $\mu \leq 1/L^2$ (each of the two sets above contains a finite number of points which does not increase with $L$). Finally, we estimate $\rho^{+-}_1$ as in (5.11) and obtain
\[
|\text{Tr}_{L^2(C_L)}(VQ^{+-}_1)| \leq \left\| \Pi^- V \frac{\Pi^+}{-\Delta_L - \mu} \right\|_{L^2} \sqrt{\text{Tr}_{L^2(C_L)}(-\Delta_L - \mu) Q}.
\]

This time we have
\[
\left\| \Pi^- V \frac{\Pi^+}{-\Delta_L - \mu} \right\|_{L^2}^2 = \frac{1}{L^{2d}} \sum_{q^2 \leq \mu \atop p^2 \leq \mu} \frac{\tilde{V}(p - q)^2}{q^2 - \mu}
\]
\[
\leq \frac{1}{L^{2d}} L^2 \sum_{q^2 \geq \mu + 1/L^2 \atop p^2 \leq \mu} |\tilde{V}(k)|^2 \leq C L^{2-d} \int_{C_L} |V|^2.
\]

Since $d \geq 2$, this completes the proof of Theorem 5.1.

### 5.2 Thermodynamic limit

With the Lieb-Thirring inequality 5.2 at hand, we can now relate the well-defined total relative energy in a large box to the one we have defined in Section 2.2. The following can therefore serve as an *a posteriori* justification of our definition of $\text{Tr}_V(-\Delta + V - \mu)Q_V$. 

\[\]
Theorem 5.3 (Thermodynamic Limit, \( d \geq 2 \)). We assume that \( d \geq 2 \) and \( \mu \geq 0 \). Let \( V \) be a real-valued function in \( L^1(\mathbb{R}^d) \cap L^{1+d/2}(\mathbb{R}^d) \). Then we have

\[
\lim_{L \to \infty} \left( -\Delta + V - \mu \right) Q_V = -\operatorname{Tr}_{L^2(C_L)}(-\Delta + V \mathbb{1}_{C_L} - \mu) - \operatorname{Tr}_{L^2(C_L)}(-\Delta_L - \mu) - \mu \int_{C_L^2} V,
\]

where the left side is defined in Definition 2.2 and \( -\Delta_L \) is the Laplacian with periodic boundary conditions on \( C_L = [-L/2, L/2]^d \).

**Sketch of the proof.** We quickly explain the main steps of the proof, which proceeds by showing an upper and a lower bound.

Let us fix a smooth finite-rank operator \( Q \in \mathcal{K} \) which we write in the form

\[
Q = \sum_{i,j=1}^K \left( q_{i,j}^{++} |u_i\rangle \langle u_j| + q_{i,j}^{+-} |u_i\rangle \langle v_j| + q_{i,j}^{-+} |v_i\rangle \langle u_j| + q_{i,j}^{--} |v_i\rangle \langle v_j| \right),
\]

where \((u_i)^K_{i=1}\) and \((v_i)^K_{i=1}\) are orthonormal systems of the kernel and the range of \( \Pi^- \), respectively. The constraint that \(-\Pi^- \leq Q = Q^* \leq 1 - \Pi^- \) is then only reflected in the coefficients \( q_{i,j}^{\pm/\pm} \) (see [10, App. B] for an explicit representation of \( q_{i,j}^{\pm/\pm} \)). By assumption, the functions \( u_i \) and \( v_i \) are all smooth. Now we build from \( Q \) a test state \( Q_L \) in the box \( C_L \), by simply replacing the \( u_i \)'s and \( v_i \)'s by orthonormal sequences \((u_i,L)^K_{i=1}\) and \((v_i,L)^K_{i=1}\), in, respectively, the kernel and the range of \( \Pi_{-L}^{-} \). We can do this in such a way that \( \mathbb{1}_{C_L} u_i \rightarrow u_i \), \( \mathbb{1}_{C_L} v_i \rightarrow v_i \), \( \mathbb{1}_{C_L} \nabla u_i \rightarrow \nabla u_i \) and \( \mathbb{1}_{C_L} \nabla v_i \rightarrow \nabla v_i \) in \( L^2(\mathbb{R}^d) \), as \( L \to \infty \). One simple way to realize that is to periodize the functions as

\[
\tilde{u}_{i,L}(x) = L^{-d} \sum_{k \in (2\pi L)^d} \tilde{u}_i(k) e^{-ik \cdot x},
\]

and then to orthonormalize the so-obtained system. Similar arguments have already been used and detailed in [4]. The test state is then defined as

