Observation of direct CP violation in the measurement of the Cabibbo-Kobayashi-Maskawa angle $\gamma$ with $B^\pm \to D^{(*)}K^{(*)\pm}$ decays

I. INTRODUCTION AND OVERVIEW

In the Standard Model (SM), the mechanism of CP violation in weak interactions arises from the joint effect...
SM [2]. In this context, the angle $\gamma$ is particularly relevant since it is the only CP-violating parameter that can be cleanly determined using tree-level $B$ meson decays [3]. In spite of a decade of successful operation and experimental efforts by the $B$ factory experiments, BABAR and Belle, $\gamma$ is poorly known due to its large statistical uncertainty. Its precise determination is an important goal of present and future flavor-physics experiments.

Several methods have been pursued to extract $\gamma$ [4–9]. Those using charged $B$ meson decays into $D^{(*)}K^{\pm}$ and $DK^{\pm}$ final states, denoted generically as $D^{(*)}K^{\pm}$, yield low theoretical uncertainties since the decays involved do not receive contributions from penguin diagrams (see Fig. 1). This is a very important distinction from most other measurements of the angles. Here, the symbol $D^{(*)}$ refers to a $D^0$ ($D^{*0}$) or a $D^\pm$ ($D^{*\pm}$) meson, and $K^{\pm}$ refers to $K^0(S)$ states, such as $K^0_S$, $K_{L/\pi}$, and $K_{S/\pi}$ states, such as $K^0_S$. In the case of a nonzero weak phase $\gamma$, the model-based approach can be used. The interference between the CKM- and color-favored $b \rightarrow c\bar{u}s$ and the suppressed $b \rightarrow u\bar{c}s$ amplitudes, which arises when the $D^0$ from a $B^- \rightarrow D^0K^-$ decay [10] (and similarly for the other related $B$ decays) is reconstructed in a final state which can be produced also in the decay of a $D^0$ originating from $B^- \rightarrow D^0K^-$ (see Fig. 1). The interference between the $b \rightarrow c\bar{u}s$ and $b \rightarrow u\bar{c}s$ tree amplitudes results in observables that depend on their relative weak phase $\gamma$, on the magnitude ratio $r_B \equiv |\mathcal{A}(b \rightarrow u\bar{c}s)/\mathcal{A}(b \rightarrow c\bar{u}s)|$, and on the relative strong phase $\delta_B$ between the two amplitudes. In the case of a nonzero weak phase $\gamma$ and a nonzero strong phase $\delta_B$, the $B^-$ and $B^+$ decay rates are different, which is a manifestation of direct CP violation. The hadronic parameters $r_B$ and $\delta_B$ are not precisely known from theory, and may have different values for $D^0K^\pm$, $D^\pm K^\pm$, and $DK^{\pm\pm}$ final states. They can be measured directly from data by simultaneously reconstructing several $D$-decay final states.

The three main approaches employed by the $B$ factory experiments are:

(i) the Dalitz plot or Giri-Grossman-Soffer-Zupan (GGSZ) method, based on three-body, self-conjugate final states, such as $K^0_S\pi^+\pi^-$ [7];

(ii) the Gronau-London-Wyler (GLW) method, based on decays to CP-eigenstate final states, such as $K^+K^-$ and $K^0_S\pi^0$ [8];

(iii) the Atwood-Dunietz-Soni (ADS) method, based on $D$ decays to doubly-Cabibbo-suppressed final states, such as $D^0 \rightarrow K^+\pi^-$ [9].

To date, the GGSZ method has provided the highest statistical power in measuring $\gamma$. The other two methods provide additional information that can further constrain the hadronic parameters and thus allow for a more robust determination of $\gamma$. The primary issue with all these methods is the small product branching fraction of the decays involved, which range from $5 \times 10^{-6}$ to $5 \times 10^{-4}$, and the small size of the interference, proportional to $r_B \approx c_F|V_{ub}V_{cb}^*|/|V_{us}V_{cb}^*| \approx 0.1$, where $c_F = 0.2$ is a color suppression factor [11–13]. Therefore a precise determination of $\gamma$ requires a very large data sample and the combination of all available methods involving different $D$ decay modes.

Recently, Belle [14] and LHCb [15] have presented the preliminary results of the combination of their measurements related to $\gamma$, yielding $\gamma$ to be $(68^{+15}_{-14})^\circ$ and $(71_{-16}^{+15})^\circ$, respectively. Attempts to combine the results by BABAR, Belle, CDF, and LHCb have been performed by the CKMfitter and UTfit groups [2]. Their most recent results are $(66 \pm 12)^\circ$ and $(72 \pm 9)^\circ$, respectively.

The BABAR experiment [16] at the PEP-II asymmetric-energy $e^+e^-$ collider at SLAC has analyzed charged $B$ decays into $D^{(*)}K^\pm$, $D^{*\pm}K^\pm$, and $DK^{\pm\pm}$ final states using the GGSZ [17–19], GLW [20–22], and ADS [22–24] methods, providing a variety of measurements and constraints on $\gamma$. The results are based on a data set collected at a center-of-mass energy equal to the mass of the $Y(4S)$ resonance, and about 10% of data collected 40 MeV below. We present herein the combination of published BABAR measurements using detailed information on correlations between parameters that we have not previously published. This combination represents the most complete study of the data sample collected by BABAR and benefits from the possibility to access and reanalyze the data sample (see Sec. II for details).

