0νββ and 2νββ nuclear matrix elements, quasiparticle random-phase approximation, and
isospin symmetry restoration

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I. INTRODUCTION

Answering the questions of whether or not total lepton number is a conserved quantity and thus whether neutrinos are massive Majorana fermions is a crucial part of the search for “physics beyond the Standard Model.” Consequently, experimental searches for the 0νββ decay are being pursued worldwide (for a recent review of the field, see, e.g., [1]). However, interpreting existing results and planning new experiments is impossible without knowledge of the corresponding nuclear matrix elements. The nuclear matrix elements M0ν of the 0νββ decay must be determined by using nuclear structure theory, and the choice of the appropriate approximations is a crucial part of that task. Some of the methods employed for evaluation of the M0ν, in particular those that begin with the transformation from particles to quasiparticles to account for the like-nucleon pairing (see, e.g., [2–8]), use wave functions that do not exactly conserve particle number. The number of protons and neutrons is usually conserved on average or, in some cases, it is restored by the particle number projection. In either case, until now no attempt was made to check that the isospin, which is known to be, to a very good approximation, a valid quantum number in nuclei, remains as such in the resulting wave functions that are obtained by solving the corresponding equations of motion.

It is well known that by the proper treatment of the quasiparticle interaction the broken symmetries can be restored. Naturally, exact calculation would restore the broken symmetries exactly. However, even with the approximate, random-phase-approximation-like treatment, it is possible to partially restore some of the broken symmetries. In this work we show, following basically the suggestions made initially in Ref. [9], how this can be done in the case of isospin, and by doing that the values of the Fermi nuclear matrix elements, both for the 2νββ and 0νββ decays, are substantially modified. Even though the resulting total M0ν nuclear matrix elements are changed only by ≈10%, it is worthwhile, and certainly more consistent, to use in future the prescriptions described below.

II. FORMALISM

Under the assumption that the 0νββ decay is caused by the exchange of light Majorana neutrinos, the half-life and the nuclear matrix element are related through

\[
\frac{1}{T_{1/2}^{0ν}} = G^{0ν}(Q, Z)|M^{0ν}|^2 |\langle m_{ββ}\rangle|^2, 
\]

where G0ν(Q, Z) is the calculable phase space factor, \langle m_{ββ}\rangle is the effective neutrino Majorana mass whose determination is the ultimate goal of the experiments, and M0ν is the nuclear matrix element consisting of Gamow-Teller (GT), Fermi, and tensor parts,

\[
M^{0ν} = M^{0ν}_{GT} - \frac{M^{0ν}_{SV}}{g_A^2} + M^{0ν}_{F} \\
\equiv M^{0ν}_{GT}(1 - \chi_F/g_A^2 + \chi_T),
\]

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where $\chi_F$ and $\chi_T$ are the matrix element ratios $\chi_F = M_{2\nu}^{GT}/M_{2\nu}^{GT}$ and $\chi_T = M_{2\nu}^{GT}/M_{2\nu}^{GT}$. (In the literature a different notation is sometimes used: $\chi_F = M_{2\nu}^{GT}/g_{\nu}^{2}\langle M_{2\nu}^{GT} \rangle$.)

The main GT part, $M_{2\nu}^{GT}$, can be somewhat symbolically written as

$$M_{2\nu}^{GT} = \langle f | \sum_{k} \sigma_{k} t_{+}^{k} t_{+}^{k} H(r_{k}, \vec{E}) | i \rangle,$$

where $H(r_{k}, \vec{E})$ is the neutrino potential described in detail in Ref. [5] and $r_{k}$ is the relative distance between the two neutrons that are transformed in the decay into the two protons.

Analogously, the Fermi matrix element is

$$M_{2\nu}^{F} = \langle f | \sum_{k} t_{+}^{k} t_{+}^{k} H(r_{k}, \vec{E}) | i \rangle.$$

Note that these $0\nu\beta\beta$ matrix elements are expressed in the closure approximation; its applicability is also discussed in Ref. [5]. However, the results reported later in this work were obtained without using closure; instead explicit summation over all virtual intermediate states was performed.