\[
Q_L := \sum_{i,j=1}^K \left( q_{i,j}^{++} |u_{i,L}\rangle \langle u_{j,L}| + q_{i,j}^{+-} |u_{i,L}\rangle \langle v_{j,L}| + q_{i,j}^{-+} |v_{i,L}\rangle \langle u_{j,L}| + q_{i,j}^{--} |v_{i,L}\rangle \langle v_{j,L}| \right),
\]

and it satisfies the constraint \(-\Pi_{-L}^- \leq Q_L \leq 1 - \Pi_{-L}^- \) by construction. We also have

\[
\lim_{L \to \infty} \left( \operatorname{Tr}(-\Delta_L - \mu) Q_L + \int_{C_L} V \rho_{Q_L} \right) = \operatorname{Tr}(-\Delta - \mu) Q + \int_{\mathbb{R}^d} V \rho Q.
\]

Because we obviously have a variational principle in the box,

\[
-\operatorname{Tr}_{L^2(C_L)}(-\Delta_L + V \mathbb{1}_{C_L} - \mu) - \operatorname{Tr}_{L^2(C_L)}(-\Delta_L - \mu) - \mu \int_{C_L} V
\]

\[
= \inf_{-\Pi_{-L}^- \leq Q \leq 1 - \Pi_{-L}^-} \operatorname{Tr}(-\Delta_L - \mu) Q + \int_{C_L} \rho Q V,
\]
we deduce the upper bound
\[
\limsup_{L \to \infty} \left( - \text{Tr}_{L^2(C_L)}(\Delta_L + V 1_{C_L} - \mu) - \mu \int_{C_L} V \right) \leq \text{Tr}(\Delta - \mu)Q + \int_{\mathbb{R}^d} V \rho_Q.
\]

From the variational principle \(2.23\) in the whole space and the density of smooth finite-rank operators in \(K\), as stated in Lemma 3.2, we conclude that
\[
\limsup_{L \to \infty} \left( - \text{Tr}_{L^2(C_L)}(\Delta_L + V 1_{C_L} - \mu) - \mu \int_{C_L} V \right) \leq \text{Tr}(\Delta + V - \mu)Q_V.
\]

In a second step we prove the reverse inequality, with the \(\limsup\) replaced by a \(\liminf\). We consider a sequence \(L_n \to \infty\) realizing this \(\liminf\). Denoting by
\[
Q_n = 1(\Delta L_n + V 1_{C_{L_n}} \leq \mu) - 1(\Delta L_n \leq \mu)
\]
the corresponding state, we know from our estimates that
\[
\| - \Delta L_n - \mu^{1/2}Q_n^{\pm} - \Delta L_n - \mu^{1/2} \|_{S^1(L^2(C_{L_n}))} \leq C,
\]
\[
\| Q_n - \Delta L_n - \mu^{1/2} \|_{S^2(L^2(C_{L_n}))} \leq C,
\]
\[
\| Q_n \| \leq 1,
\]
and that \(1_{C_{L_n}} \rho_{Q_n} = \rho_n^1 + \rho_n^2\) where \(\| \rho_n^2 \|_{L^\infty} \leq C \mu^{(d-1)/2}(L_n)^{-1}\) and \(\rho_n^1\) is bounded in \(L^2(\mathbb{R}^d) + L^{1+2/d}(\mathbb{R}^d)\). Passing to weak limits, using \(V \in L^1(\mathbb{R}^d) \cap L^{1+2/d}(\mathbb{R}^d)\), we deduce that
\[
\liminf_{L \to \infty} \left( - \text{Tr}_{L^2(C_L)}(\Delta_L + V 1_{C_L} - \mu) - \mu \int_{C_L} V \right) \geq \text{Tr}(R^{++} + R^{--}) + \int_{\mathbb{R}^d} V \rho
\]
where \(1_{C_{L_n}} \rho_{Q_n} \to \rho\) and \(1_{C_{L_n}} | - \Delta L_n - \mu^{1/2}Q_n^{\pm} - \Delta L_n - \mu^{1/2}1_{C_{L_n}} \to R^{\pm}\).
Similarly, we have \(1_{C_{L_n}} Q_n - \Delta L_n - \mu^{1/2}1_{C_{L_n}} \to S\) weakly-\(\ast\) in \(S^2(L^2(\mathbb{R}^d))\) and \(1_{C_{L_n}} Q_n 1_{C_{L_n}} \to Q\) weakly-\(\ast\) in \(B\). We now claim that \(Q \in K\), \(\rho_Q = \rho\), \(S = Q - \Delta - \mu^{1/2}\), and \(R^{\pm} = | - \Delta - \mu^{1/2}Q^{\pm} - \Delta - \mu^{1/2}|.\) All this can be seen by testing against smooth functions of compact support, and we skip the details. We conclude that
\[
\liminf_{L \to \infty} \left( - \text{Tr}_{L^2(C_L)}(\Delta_L + V 1_{C_L} - \mu) - \mu \int_{C_L} V \right) \geq \text{Tr}(\Delta + V - \mu)Q_V
\]
by the variational principle \(2.23\). This completes our sketch of the proof of Theorem 5.3. \(\square\)
5.3. **Extension to positive temperature.** In this section we extend our results to smooth partition functions, following [7]. This means we consider a smooth function \( f : \mathbb{R} \to \mathbb{R} \) tending to zero at infinity, and we look for a lower bound on the formal expression