Other analyses related to $\gamma$ [25–27] or $2\beta + \gamma$ [28,29] have not been included, because the errors on the experimental measurements are too large.

**II. INPUT MEASUREMENTS**

In the GGSZ approach, where $D$ mesons are reconstructed into the $K^0_S\pi^+\pi^-$ and $K^0_SK^+K^-$ final states [17–19], the signal rates for $B^- \rightarrow D^{(*)}K^\pm$ and $B^+ \rightarrow DK^{\pm\pm}$ decays are analyzed as a function of the position in the Dalitz plot of squared invariant masses $m^2 \equiv m^2(K^0_S h^\pm)$, $m^2 \equiv m^2(K^0_S h^\perp)$, where $h$ is either a charged pion or a kaon ($h = \pi, K$). We assume no CP violation in the neutral $D$ and $K$ meson systems and neglect small $D^0 - \bar{D}^0$ mixing effects [30,31], leading to $\mathcal{A}(m^2, m^2 \equiv m^2(K^0_S h^\perp)) \equiv \mathcal{A}(m^2, m^2 \equiv m^2(K^0_S h^\perp))$, where $\mathcal{A}$ ($\bar{\mathcal{A}}$) is the $D^0$ ($\bar{D}^0$) decay amplitude. In this case, the signal decay rates can be written as [32]...
with \( \mathcal{A}_+ \equiv \mathcal{A}(m_{Z}^{2}, m_{Z}^{2}) \) and \( \mathcal{A}_+^\dagger \) is the complex conjugate of \( \mathcal{A}_+ \). The symbol \( \lambda \) for \( B^\pm \to D^0 K^\pm \) accounts for the different CP parity of the \( D^* \) when it is reconstructed into \( D^0 \pi \) (\( \lambda = +1 \)) and \( D\gamma \) (\( \lambda = -1 \)) final states, as a consequence of the opposite \( CP \) eigenvalue of the \( \pi^0 \) and the photon [33]. Here, \( r_B^{(s)} \) and \( r_{rs} \) are the magnitude ratios between the \( b \to u\bar{c}s \) and \( b \to c\bar{u}s \) amplitudes for \( B^\pm \to D^{(*)}\!K^\pm \) and \( B^\pm \to D^*K^{\pm*} \) decays, respectively, and \( \delta_B^{(s)} \), \( \delta_s \) are their relative strong phases. The analysis extracts the \( CP \)-violating observables [19]

\[
Z_B^{(s)} = x_B^{(s)} + iy_B^{(s)}, \quad Z_s = x_s + iy_s,
\]

defined as the suppressed-to-favored complex amplitude ratios \( Z_B^{(s)} = r_B^{(s)} \, e^{i(\delta_B^{(s)} - \gamma)} \) and \( Z_s = k r_s \, e^{i(\delta_s - \gamma)} \), for \( B^\pm \to D^{(*)}\!K^\pm \) and \( B^\pm \to D^*K^{\pm*} \) decays, respectively. The hadronic parameter \( \kappa \) is defined as

\[
\kappa e^{i\delta_\kappa} = \frac{\int A_s(p)A_t(p)e^{i\delta(p)}dp}{\sqrt{\int A_s^2(p)dp \int A_t^2(p)dp}},
\]

where \( A_s(p) \) and \( A_t(p) \) are the magnitudes of the \( b \to c\bar{u}s \) and \( b \to u\bar{c}s \) amplitudes as a function of the \( B^\pm \to D^{(*)}\!K^\pm \) phase-space position \( p \), and \( \delta(p) \) is their relative strong phase. This coherence factor, with \( 0 < \kappa < 1 \) in the most general case and \( \kappa = 1 \) for two-body \( B \) decays, accounts for the interference between \( B^\pm \to D^*K^{\pm*} \) and other \( B^\pm \to D^{(*)}\!K^\pm \) decays, as a consequence of the \( K^{*}\! \) natural width [12]. In our analysis, \( \kappa \) has been fixed to 0.9 and a systematic uncertainty has been assigned, varying its value by \( \pm 0.1 \), as estimated using a Monte Carlo simulation based on the Dalitz plot model of \( B^\pm \to D^{(*)}\!K^\pm \) decays [18]. Thus the parameter \( \delta_\kappa \) is an effective strong-phase difference averaged over the phase space.

Table I summarizes our experimental results for the \( CP \)-violating parameters \( Z_B^{(s)} \) and \( Z_s \). Complete \( 12 \times 12 \) covariance matrices for statistical, experimental systematic, and amplitude-model uncertainties are reported in Ref. [17]. The \( Z_B^{(s)} \) and \( Z_s \) observables are unbiased and have Gaussian behavior with small correlations, even for low values of \( r_B^{(s)} \), \( k r_s \), and relatively low-statistics samples. Furthermore, their uncertainties have a minimal dependence on their central values and are free of physical bounds [19]. These good statistical properties allow for easier combination of several measurements into a single result. For example, the rather complex experimental GGSZ likelihood function can be parametrized by a 12-dimensional (correlated) Gaussian probability density function (P.D.F.), defined in the space of the \( Z_B^{(s)} \) and \( Z_s \) measurements from Table I. After this combination has been performed, the values of \( \gamma \) and of the hadronic parameters \( r_B^{(s)} \), \( k \), \( \delta_B^{(s)} \), and \( \delta_s \) can be obtained.