The half-life of the experimentally well studied $2\nu\beta\beta$ decay depends formally on two nuclear matrix elements:

$$\frac{1}{T_{1/2}} = G^{2\nu}(Q, Z) \left[ M_{2\nu}^{GT} + \frac{S_{T}^{2}}{S_{A}^{2}} M_{2\nu}^{F} \right]^{2},$$

The Gamow-Teller $2\nu\beta\beta$ matrix element is

$$M_{2\nu}^{GT} = \sum_{m} \langle f | \sum_{k} \sigma_{k} t_{+}^{k} \langle m | \sum_{k} \sigma_{k} t_{+}^{k} | i \rangle \rangle_{m} \frac{1}{E_{m} - (M_{i} + M_{f})/2},$$

where the summation extends over all $1^{+}$ virtual intermediate states. In that case the closure approximation is not a valid approach but can be formally introduced by defining the corresponding closure matrix element $M_{2\nu}^{GT}$ when replacing the energies $E_{m}$ by the proper average value $\bar{E}_{2\nu}$. Thus,

$$M_{2\nu}^{GT} = \sum_{m} \langle f | \sum_{k} \sigma_{k} t_{+}^{k} | i \rangle,$$

$$M_{2\nu}^{GT} = M_{2\nu}^{GT}(c) = \bar{E}_{2\nu} - (M_{i} + M_{f})/2).$$

Formally, in the description of the $2\nu\beta\beta$ decay also appears the Fermi matrix element

$$M_{2\nu}^{F} = \sum_{m} \langle f | \sum_{k} t_{+}^{k} | m \rangle | \sum_{k} t_{+}^{k} | i \rangle \rangle_{m} \frac{1}{E_{m} - (M_{i} + M_{f})/2},$$

where the summation extends over all $0^{+}$ virtual intermediate states, and its closure form is

$$M_{2\nu}^{GT}(c) = \sum_{m} \langle f | \sum_{k} t_{+}^{k} \rangle,$$

$$M_{2\nu}^{GT}(c) = M_{2\nu}^{GT} \bar{E}_{2\nu} - (M_{i} + M_{f})/2).$$

The ground state $|i\rangle$ of the initial nucleus has isospin $T = T_{i} = (N - Z)/2$ while the final state $|f\rangle$ has isospin $T = 2 - T_{i} = (N - Z - 4)/2$. Since the operator $\Sigma_{k} t_{+}^{k}$ just changes the isospin projection and cannot change the total isospin, it is obvious that when isospin is a good quantum number both Fermi matrix elements must vanish,

$$M_{2\nu}^{F} = M_{2\nu}^{GT}(c) = 0,$$

since the average energy denominators in Eq. (9) are nonvanishing.

Until now, within the quasiparticle random-phase approximation (QRPA), projected Hartree-Fock-Bogoliubov (PHFB), energy-density functional (EDF), and interacting boson model (IBM-2) methods the validity of Eq. (10) has not been usually tested [2–8]. When results were published, $M_{2\nu}^{GT}$ and $M_{2\nu}^{GT}(c)$ do not vanish and are, in fact, comparable to $M_{2\nu}^{GT}$ and $M_{2\nu}^{GT}(c)$ respectively. Despite that, when evaluating the $2\nu\beta\beta$ half-life the Fermi matrix element was usually simply neglected.

As we show further, in the usual application of the QRPA Eq. (10) is not obeyed. Instead, the magnitude of $M_{2\nu}^{GT}$ is numerically comparable to the magnitude of $M_{2\nu}^{GT}$ as just pointed out. In addition, for the $0\nu\beta\beta$ decay, within the QRPA the ratio $\chi_T \approx 0.5$ while in the nuclear shell model, where isospin is a good quantum number by construction, Eq. (10) is, naturally, obeyed and $\chi_T \approx (0.2–0.3)$ [10].