\[
(5.17) \quad \text{Tr} \left( f(-\Delta + V) - f(-\Delta) - f'(-\Delta)V \right).
\]

Our results above dealt with the function \( f_{0,\mu}(x) = -(x - \mu)_- \). Here we typically think of the free energy for a Fermi-Dirac distribution at positive temperature \( T \) and chemical potential \( \mu \), corresponding to

\[
(5.18) \quad f_{T,\mu}(x) = -T \log \left( 1 + e^{-(x - \mu)/T} \right),
\]

which converges to \( f_{0,\mu} \) in the limit \( T \to 0 \). We will, however, be able to treat general functions \( f \), provided they are concave and decay fast enough at infinity. The trick is to write \( f \) as an average of the reference functions \( f_{0,\mu} \) as

\[
(5.19) \quad f(x) = \int_{\mathbb{R}} (x - \lambda)_{-} f''(\lambda) \, d\lambda,
\]

leading to the formal expression

\[
(5.20) \quad \text{Tr} \left( f(-\Delta + V) - f(-\Delta) - f'(-\Delta)V \right) = -\int_{\mathbb{R}} \text{Tr} V (-\Delta + V - \lambda) Q_{\lambda,V} \, f''(\lambda) \, d\lambda,
\]

where

\[
Q_{\lambda,V} := 1(-\Delta + V \leq \lambda) - 1(-\Delta \leq \lambda).
\]

When \( f \) is concave, the integrand in the right side of (5.20) is \( \geq 0 \) since \( \text{Tr} V (-\Delta + V - \lambda) Q_{\lambda,V} \leq 0 \), hence the integral always makes sense in \( \mathbb{R}^+ \cup \{+\infty\} \). We may thus use this as a definition for the left side. In the following result we justify this formal calculation by a thermodynamic limit, and we state the corresponding Lieb-Thirring inequality.

**Theorem 5.4** (Lieb-Thirring inequality for smooth partition functions, \( d \geq 2 \)).

*Let \( f : \mathbb{R} \to \mathbb{R} \) be a concave function such that \( f'' \in L^\infty_{\text{loc}}(\mathbb{R}) \) and

\[
(5.21) \quad \int_{0}^{\infty} \lambda^{1+\frac{d}{2}} |f''(\lambda)| \, d\lambda < \infty
\]

for some \( d \geq 2 \). Then, for \( V \in L^1(\mathbb{R}^d) \cap L^{1+d/2}(\mathbb{R}^d) \), we have

\[
(5.22) \quad \lim_{L \to \infty} \left\{ \text{Tr}_{L^2(C_L)} f \left( -\Delta_L + V \mathds{1}_{C_L} \right) - \text{Tr}_{L^2(C_L)} f(-\Delta_L) - \int_{C_L} \rho \left( -\Delta_L \right) V \right\}
\]

\[
= -\int_{\mathbb{R}} \text{Tr} V (-\Delta + V - \lambda) Q_{\lambda,V} \, f''(\lambda) \, d\lambda,
\]

where, as before, \( -\Delta_L \) is the Laplacian with periodic boundary conditions on \( C_L = [-L/2, L/2]^d \). Moreover, we have the following inequality

\[
(5.23) \quad \int_{\mathbb{R}} \text{Tr} V (-\Delta + V - \lambda) Q_{\lambda,V} \, f''(\lambda) \, d\lambda
\]

\[
\leq L(d) \int_{\mathbb{R}} d\lambda f''(\lambda) \int_{\mathbb{R}^d} dx \left( (V(x) - \lambda)^{1+\frac{d}{2}} - \lambda^{1+\frac{d}{2}} + \frac{2 + d}{2} \lambda^\frac{d}{2} V(x) \right).
\]
rem 5.4 already applies to the Fermi-Dirac free energy that the right side of (5.22) is interpreted in a suitable manner. As such, Theorem 5.3 already applies to the Fermi-Dirac free energy \( f_{T,\mu} \) as given in (5.18), since we have

\[
f''_{T,\mu}(\lambda) = -\frac{e^{(\lambda-\mu)/T}}{T (1 + e^{(\lambda-\mu)/T})^2}
\]

in this case.