The \( D \) decay amplitudes \( \mathcal{A}_\pm \) have been determined from Dalitz plot analyses of tagged \( D^0 \) mesons from \( D^{(*)}\!\pi^\pm \) decays produced in \( e^+e^- \to c\bar{c} \) events [18,34], assuming an empirical model to describe the variation of the amplitude phase as a function of the Dalitz plot variables. A model-independent, binned approach also exists [7,35], which optimally extracts information on \( \gamma \) for higher-statistics samples than the ones available. This type of analysis has been performed as a proof of principle by the Belle collaboration [36], giving consistent results to the model-dependent approach [37]. The LHCb collaboration has also released results of a model-independent GGSZ analysis [38].

In order to determine \( \gamma \) with the GLW method, the analyses measure the direct \( CP \)-violating partial decay rate asymmetries

\[
A_{CP}^{(s)} = \frac{\Gamma(B^- \to D^{(*)}\!K^-)}{\Gamma(B^- \to D^{(*)}\!K^+)} - \frac{\Gamma(B^+ \to D^{(*)}\!K^+)}{\Gamma(B^+ \to D^{(*)}\!K^-)},
\]

and the ratios of charge-averaged partial rates using \( D \) decays to \( CP \) and flavor eigenstates,

\[
R_{CP}^{(s)} = 2 \frac{\Gamma(B^- \to D^{(*)}\!K^-)}{\Gamma(B^- \to D^{(*)}\!K^+)} + \frac{\Gamma(B^+ \to D^{(*)}\!K^+)}{\Gamma(B^+ \to D^{(*)}\!K^-)},
\]

where \( D^{(*)}\!K \) refers to the \( CP \) eigenstates of the \( D^{(*)} \) meson system. We select \( D \) mesons in the \( CP \)-even eigenstates \( \pi^-\pi^+ \) and \( K^-K^+ \) (\( D_{CP}^- \)), in the \( CP \)-odd eigenstates \( K_S^0\pi^0 \), \( K^0_s\phi \), and \( K^0_s\omega \) (\( D_{CP}^0 \)), and in the non-\( CP \) eigenstate \( K^-\pi^+ \) (\( D^0 \) from \( B^- \to D^0\!h^- \)) or \( K^+\pi^- \) (\( D^0 \) from \( B^+ \to D^0\!h^+ \)). We reconstrue \( D^* \) mesons in the states \( D\pi^0 \) and \( D\gamma \). The observables \( A_{CP}^{(*)} \) and \( R_{CP}^{(*)} \) for \( B^\pm \to D^*K^{\pm*} \) decays are defined similarly.
For later convenience, the GLW observables can be related to $Z^{(s)}_\pm$ and $Z_{\pm}$ (neglecting mixing and CP violation in neutral $D$ decays) as

$$A_{CP\pm}^{(s)} = \pm \frac{x^{(s)}_\mp - x^{(s)}_+}{1 + |z^{(s)}_\pm|^2 \pm (x^{(s)}_\mp + x^{(s)}_+)}$$  \hspace{1cm} (6)$$

and

$$R_{CP\pm}^{(s)} = 1 + |z^{(s)}_\pm|^2 \pm (x^{(s)}_\mp + x^{(s)}_+).$$  \hspace{1cm} (7)$$

where $|z^{(s)}|^2$ is the average value of $|Z^{(s)}_\pm|^2$ and $|Z^{(s)}_\pm|^2$. For $B^\pm \to DK^{\mp\mp}$ decays, similar relations hold to Eqs. (6) and (7), with $\kappa = 1$, since the effects of the non-$K^*$ $B \to DK\pi$ events and the width of the $K^*$ are incorporated into the systematic uncertainties of the $A_{CP\pm}^{(s)}$ and $R_{CP\pm}^{(s)}$ measurements [22].

Table II summarizes the results obtained for the GLW observables. In order to avoid overlaps with the samples selected in the Dalitz plot analysis, the results for $B^\pm \to D_{CP\mp} K^{\mp\mp}$ decays are corrected by removing the contribution from $D_{CP\mp} \to K^0_S \phi, \phi \to K^- K^-$ candidates [20]. For the decays $B^\pm \to D^{*\mp}_{CP\mp} \to K^0_S \phi, \phi \to K^- K^-$, where the favored decays $B^- \to [K^- \pi^-]_D K^-$, $B^- \to [K^+ \pi^-]_D K^-$, $B^- \to [K^- \pi^-]_D K^+$, and $B^- \to [K^+ \pi^-]_D K^-$ serve as normalization so that many systematic uncertainties cancel. The rates in Eq. (8) depend on $\gamma$ and the $B$-decay hadronic parameters. They are related to $Z^{(s)}_\pm$ and $Z_{\pm}$ through