Where does this problem in the QRPA method originate? The method begins with the Bogoliubov transformation relating the particle creation and annihilation operators $a_{jm}^{\dagger}, \bar{a}_{jm}$ with the quasiparticle creation and annihilation operators $c_{jm}^{\dagger}, \bar{c}_{jm}$. By solving the BCS equations one includes the neutron-neutron and proton-proton isovector pairing interactions.

At this stage several symmetries are broken. The numbers of protons, $Z$, and neutrons, $N$, are no longer exact but are valid only on average. In addition, since the neutron-proton part of the isovector pairing interaction is neglected, an additional source of isospin violation is introduced. It turns out that it is relatively easy to remedy this additional effect and restore isospin conservation, at least in part, as explained further here. Because the random-phase approximation (RPA) (as well as the QRPA) is derived from the equation of motion for bifermionic operators (treated in the quasiboson approximation), symmetries of the model Hamiltonian can naturally be fulfilled in that approximation.

To proceed further, the equations of motion need to be solved. Within the QRPA method the forward- and backward-going amplitudes $X$ and $Y$ that are needed for the evaluation of the nuclear matrix elements, as well as the corresponding energy eigenvalues $\omega_{sn}$, are determined by solving the eigenvalue equations of motion for each angular momentum and parity $J^{\pi}$,

$$\begin{pmatrix} A & B \\ -B & A \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = \omega \begin{pmatrix} X \\ Y \end{pmatrix}.$$

The matrices $A$ and $B$ are (see, e.g., [11])

$$A_{pn, p'n'}^{J} = (\langle c_{p} c_{n}^{J} | [\hat{H}, (c_{p} c_{n}^{J})] | O \rangle)$$

$$= \delta_{pp'} \delta_{nn'} (E_{p} + E_{n}) - (u_{p} v_{n} u_{p'} v_{n'} + u_{p} u_{n} v_{p'} v_{n'})$$

$$\times 2 g_{ph} \langle p | n^{-1} \rangle, J | V | p' n' \rangle, J - (u_{p} u_{n} u_{p'} u_{n'})$$

$$+ (u_{p} v_{n} v_{p'} v_{n'}) \times 2 g_{pp} \langle p | n \rangle | V | p' n' \rangle, J, J$$

and

$$B_{pn, p'n'}^{J} = (\langle c_{p} c_{n}^{J} | [\hat{H}, (c_{p} c_{n}^{J})] | O \rangle)$$

$$= - (u_{p} v_{n} u_{p'} v_{n'} + v_{p} u_{n} v_{p'} u_{n'})$$

$$\times 2 g_{ph} \langle p | n^{-1} \rangle, J | V | p' n' \rangle, J$$

$$+ (u_{p} u_{n} v_{p'} v_{n'}) \times 2 g_{pp} \langle p | n \rangle | V | p' n' \rangle, J.$$

where $E_{p}$ and $E_{n}$ are the quasiparticle energies.
The definitions, Eqs. (12) and (13), contain two renormalization adjustable parameters: $g^0_{\text{ph}}$ for the particle-hole interaction and $g_{\text{pp}}$ for the particle-particle interaction. While $g^0_{\text{ph}} = 1.0$ is typically used, it is customary to adjust $g_{\text{pp}}$ so that the experimentally known half-life of the $2\nu\beta\beta$ decay is correctly reproduced [2]. But the particle-particle neutron-proton interaction governed by $g_{\text{pp}}$ actually contains two kinds of interaction matrix elements, isovector and isoscalar. Thus, to be consistent with the treatment of the like-particle pairing, one should separate the $T = 1$ part from the $T = 0$ part, i.e., replace

$$g_{\text{pp}}(pn, J)\,|V|\,p'n', J) \to g^{T=1}_{\text{pp}}(pn, J, T = 1)|V|p'n', J, T = 1)$$