**Proof.** The Lieb-Thirring inequality (5.23) is an immediate consequence of (5.22) and we only explain the thermodynamic limit (5.22). First, it follows from the integral formula (5.19) and our assumption (5.21), that \( f(-\Delta_L) \) and \( f(-\Delta_L + V\mathbf{1}_{C_L}) \) are both trace-class. Using (5.19), we obtain the identity

\[
\text{Tr}_{L^2(C_L)} f(-\Delta_L + V\mathbf{1}_{C_L}) - \text{Tr}_{L^2(C_L)} f(-\Delta_L) \geq \int_{C_L} \rho f'(-\Delta_L) V dx
\]

with \( Q_{\lambda,V,L} = 1(\lambda(\Delta_L + V\mathbf{1}_{C_L}) \leq \lambda) - 1(\lambda(\Delta_L) \leq \lambda) \). By the Lieb-Thirring inequality (5.22) in the box, we have for \( \lambda \geq 1/L^2 \)

\[
- \text{Tr}_{L^2(C_L)}(-\Delta_L + V\mathbf{1}_{C_L} - \lambda)Q_{\lambda,V,L}
\]

\[
\leq \tilde{L}(d) \int_{C_L} \left( (V(x) - \lambda)^{1+\frac{d}{2}} - \lambda^{1+\frac{d}{2}} + \frac{2 + d}{2} \lambda^{\frac{d}{2}} V(x) + \frac{\lambda^{\frac{d-2}{2}}}{L} |V(x)| \right) dx
\]

\[
\leq C \left( 1 + \lambda^{\frac{d-2}{2}} + \frac{\lambda^{\frac{d-1}{2}}}{L} \right).
\]

The last estimate is obtained by first replacing the domain of integration \( C_L \) by \( \mathbb{R}^3 \) (the integrand being \( \geq 0 \)), and then using that

\[
\int_{\mathbb{R}^3} \left( (V(x) - \lambda)^{1+\frac{d}{2}} - \lambda^{1+\frac{d}{2}} + \frac{2 + d}{2} \lambda^{\frac{d}{2}} V(x) \right) dx \sim \frac{d(d+2)}{8} \lambda^{\frac{d+2}{2}} \int_{\mathbb{R}^3} |V|^2.
\]

For \( 0 \leq \lambda \leq 1/L^2 \), we use (5.3) instead and obtain

\[
- \text{Tr}_{L^2(C_L)}(-\Delta_L + V\mathbf{1}_{C_L} - \lambda)Q_{\lambda,V,L} \leq C (1 + L^{-d})
\]

Finally, for \( \lambda < 0 \), we simply note that

\[
- \text{Tr}_{L^2(C_L)}(-\Delta_L + V\mathbf{1}_{C_L} - \lambda)Q_{\lambda,V,L} = \text{Tr}(-\Delta_L + V\mathbf{1}_{C_L} - \lambda)_-
\]

This last term vanishes when \( \lambda \leq \inf \sigma(-\Delta_L + V\mathbf{1}_{C_L}) \) and it is bounded by \( \text{Tr}(-\Delta_L + V\mathbf{1}_{C_L}) - \leq C(1 + L^{-d}) \) otherwise. As a conclusion, for \( L \) large enough we have a uniform bound

\[
(5.24) \quad - \text{Tr}_{L^2(C_L)}(-\Delta_L + V\mathbf{1}_{C_L} - \lambda)Q_{\lambda,V,L} \leq C \left( 1 + \lambda^{\frac{d}{d-1}} \right) \mathbf{1}(\lambda \geq -M)
\]

with \( M < \lim \inf_{L \to \infty} \inf \sigma(-\Delta_L + V\mathbf{1}_{C_L}) \). On the other hand we know by Theorem 5.3 that

\[
\lim_{L \to \infty} \text{Tr}_{L^2(C_L)}(-\Delta_L + V\mathbf{1}_{C_L} - \lambda)Q_{\lambda,V,L} = \text{Tr}_V(-\Delta + V - \lambda)Q_{\lambda,V}
\]

for every fixed \( \lambda \). Now (5.22) simply follows from Lebesgue’s dominated convergence theorem. \( \square \)
6. Extension to more general background operators

In the previous sections, we have considered perturbations of a constant density $\rho_0$. Our approach is, in fact, more general and we explain now how to handle other background densities. We typically think of a periodic background but, since we actually need very few assumptions, we state below an abstract theorem. We comment on the assumptions in the periodic case in Section 6.2.