$$R^{(s)}_\pm = r^{(s)}_B^2 + r_D^2 + 2\lambda r_D \left[ x^{(s)}_\pm \cos \delta_D - y^{(s)}_\pm \sin \delta_D \right],$$  \hspace{1cm} (9)$$

where $r_D = |A(D^0 \to K^+ \pi^-)/A(D^0 \to K^- \pi^+)|$ and $\delta_D$ are the ratio between magnitudes of the suppressed and favored $D$-decay amplitudes and their relative strong phase, respectively. As in Eq. (1), the symbol $\lambda$ for $B^\pm \to D^{*\mp} K^{\mp\mp}$ decays accounts for the different CP parity of $D^* \to D \pi^0$ and $D^* \to D \gamma$. The values of $r_D$ and $\delta_D$ are taken as external constraints in our analysis. As for the GLW method, the effects of other $B^\pm \to D K_S^{\mp\mp}$ events, not going through $K^{\mp\mp}$, and the $K^{\mp\mp}$ width, are incorporated in the systematic uncertainties on $R^{(s)}_\pm$. Thus similar relations hold for these observables with $\kappa = 1$.

The choice of the observables $R_\pm$ (and similarly for $R^{(s)}_\pm$) rather than the original $R^{(s)}_+ - R^{(s)}_-$) is well-behaved since the uncertainty on $A_{ADS}$ depends on the central value of $R_{ADS}$, while $R_+ - R_-$ are statistically independent observables. Although systematic uncertainties are largely correlated, the measurements of $R_+$ and $R_-$ are effective uncorrelated since the total uncertainties are dominated by the statistical component.

We have also reconstructed $B^\pm \to [K^\mp \pi^\mp \pi^0]_D K^{\mp}$ decays [24] from which the observables $R_{K^\mp \pi^\mp \pi^0}$ have been measured, which are related to the GGSZ observables as

$$R_{K^\mp \pi^\mp \pi^0} = r^{(s)}_B^2 + r_D^2 + 2\kappa_{K^\mp \pi^\mp \pi^0} r^{K^\mp \pi^\mp \pi^0} [x^{(s)}_\pm \cos \delta_{K^\mp \pi^\mp \pi^0} - y^{(s)}_\pm \sin \delta_{K^\mp \pi^\mp \pi^0}],$$  \hspace{1cm} (9)$$

where $\kappa_{K^\mp \pi^\mp \pi^0}$ is a $D$ decay coherence factor similar to that defined in Eq. (3) for the $B^\pm \to D K_S^{\mp\mp}$ decay, and where $r^{K^\mp \pi^\mp \pi^0}$ and $\delta_{K^\mp \pi^\mp \pi^0}$ are hadronic parameters for $D^0 \to K^{\mp} \pi^\mp \pi^0$ decays analogous to $r_D$ and $\delta_D$. In the ADS method, the $D^0$ meson from the favored $b \to c \bar{u} s$ amplitude is reconstructed in the doubly-Cabibbo-suppressed decay $K^+ \pi^-$, while the $D^0$ from the $b \to u \bar{c} \bar{s}$ suppressed amplitude is reconstructed in the favored decay $K^+ \pi^+$ [22,23]. The product branching fractions for these final states, which we denote as $B^\pm \to [K^{\mp} \pi^-]_D K^\mp$, $B^\pm \to [K^+ \pi^-]_D K^\mp$, $B^\pm \to [K^- \pi^-]_D K^+$, and $B^\pm \to [K^+ \pi^-]_D K^+$, and their CP conjugates, are small ($\sim 10^{-7}$). However, the two interfering amplitudes are of the same order of magnitude, allowing for possible large CP asymmetries. We measure charge-specific ratios for $B^+$ and $B^-$ decay rates to the ADS final states, which are defined as

$$R^{(s)}_\pm = \frac{\Gamma(B^\pm \to [K^\mp \pi^-]_D K^\mp)}{\Gamma(B^\pm \to [K^\mp \pi^\mp]_D K^{\mp\mp})},$$  \hspace{1cm} (8)$$

and similarly for $R^{(s)}_+$.
TABLE III. ADS observables included into the combination for $B^+ \rightarrow D K^-$ with $D \rightarrow K \pi$ (based on 467 million $B \bar{B}$ pairs) and $D \rightarrow K \pi \pi^0$ (based on 474 million $B \bar{B}$ pairs), $B^+ \rightarrow D' K^-$ (467 million $B \bar{B}$ pairs), and $B^+ \rightarrow D^+ K^-$ (379 million $B \bar{B}$ pairs) decays [22–24]. The first uncertainty is statistical, the second is systematic.

\[
\begin{array}{ccc}
B^+ & B^- \\
R_\pi & \frac{B \bar{B}}{(\text{pairs})} & \frac{B \bar{B}}{(\text{pairs})} \\
R_\pi^+ [D \pi^0] & 0.022 \pm 0.009 \pm 0.003 & 0.007 \pm 0.016 \pm 0.007 \\
R_\pi^- [D \gamma] & 0.037 \pm 0.018 \pm 0.009 & 0.019 \pm 0.023 \pm 0.012 \\
R_\pi^0 [D \bar{q} \bar{q}] & 0.054 \pm 0.049 \pm 0.011 & 0.012 \pm 0.012 \pm 0.002 \\
R_\pi [K \pi^0] & 0.005 \pm 0.012 \pm 0.001 & 0.005 \pm 0.012 \pm 0.001 \\
\end{array}
\]