and adjust the parameters $g^{T=1}_{\text{pp}}$ and $g^{T=0}_{\text{pp}}$ independently. To partially restore isospin symmetry and achieve Eq. (10) being obeyed, it is sufficient to choose $g^{T=1}_{\text{pp}} \approx 8_{\text{pair}}$. (That the coupling constant of the isovector proton-neutron particle-particle force should be close, or identical, to the pairing strength constant, was recognized already in the early works on the QRPA application to $\beta\beta$ decay that used a schematic, $\delta$-force interaction; see Ref. [12].)

### III. Determination of the Parameter $g^{T=1}_{\text{pp}}$

When solving the BCS pairing equations, it is customary to slightly renormalize the strength of the pairing part of the realistic nucleon-nucleon interaction so that experimental pairing gaps are correctly reproduced. Thus, four adjusted parameters $(d^{(0)}_{\text{pp}}, d^{(f)}_{\text{pp}}, d^{(f)}_{\text{nn}}, d^{(f)}_{\text{nn}})$ are introduced (see, e.g., [2–5]) to represent the adjustments needed to describe the neutron and proton pairing gaps in the initial and final nuclei. The values of these parameters as well as their averages for selected $\beta\beta$-decay candidate nuclei are displayed in Table I. [The table entries are for two variants of the nucleon-nucleon interaction and one choice, of large size (21 or 23 levels, for oscillator shells $N = 0–5$ with the addition of the $i_{13/2}$ and $i_{11/2}$ for nuclei heavier than $^{124}$Sn), of the single-particle level scheme. The results for other choices are not very different.]

In several cases in Table I we encounter magic numbers of neutrons or protons. In those cases the BCS treatment is inappropriate and hence the corresponding entries are missing there. For the case of $^{48}$Ca we considered two variants. In the listed one we assumed that there is no pairing in the doubly magic $^{48}$Ca. In the other variant we assumed that the usual odd-even mass difference with the five-point formula represent the pairing gaps; the resulting $g^{T=1}_{\text{pp}}$ is rather similar to the values listed in Table I.

The example in Fig. 1 shows how the matrix elements $M^{2\nu}_{\text{GT}}$ and $M^{2\nu}_{\text{GT}}$ behave when the isovector coupling constant $g^{T=1}_{\text{pp}}$ is changed while the isoscalar $g^{T=0}_{\text{pp}}$ is kept constant. As one can see, the Fermi matrix element $M^{2\nu}_{\text{Fermi}}$ decreases and crosses zero, with increasing $g^{T=1}_{\text{pp}}$ while the Gamow-Teller matrix element remains constant. This is a typical case, and we can

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**TABLE I. Renormalization parameters of the pairing interaction, their average, and the $T = 1$ renormalization constant $g^{T=1}_{\text{pp}}$ adjusted such that $M^{2\nu}_{\text{Fermi}}$ and $M^{2\nu}_{\text{GT}}$ vanish.**