6.1. An abstract Lieb-Thirring inequality with positive background. We consider a bounded-below self-adjoint operator $H$ in $L^2(\mathbb{R}^d, \mathbb{C}^d)$, with $d \geq 2$, and we fix a real number $\mu \in \mathbb{R}$. We assume that there is a constant $C$ and an $\varepsilon > 0$ such that

(A1) $\rho_1(|H - \mu| \leq E)(x) \leq C \left( E + E^{d/2} \right)$ for all $E \geq 0$ and a.e. $x \in \mathbb{R}^d$;

(A2) $\left\| \mathbf{1}_{\{\mu \leq H \leq \mu + \varepsilon\}} \right\|_{L^1(\mathbb{R}^d)} \leq C \left\| V \mathbf{1}_{\{\mu - \varepsilon \leq H \leq \mu\}} \right\|_{L^2(\mathbb{R}^d)}$ for all $V \in L^2(\mathbb{R}^d)$;

(A3) $\rho_0(x) := \rho_1(H \leq \mu)(x) \geq \varepsilon$ for a.e. $x \in \mathbb{R}^d$.

We define

$$\Pi^- := \mathbf{1}_{(-\infty, \mu]}(H) \quad \text{and} \quad \Pi^+ = 1 - \Pi^-.$$ 

We emphasize that (A1) implies that $\rho_1(H = \mu) \equiv 0$, hence that $\mu$ is not an eigenvalue of $H$. With respect to the projections $\Pi^-$ and $\Pi^+$ we can decompose any bounded operator $Q = Q^{++} + Q^{+-} + Q^{-+} + Q^{--}$. Similarly as in Definition 2.1, we define the relative kinetic energy by

$$\text{Tr}_0(H - \mu)Q = \text{Tr} |H - \mu|^{1/2} (Q^{++} - Q^{--}) |H - \mu|^{1/2}$$

for any bounded self-adjoint operator $Q$ such that $|H - \mu|^{1/2} Q^{\pm \pm} |H - \mu|^{1/2}$ are trace-class.

**Theorem 6.1** (Abstract Lieb-Thirring inequality, density version, $d \geq 2$). We assume that the bounded-below self-adjoint operator $H$ satisfies (A1)–(A3). Let $Q$ be a self-adjoint operator such that $-\Pi^- \leq Q \leq \Pi^+$ and such that $|H - \mu|^{1/2} Q^{\pm \pm} |H - \mu|^{1/2}$ are trace-class. Then $Q$ is locally trace-class and the corresponding density satisfies

$$\rho_Q \in L^{1 + \frac{2}{d}}(\mathbb{R}^d) + L^2(\mathbb{R}^d).$$

Moreover, there exists a positive constant $K$ (depending only on $d$, $q$, $\mu$, $C$ and $\varepsilon$) such that

$$\text{Tr}_0(H - \mu)Q \geq K \int_{\mathbb{R}^d} \left( \rho_0(x) + \rho_Q(x) \right)^{1 + \frac{2}{d}} - \rho_0(x)^{1 + \frac{2}{d}} - \frac{2 + d}{d} \rho_0(x)^{\frac{2}{d}} \rho_Q(x) \, dx$$

with $\rho_0(x)$ the background density of $\Pi^-$, defined above in (A3).

**Remark 6.1.** For simplicity we restrict our attention to $d \geq 2$ but, with appropriate modifications, a similar result holds for $d = 1$. If in Assumption (A1) the exponent $d/2$ is replaced by $\delta \geq 1$, our method still applies but the resulting lower bound is of course different. For instance, if relativistic effects are taken into account, $d/2$ should be $d$ in (A1), in which case the exponent $1 + 2/d$ in (6.2) becomes $1 + 1/d$. 
Remark 6.2. Similarly to Theorem 2.3 it is possible to deduce from (6.2) a dual estimate on \( \text{Tr}_V (H + V - \mu) Q_V \), where \( Q_V = 1 (H + V \leq \mu) - \Pi^- \), for any potential \( V \in L^2 (\mathbb{R}^d) \cap L^{1+d/2} (\mathbb{R}^d) \). For brevity we will not discuss this here.