Table III summarizes the measurements of the ADS charge-specific ratios for the different final states. Contrary to the case of the GGSZ and GLW observables, $R_\pi^+$, $R_\pi^-$, and $R_\pi^0$ do not have Gaussian behavior. The experimental likelihood function for each of the four decay modes, shown in Fig. 2 for $B^+ \rightarrow D K^-$ and $B^+ \rightarrow D^+ K^-$ decays, is well described around the best solution by an analytical P.D.F. composed of the sum of two asymmetric Gaussian functions. For the $B^+ \rightarrow D K^-$ channel, we use instead a simple Gaussian approximation since in this case the experimental likelihood scans are not available. The effect of this approximation has been verified to be negligible, given the small statistical weight of this sample in the combination. Measurements using the ADS technique have also been performed by the Belle [42,43], CDF [44], and LHCb collaborations [41], with consistent results.

III. OTHER MEASUREMENTS

Similar analyses related to $\gamma$ measurement have been carried out using the decay $B^+ \rightarrow D K^-$ with the $D \rightarrow \pi^+ \pi^- \pi^0$ final state [25], and the neutral $B$ decay $B^0 \rightarrow D K^0 \pi^0$, with $D \rightarrow K^0 \pi^+ \pi^-$ [26] and $D \rightarrow K^- \pi^0$, $K^0 \pi^+ \pi^0$, $K^0 \pi^+ \pi^- \pi^0$ [27]. For neutral $B$ decays, $r_B$ is naively expected to be larger ($\approx 0.3$) because both interfering amplitudes are color suppressed and thus $c_F \approx 1$. However, the overall rate of events is smaller than for $B^+ \rightarrow D K^-$ decays. The flavor of the neutral $B$ meson is tagged by the charge of the kaon produced in the $K^0 \pi^0$ decay, $K^+ \pi^- \pi^0$.

Experimental analyses of the time-dependent decay rates of $B \rightarrow D^{(*)+} \pi^-$ and $B \rightarrow D^{(*)+} \rho(770)^-$ have also been used to constrain $\gamma$ [28,29]. In these decays, the interference occurs between the favored $b \rightarrow c \bar{u}d$ and the suppressed $b \rightarrow u \bar{c}d$ tree amplitudes with and without $B^0 - \bar{B}^0$ mixing, resulting in a total weak-phase difference $2\beta + \gamma$ [45], where $\beta$ is the angle of the unitarity triangle defined as $\arg(-V_{cb} V_{tb}^* / V_{ub} V_{tb}^*)$. The magnitude ratios between the suppressed and favored amplitudes $r_{f}^* \pi$ and $r_{f} \rho$ are expected to be $\approx 2\%$, and have to be estimated either by analyzing suppressed charged $B$ decays (e.g., $B^+ \rightarrow D^+ \pi^0$) with an isospin assumption or from self-tagging neutral $B$ decays to charmed-strange mesons (e.g., $B^0 \rightarrow D_s^+ \pi^-$) assuming SU(3) flavor symmetry and neglecting contributions from $W$-exchange diagrams [45].

Performing a time-dependent Dalitz plot analysis of $B \rightarrow D^+ K^0 \pi^\pm$ decays [46] could in principle avoid the problem of the smallness of $r$. In these decays the two interfering amplitudes are color suppressed, and it is expected to be $\approx 0.3$, but the overall rate of events is too small with the current data sample.

In both cases, the errors on the experimental measurements are too large for a meaningful determination of $\gamma$, and have not been included in the combined determination of $\gamma$ reported in this paper. However, these decay channels might provide important information in future experiments.
IV. COMBINATION PROCEDURE

We combine all the GGSZ, GLW, and ADS observables (34 in total) to extract $\gamma$ in two different stages. First, we extract the best-fit values for the CP-violating quantities $z^{(s)}$ and $z^{+}$, whose definitions correspond to those for the quantities $z^{(s)}$ and $z^{+}$ of the GGSZ analysis given in Eq. (2).

Their best-fit values are obtained by maximizing a combined likelihood function constructed as the product of partial likelihood P.D.F.s for GGSZ, GLW, and ADS measurements. The GGSZ likelihood function uses a 12-dimensional Gaussian P.D.F. with measurements $z^{(s)}$ and $z^{+}$ and their covariance matrices for statistical, experimental systematic and amplitude-model uncertainties, and mean (expected) values $\bar{z}^{(s)}$ and $\bar{z}^{+}$. Similarly, the GLW likelihood is formed as the product of four-dimensional Gaussian P.D.F.s for each $B$ decay with measurements $A_{CP}^{s}$, $A_{CP}^{+}$, $B_{CP}^{s}$, and $B_{CP}^{+}$, and their covariance matrices, and expected values given by Eqs. (6) and (7) after replacing the $z^{(s)}$ and $z^{+}$ observables by the $\bar{z}^{(s)}$ and $\bar{z}^{+}$ parameters. Finally, the ADS P.D.F. is built from the product of experimental likelihoods shown in Fig. 2. With this construction, GGSZ, GLW, and ADS observables are taken as uncorrelated. Similarly, the individual measurements are considered uncorrelated as the experimental uncertainties are dominated by the statistical component.