In applications, we typically think of \( H = -\Delta + W (x) \) where \( W \) is a sufficiently regular function, and of \( \mu \) strictly above the infimum of the essential spectrum of \( H \). In Assumption (A1), the \( E^{d/2} \) behavior of the density for large \( E \) is a rather general fact which we discuss below. On the other hand, the small \( E \) behavior in (A1) as well as (A2) are assumptions on \( H \) close to the Fermi surface. Vaguely speaking, (A1) is a (rather weak) assumption on the regularity of the spectral projections uniformly in \( x \)-space, whereas (A2) controls the interactions between particles inside and outside the Fermi sea.

Next we show how to verify the large \( E \) behavior in (A1), under the assumption that \( H = -\Delta + W (x) \) with \( W \) bounded from below.

Lemma 6.1. Let \( W \in L^1_{\text{loc}} (\mathbb{R}^d) \) such that \( W^- \in L^\infty (\mathbb{R}^d) \), and consider the Friedrichs self-adjoint extension of \( -\Delta + W \) on \( C_c^\infty (\mathbb{R}^d) \). Then \( \rho_t (-\Delta + W \leq E) \) is uniformly bounded on \( \mathbb{R}^d \) for every \( E \in \mathbb{R} \)

\[
\rho \left[ \mathbbm{1} (-\Delta + W \leq E) \right] \leq \left( \frac{e}{2\pi d} \right)^{d/2} (|W^-|_{L^\infty} + E)^{d/2},
\]

holds almost everywhere.

Proof. Since \( W \) is bounded from below, we have, by the Feynman-Kac formula,

\[
\rho \left[ e^{-t(-\Delta + W)} \right] \leq \frac{e^{t|W^-|_{L^\infty}}}{(4\pi t)^{d/2}}
\]

where \( \rho [A] (x) = A (x, x) \) denotes the density of an operator \( A \). Using the inequality \( \mathbbm{1} (x \leq E) \leq e^{-t(x-E)} \) we deduce that

\[
\rho \left[ \mathbbm{1} (-\Delta + W \leq E) \right] \leq e^{tE} \rho \left[ e^{-t(-\Delta + W)} \right] \leq \frac{e^{t(|W^-|_{L^\infty} + E)}}{(4\pi t)^{d/2}}.
\]

Optimizing this bound with respect to \( t \) gives the result. \( \square \)

6.2. Application to periodic backgrounds. In this section we restrict ourselves to periodic systems, that is, we take

\[
H = -\Delta + W (x)
\]

where \( W \) is a \( \mathbb{Z}^d \)-periodic function which we assume to be sufficiently regular. Of course, we could as well consider other lattices than \( \mathbb{Z}^d \). It is well known, see, e.g., [31], Sec. XIII.16, that the spectrum of \( H \) is the union of bands

\[
\sigma (H) = \sigma_{\text{ess}} (H) = \bigcup_{n \geq 1} \{ \lambda_n (\xi), \xi \in [-\pi, \pi]^d \},
\]

where \( \lambda_n (\xi) \) denotes the sequence of Bloch-Floquet eigenvalues of \( H \) with corresponding eigenvectors \( u_n (\xi, x) \). Each \( \lambda_n \) is a periodic Lipschitz function of \( \xi \), but the map \( \xi \mapsto u_n (\xi) \in L^2 ((0, 1)^d) \) is only piecewise smooth because of possible degeneracies. Writing for instance \( H = -\Delta/2 + (-\Delta/2 + W) \geq -\Delta/2 - C \) and comparing the \( \lambda_n (\xi) \) with the eigenvalues of the periodic Laplacian in each Bloch sector, it can be seen that

\[
\lambda_n (\xi) \geq a n^{2/d} - b
\]
for some constants $a, b > 0$ independent of $\xi$. Hence for every fixed $\mu \in \mathbb{R}$, there is only a finite number of $n$'s such that $\lambda_n(\xi) = \mu$ for some $\xi \in [-\pi, \pi]^d$.

Let us fix $\mu > \inf \sigma_{\text{ess}}(-\Delta + W)$. Then we have

$$\rho_0(x) = (2\pi)^{-d} \sum_{n \geq 1} \int_{[-\pi, \pi]^d} d\xi \, 1(\lambda_n(\xi) \leq \mu) |u_n(\xi, x)|^2.$$ 

Since $u_1(0, x)$ is strictly positive, we easily conclude, by continuity in $\xi$, that $\rho_0(x) \geq \varepsilon > 0$, and hence that (A3) is verified.