The combination requires external inputs for the $D$ hadronic parameters $r_\phi$, $\delta_\phi$, $r_{K\pi\pi}$, $\delta_{K\pi\pi}$, and $\kappa_{K\pi\pi}$. We assume Gaussian P.D.F.s for $r_\phi = 0.0575 \pm 0.0007$ [30] and $r_{K\pi\pi} = 0.0469 \pm 0.0011$ [47], while for the other three we adopt asymmetric Gaussian parameterizations based on the experimental likelihoods available either from world averages for $\delta_\phi = (202.0^{+9.9}_{-11.2})^\circ$ [30] or from the CLEO collaboration for $\delta_{K\pi\pi} = (47_{-17}^{+14})^\circ$ and $\kappa_{K\pi\pi} = 0.84 \pm 0.07$ [48]. The values of $\delta_\phi$ and $\delta_{K\pi\pi}$ have been corrected for a shift of 180° in the definition of the phases between Refs. [23,24] and Refs. [30,48]. The correlations between $r_\phi$ and $\delta_\phi$, and between $K_{K\pi\pi}$ and $\delta_{K\pi\pi}$, are small and have been neglected. All five external observables are assumed to be uncorrelated with the rest of the input observables.

The results for the combined CP-violating parameters $z^{(s)}$ and $z^{+}$ are summarized in Table IV. Figure 3 shows comparisons of two-dimensional regions corresponding to one-, two-, and three-standard-deviation regions in the $z_{+}$, $z_{+}$, and $z_{+}$ planes, including statistical and systematic uncertainties for GGSZ only, GGSZ and GLW methods combined, and the overall combination. These contours have been obtained using the likelihood ratio method, $-2\Delta \ln \mathcal{L} = \chi^2$, where $\chi^2$ is the number of standard deviations, where $\Delta \ln \mathcal{L}$ represents the variation of the combined log-likelihood with respect to its maximum value [47]. With this construction, the approximate confidence level (C.L.) in two dimensions for each pair of variables is 39.3%, 86.5%, and 98.9%. In these two-dimensional regions, the separation of the $B^-$ and $B^+$ positions is equal to $2r_\phi|\sin \gamma|$, $2r_\phi|\sin \gamma|$, $2r_\phi|\sin \gamma|$, and is a measurement of direct CP violation, while the angle between the lines connecting the $B^-$ and $B^+$ centers with the origin $(0,0)$ is equal to $2\gamma$. Therefore, the net difference between $\bar{x}^+$ and $\bar{x}^-$ observed in Table IV and Fig. 3 is clear evidence for direct CP violation in $B^+ \rightarrow DK^+$ decays.

In Fig. 3, we notice that when the information from the GLW measurements is included the constraints on the best-fit values of the parameters are improved. However, the constraints on $\bar{y}_+$ are poor due to the quadratic dependence and the fact that $r_\phi \ll 1$. This is the reason why the GLW method alone can hardly constrain $\gamma$. Similarly, Eq. (9) for the ADS method represents two circles in the $(\bar{x}_+, \bar{y}_+)$ plane centered at $(r_\phi \cos \delta_\phi, r_\phi \sin \delta_\phi)$ and with radii $\sqrt{r_\phi^2}$. It is not possible to determine $\gamma$ with only ADS observables because the true $(\bar{x}_+, \bar{y}_+)$ points are distributed over two circles [49]. Therefore, while the GLW and ADS methods alone can hardly determine $\gamma$, when combined with the GGSZ measurements they help to improve significantly the constraints on the CP-violating parameters $z_{+}$, $z_{+}$, and $z_{+}$.

V. INTERPRETATION OF RESULTS

In a second stage, we transform the combined $(\bar{x}_+, \bar{y}_+)$, $(\bar{x}_+, \bar{y}_+)$, and $(\bar{x}_+, \bar{y}_+)$ measurements into the physically relevant quantities $\gamma$ and the set of hadronic parameters $u = (r_\phi, r_\phi, r_\phi, \delta_\phi, \delta_\phi, \delta_\phi)$. We adopt a frequentist procedure [50] to obtain one-dimensional confidence intervals of well-defined C.L. that takes into account non-Gaussian effects due to the nonlinearity of the relations between the observables and physical quantities. This procedure is identical to that used in Refs. [17,18,20,22,23].