Now we give some ideas on how one can verify Assumptions (A1) and (A2) in practice. First, we have

$$\rho_1([-\Delta + W - \mu|E|](x) = (2\pi)^{-d} \sum_{n \geq 1} \int_{[-\pi, \pi]^d} d\xi \, 1(|\lambda_n(\xi) - \mu| \leq E) |u_n(\xi, x)|^2.$$ 

Under suitable assumptions on $W$, $u_n(\cdot, \xi)$ is bounded in $L^\infty(\mathbb{R}^d)$, uniformly with respect to $\xi$, for each fixed $n \geq 1$. In this case, Assumption (A2) follows if the eigenvalues satisfy the following property:

$$(6.3) \quad |\{\xi \in [-\pi, \pi]^d, |\lambda_n(\xi) - \mu| \leq E\}| \leq CE.$$ 

This is generically true: If there is a unique $n$ such that the graph of $\lambda_n$ crosses $\mu$, and if $\nabla \lambda_n(\xi) \neq 0$ for all $\xi$ with $\lambda_n(\xi) = \mu$, one can easily verify that (6.3) is satisfied ($\mu_1$ in Figure 1). At a point $\xi$ such that $\nabla \lambda_n(\xi) = 0$, the validity of (6.3) depends on the order of vanishing at this point. If, for instance, the second derivative is invertible, then (6.3) holds in any dimension $d \geq 2$ ($\mu_5$ in Fig. 1). Only the $\xi$'s which have a high (depending on the dimension $d$) order of vanishing can make (6.3) fail. When the Fermi surface is disconnected, each component being as before, the result is the same ($\mu_3$ in Fig. 1). Finally, if $\lambda_n(\xi) = \lambda_m(\xi) = \mu$ for $n \neq m$, the analysis is similar. For instance, transversal crossing of surfaces ($\mu_4$ in Fig. 1) as well as Dirac-type cone singularities ($\mu_2$ in Fig. 1) are allowed.

Verifying (A2) is much more subtle and requires a detailed analysis of the bands close to the Fermi surface. An exception is when $\mu$ lies in or at the edge of a gap, in which case (A2) is trivially satisfied (the estimate on $\rho_{Q_{\pm}}$ in $L^2$ was already obtained in this case in [3]). In the case where $\mu$ lies in the interior of the essential spectrum, we expect (A2) to be true, as soon as the Fermi surface is sufficiently regular. To make this intuition precise, a possible line of attack could be as follows. We assume again, for simplicity, that there is a unique $n$ such that the graph of $\lambda_n$ crosses $\mu$. Then we have to prove that the operator whose kernel is

$$(6.4) \quad \int_{\mu - \varepsilon \leq \lambda_n(\xi) \leq \mu} \int_{\mu \leq \lambda_n(\xi') \leq \mu + \varepsilon} d\xi \, u_n(\xi, x) u_n(\xi', x') u_n(\xi', x') \frac{u_n(\xi', x) u_n(\xi', x')}{\sqrt{\mu - \lambda_n(\xi') \sqrt{\lambda_n(\xi') - \mu}}}$$ 

is bounded on $L^2(\mathbb{R}^d)$. The main idea is now that, for the question of boundedness, each Bloch function $u_n(\xi, x)$ can be replaced by the corresponding plane wave $\exp(i x \cdot \xi)$. Arguments of this sort have been carried out in a similar context in [3] [2] [8], for instance. When $\nabla \lambda_n(\xi) \neq 0$ for all $\xi$ such that $\lambda_n(\xi) = \mu$, the Fermi surface is smooth and can be locally replaced by a sphere. This reduces the computation to what we have done in Lemma 3.4 in the translation-invariant case.

This concludes our intuitive description of how to verify Assumptions (A1) and (A2) for periodic backgrounds. Rendering all this rigorous is beyond the scope of this paper, however.
Figure 1. Typical Fermi surfaces for a two-dimensional periodic Schrödinger operator.

6.3. **Sketch of the proof of Theorem 6.1** The proof follows again the same two steps as that of Theorem 2.1.

**Step 1. Estimate on $Q_{\pm \pm}$.** We start by estimating the diagonal densities $\rho_{Q_{\pm \pm}}$. Following the proof of Lemma 3.3, Assumption (A1) implies that

$$\text{Tr} \left| H - \mu \right| \gamma \geq \int_{\mathbb{R}^d} \hat{R}(\rho_{\gamma}(x)) \, dx$$

for every $0 \leq \gamma \leq 1$, where

$$\hat{R}(\rho) = \int_0^\infty \left( \sqrt{\rho} - \sqrt{Cq(e^{\epsilon/2} + e^{d/2})} \right)^2 \, de.$$

One has

$$\hat{R}(\rho) \sim_{0} \frac{\rho^2}{6Cq} \quad \text{and} \quad \hat{R}(\rho) \sim_{\infty} \frac{d^2}{(d+4)(d+2)(Cq)^2/d} \rho^{1+2/d}.$$

Since $\epsilon < \rho_0 \leq M$ by Assumptions (A1) and (A3), we therefore also have a lower bound

$$\int_{\mathbb{R}^d} \hat{R}(\rho_{\gamma}(x)) \, dx \geq c \int_{\mathbb{R}^d} \left( (\rho_0(x) + \rho_{\gamma}(x))^{1+\frac{2}{d}} - \rho_0(x)^{1+\frac{2}{d}} - \frac{2 + d}{d} \rho_0(x)^{\frac{2}{d}} \rho_{\gamma}(x) \right) \, dx$$

with a constant $c$ only depending on $d$, $\epsilon$, $M$ and $Cq$. Applying these estimates to $\gamma = \pm Q_{\pm \pm}$, we obtain the estimate analogous to (3.9).
Step 2. Estimate on $Q^{\frac{1}{2}}$. Following the corresponding step in the proof of Theorem 2.1 a bound of the form

$$\int_{\mathbb{R}^d} |\rho_{Q^\pm}|^2 \leq c \text{Tr}_0(H - \mu)Q$$

can be derived from the estimate

$$(A2) \left\| \frac{\Pi^+}{|H - \mu|^{1/4}} V \frac{\Pi^-}{|H - \mu|^{1/4}} \right\|_{\mathcal{S}_2} \leq C \|V\|_{L^2}$$

for all $V \in L^2(\mathbb{R}^d)$.

We now explain how to derive this bound from (A1) and (A2).

To this end, we split

$$\Pi^- = \Pi^-_\leq + \Pi^-_\geq = \mathbb{1}(\mu - \varepsilon \leq H \leq \mu) + \mathbb{1}(H < \mu - \varepsilon),$$

$$\Pi^+ = \Pi^+_\leq + \Pi^+_\geq = \mathbb{1}(\mu \leq H \leq \mu + \varepsilon) + \mathbb{1}(H > \mu + \varepsilon)$$

and estimate each term separately. The bound on $\Pi^+_\leq V \Pi^-_\geq$ is exactly our assumption (A2). For $\Pi^+_\leq V \Pi^-_\geq$, we write that

$$(6.5) \left\| \frac{\Pi^+_\leq}{|H - \mu|^{1/4}} V \frac{\Pi^-_\geq}{|H - \mu|^{1/4}} \right\|^2 \leq \frac{1}{\sqrt{\varepsilon}} \left\| V \frac{\Pi^-_\geq}{|H - \mu|^{1/4}} \right\|^2 = \frac{1}{\sqrt{\varepsilon}} \int_{\mathbb{R}^d} |V|^2 \rho\left(\frac{\Pi^-_\geq}{|H - \mu|^{1/2}}\right).$$

The density appearing on the right side is uniformly bounded on $\mathbb{R}^d$. Indeed, one has, more generally,

$$(6.6) \rho\left(\frac{\mathbb{1}(|H - \mu| \leq a)}{|H - \mu|^{1/2}}\right) \in L^\infty(\mathbb{R}^d)$$

for every fixed $a > 0$. To see this, we write

$$\frac{\mathbb{1}(|x| \leq a)}{|x|^{1/2}} = \frac{1}{2} \int_0^\infty \mathbb{1}(|x| \leq \min(y, a)) \frac{dy}{y^{3/2}}$$

which, by (A1), implies the uniform bound

$$\rho\left(\frac{\mathbb{1}(|H - \mu| \leq a)}{|H - \mu|^{1/2}}\right) \leq C \frac{1}{2} \int_0^\infty \left(\min(y, a) + \min(y, a)^{d/2}\right) \frac{dy}{y^{3/2}}.$$

Inserting (6.6) in (6.5) and using the fact that $H$ is bounded from below, we obtain the desired estimate for $\Pi^+_\leq V \Pi^-_\geq$. The term corresponding to $\Pi^+_\leq V \Pi^-_\geq$ is estimated similarly, using the fact that $\Pi^-_\leq |H - \mu|^{-1/4} \leq \varepsilon^{-1/4}$.

This completes our sketch of the proof of Theorem 6.1. $\square$

REFERENCES


A POSITIVE DENSITY ANALOGUE OF THE LIEB-THIRRING INEQUALITY


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