We define a $\chi^2$ function as

$$\chi^2(\gamma, u) = -2\Delta \ln \mathcal{L}(\gamma, u) = -2[\ln \mathcal{L}(\gamma, u) - \ln \mathcal{L}_{\text{max}}].$$

(11)

where $2\Delta \ln \mathcal{L}(\gamma, u)$ is the variation of the combined log-likelihood with respect to its maximum value, with the $z^{(s)}$
and $\bar{Z}_{\pm}$ expected values written in terms of $\gamma$ and $u$, i.e., replacing $Z_{\pm}^{(s)}$ and $\bar{Z}_{\pm}$ by $r_B^{(s)} e^{i(\delta^{(s)} - \gamma)}$ and $\kappa r_s e^{i(\delta_s - x)}$, respectively. To evaluate the C.L. of a certain parameter (for example $\gamma$) at a given value ($\gamma_0$), we consider the value of the $\chi^2$ function at the new minimum, $\chi^2_{\text{min}}(\gamma_0, u_0)$, satisfying $\Delta \chi^2(\gamma_0) = \chi^2(\gamma_0, u_0) - \chi^2_{\text{min}} \geq 0$. In a purely Gaussian situation, the C.L. is given by the probability that $\chi^2(\gamma_0)$ is exceeded for a $\chi^2$ distribution with one degree of freedom, $1 - \text{C.L.} = \text{Prob}[\Delta \chi^2(\gamma_0); \nu = 1]$, where $\text{Prob}[\Delta \chi^2(\gamma_0); \nu = 1]$ is the corresponding cumulative distribution function (this approach is later referred to as “Prob method”) [47]. In a non-Gaussian situation one has to consider $\Delta \chi^2(\gamma_0)$ as a test statistic, and rely on a Monte Carlo simulation to obtain its expected distribution. This Monte Carlo simulation is performed by generating more than $10^9$ samples (sets of the 39 GGSZ, GLW, ADS, and $D$-decay observable values), using the combined likelihood evaluated at values $(\gamma_0, u_0)$, i.e., $L(\gamma_0, u_0)$. The confidence level C.L. is determined from the fraction of experiments for which $\Delta \chi^2(\gamma_0) > \chi^2_{\text{min}}$ for each simulated experiment is determined as in the case of the actual data sample. We adopt the Monte Carlo simulation method as baseline to determine the C.L., and allow $0 \leq r_B^{(s)}, \kappa r_s \leq 1$ and $-180^\circ \leq \gamma, \delta_B^{(s)}, \delta_s \leq 180^\circ$.

Figure 4 illustrates $1 - \text{C.L.}$ as a function of $\gamma$, $r_B^{(s)}$, $\kappa r_s$, $\delta_B^{(s)}$, and $\delta_s$, for each of the three $B$-decay channels separately and, in the case of $\gamma$, their combination. The combination has the same twofold ambiguity in the weak and strong phases as that of the GGSZ method.
From these distributions, we extract one- and two-standard-deviation intervals as the sets of values for which $1 - \text{C.L.}$ is greater than 31.73% and 4.55%, respectively, as summarized in Table V. When comparing these intervals to those obtained with the GGSZ method only, also shown in Table V, we observe that the combination helps to improve the constraints on $r_B$ and $\kappa_r$, but not those on $\gamma$. To assess the impact of the GLW and ADS observables in the determination of $\gamma$, we compare $1 - \text{C.L.}$ as a function of $r^{(*)}_{B}$ and $\gamma$ for all $B$-decay channels combined using the GGSZ method alone, the combination with the GLW measurements, and the global combination, as shown in Fig. 5. While the constraints on $r_B$ are clearly improved at the one- and two-standard-deviation level, and to a lesser extent on $r^{(*)}_{B}$, their best (central) values move towards slightly lower values. Since the uncertainty on $\gamma$ scales approximately as $1/r_B^{(*)}$, the constraints on $\gamma$ at 68.3% and 95.4% C.L. do not improve, in spite of the tighter constraints on the combined measurements shown in Fig. 3. However, adding GLW and ADS information reduces the confidence intervals for smaller $1 - \text{C.L.}$, as a consequence of the more Gaussian behavior when the significance of excluding $r^{(*)}_{B} = 0$ increases. Thus, for example, in the region close to four standard deviations, the GGSZ method alone does not constrain $\gamma$, while the combination is able to exclude large regions.

The significance of direct $CP$ violation is obtained by evaluating $1 - \text{C.L.}$ for the most probable $CP$ conserving point, i.e., the set of hadronic parameters $u$ with $\gamma = 0$. Including statistical and systematic uncertainties, we obtain $1 - \text{C.L.} = 3.4 \times 10^{-7}$, $2.5 \times 10^{-3}$, and $3.6 \times 10^{-2}$, corresponding to $5.1$, $3.0$, and $2.1$ standard deviations, for $B^+ \rightarrow DK^\pm$, $B^0 \rightarrow D^*K^\pm$, and $B^\pm \rightarrow DK^{\pm\pm}$ decays, respectively. For the combination of the three decay modes we obtain $1 - \text{C.L.} = 3.1 \times 10^{-9}$, corresponding to $5.9$ standard deviations. For comparison, the corresponding significances with the GGSZ method alone are $2.9$, $2.8$, $1.5$, and $4.0$ standard deviations [51], while with the GGSZ and GLW combination they are $4.8$, $2.7$, $1.8$, and $5.4$, respectively.

The frequentist procedure used to obtain $\gamma$ and the hadronic parameters $u$ is not guaranteed to have perfect coverage, especially for low values of $r^{(*)}_{B}$ and $r_B$. This is due to the treatment of nuisance parameters [50]. Instead of scanning the entire parameter space defined by $\gamma$ and $u$ (seven dimensions), we perform one-dimensional scans, in which, during MC generation, the nuisance parameters are set to their reoptimized best-fit values at each scan point. In order to evaluate the coverage properties of our procedure, we generate more than $10^9$ samples with true values of $(\gamma, u)$.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|}
\hline
Parameter Combination & GGSZ & Combination & GGSZ \\
\hline
$\gamma (\text{deg})$ & $69^{+17}_{-16}$ & $68^{+15}_{-14}$ & $[41, 102]$ & $[39, 98]$ \\
$r_B (\%)$ & $9.2^{+1.3}_{-1.2}$ & $9.6 \pm 2.9$ & $[6.0, 12.6]$ & $[3.7, 15.5]$ \\
$r^{(*)}_B (\%)$ & $10.6^{+1.9}_{-1.6}$ & $13.3^{+4.2}_{-2.3}$ & $[3.0, 14.7]$ & $[4.9, 21.5]$ \\
$\kappa_r (\%)$ & $14.3^{+2.8}_{-2.3}$ & $14.9^{+6.6}_{-0.2}$ & $[3.3, 25.1]$ & $<28.0$ \\
$\delta_B (\text{deg})$ & $105^{+17}_{-16}$ & $119^{+19}_{-20}$ & $[72, 139]$ & $[75, 157]$ \\
$\delta^{(*)}_B (\text{deg})$ & $-66^{+3}_{-1}$ & $-82 \pm 21$ & $[-132, -26]$ & $[-124, -38]$ \\
$\delta_i (\text{deg})$ & $101 \pm 43$ & $111 \pm 32$ & $[32, 166]$ & $[42, 178]$ \\
\hline
\end{tabular}
\caption{68.3% and 95.5% one-dimensional C.L. regions, equivalent to one- and two-standard-deviation intervals, for $\gamma$, $\delta^{(*)}_B$, $\delta_\gamma$, $r^{(*)}_B$, $\kappa_r$, including all sources of uncertainty, obtained from the combination of GGSZ, GLW, and ADS measurements. The combined results are compared to those obtained using the GGSZ measurements only, taken from Ref. [17]. The results for $\gamma$, $\delta^{(*)}_B$, and $\delta_\gamma$ are given modulo a 180° phase.}
\end{table}
VI. SUMMARY

In summary, using up to $474 \times 10^6 B\bar{B}$ decays recorded by the $BABAR$ detector, we have presented a combined measurement of the $CP$-violating ratios between the $b \rightarrow u\bar{c}s$ and $b \rightarrow c\bar{u}s$ amplitudes in processes $B^\pm \rightarrow D^{(s)}K^\pm$ and $B^\pm \rightarrow DK^{\ast \pm}$. The combination procedure maximizes the information provided by the most sensitive $\gamma$ measurements and analysis techniques that exploit a large number of $D$-decay final states, including three-body self-conjugate, $CP$, and doubly-Cabibbo-suppressed states, resulting in the most precise measurement of these ratios. From the measurements of these ratios we determine $\gamma = (69^{+17}_{-16})^\circ$ modulo 180°, where the total uncertainty is dominated by the statistical component, with the experimental and amplitude-model systematic uncertainties amounting to $\pm 4^\circ$. We also derive the most precise determinations of the magnitude ratios $r_B$ and $\kappa r_s$. The two-standard-deviation region for $\gamma$ is $41^\circ < \gamma < 102^\circ$. The combined significance of $\gamma \neq 0$ is $1 - C.L. = 3.1 \times 10^{-9}$, corresponding to 5.9 standard deviations, meaning observation of direct $CP$ violation in the measurement of $\gamma$. These results supersede our previous constraints based on the GGSZ, GLW, and ADS analyses of charged $B$ decays [17–21,23,24], and are consistent with the range of values implied by other experiments [36–44].

ACKNOWLEDGMENTS

We are grateful for the extraordinary contributions of our PEP-II colleagues in achieving the excellent luminosity and machine conditions that have made this work possible. The success of this project also relies critically on the expertise and dedication of the computing organizations that support $BABAR$. The collaborating institutions wish to thank SLAC for its support and the kind hospitality extended to them. This work is supported by the US Department of Energy and National Science Foundation, the Natural Sciences and Engineering Research Council (Canada), the Commissariat à l’Energie Atomique and Institut National de Physique Nucléaire et de Physique des Particules (France), the Bundesministerium für Bildung und Forschung and Deutsche Forschungsgemeinschaft (Germany), the Istituto Nazionale di Fisica Nucleare (Italy), the Foundation for Fundamental Research on Matter (The Netherlands), the Research Council of Norway, the Ministry of Education and Science of the Russian Federation, Ministerio de Economía y Competitividad (Spain), and the Science and Technology Facilities Council (United Kingdom). Individuals have received support from the Marie-Curie IEF program (European Union) and the A. P. Sloan Foundation (USA).
[10] Charge conjugate modes are implicitly included unless otherwise stated.
[32] There is a misprint in the analogous decay rate written in Ref. [17].
[51] These values supersede those given in Ref. [17], which were slightly underestimated.