<table>
<thead>
<tr>
<th>Nucleus</th>
<th>NN potential</th>
<th>Number of single-particle levels</th>
<th>$d^{(0)}_{\text{pp}}$</th>
<th>$d^{(f)}_{\text{pp}}$</th>
<th>$d^{(f)}_{\text{nn}}$</th>
<th>$d^{(f)}_{\text{nn}}$</th>
<th>$\tilde{a}$</th>
<th>$g^{T=1}_{\text{pp}}$</th>
</tr>
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<tbody>
<tr>
<td>$^{40}$Ca</td>
<td>Argonne</td>
<td>21</td>
<td>–</td>
<td>–</td>
<td>1.075</td>
<td>0.988</td>
<td>1.034</td>
<td>1.031</td>
</tr>
<tr>
<td></td>
<td>CD-Bonn</td>
<td>21</td>
<td>–</td>
<td>–</td>
<td>0.985</td>
<td>0.903</td>
<td>0.944</td>
<td>0.944</td>
</tr>
<tr>
<td>$^{76}$Ge</td>
<td>Argonne</td>
<td>21</td>
<td>0.930</td>
<td>1.074</td>
<td>0.970</td>
<td>1.106</td>
<td>1.020</td>
<td>1.038</td>
</tr>
<tr>
<td></td>
<td>CD-Bonn</td>
<td>21</td>
<td>0.863</td>
<td>0.983</td>
<td>0.899</td>
<td>1.013</td>
<td>0.940</td>
<td>0.958</td>
</tr>
<tr>
<td>$^{82}$Se</td>
<td>Argonne</td>
<td>21</td>
<td>0.869</td>
<td>1.085</td>
<td>0.930</td>
<td>1.131</td>
<td>1.004</td>
<td>1.032</td>
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<tr>
<td></td>
<td>CD-Bonn</td>
<td>21</td>
<td>0.808</td>
<td>0.995</td>
<td>0.864</td>
<td>1.038</td>
<td>0.926</td>
<td>0.955</td>
</tr>
<tr>
<td>$^{96}$Zr</td>
<td>Argonne</td>
<td>21</td>
<td>0.923</td>
<td>0.768</td>
<td>1.000</td>
<td>0.962</td>
<td>0.913</td>
<td>0.984</td>
</tr>
<tr>
<td></td>
<td>CD-Bonn</td>
<td>21</td>
<td>0.856</td>
<td>0.704</td>
<td>0.926</td>
<td>0.881</td>
<td>0.842</td>
<td>0.907</td>
</tr>
<tr>
<td>$^{100}$Mo</td>
<td>Argonne</td>
<td>21</td>
<td>1.019</td>
<td>0.960</td>
<td>1.041</td>
<td>0.979</td>
<td>1.000</td>
<td>1.008</td>
</tr>
<tr>
<td></td>
<td>CD-Bonn</td>
<td>21</td>
<td>0.946</td>
<td>0.883</td>
<td>0.966</td>
<td>0.900</td>
<td>0.924</td>
<td>0.933</td>
</tr>
<tr>
<td>$^{110}$Pd</td>
<td>Argonne</td>
<td>21</td>
<td>1.000</td>
<td>0.975</td>
<td>1.025</td>
<td>0.945</td>
<td>0.986</td>
<td>0.979</td>
</tr>
<tr>
<td></td>
<td>CD-Bonn</td>
<td>21</td>
<td>0.930</td>
<td>0.895</td>
<td>0.954</td>
<td>0.871</td>
<td>0.913</td>
<td>0.908</td>
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<tr>
<td>$^{116}$Cd</td>
<td>Argonne</td>
<td>21</td>
<td>1.017</td>
<td>0.971</td>
<td>–</td>
<td>0.919</td>
<td>0.969</td>
<td>0.922</td>
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<tr>
<td></td>
<td>CD-Bonn</td>
<td>21</td>
<td>0.949</td>
<td>0.895</td>
<td>–</td>
<td>0.847</td>
<td>0.897</td>
<td>0.852</td>
</tr>
<tr>
<td>$^{124}$Sn</td>
<td>Argonne</td>
<td>23</td>
<td>–</td>
<td>–</td>
<td>1.001</td>
<td>0.929</td>
<td>1.000</td>
<td>0.977</td>
</tr>
<tr>
<td></td>
<td>CD-Bonn</td>
<td>23</td>
<td>–</td>
<td>–</td>
<td>0.918</td>
<td>0.860</td>
<td>0.917</td>
<td>0.989</td>
</tr>
<tr>
<td>$^{128}$Te</td>
<td>Argonne</td>
<td>23</td>
<td>0.881</td>
<td>0.968</td>
<td>0.926</td>
<td>0.999</td>
<td>0.944</td>
<td>0.988</td>
</tr>
<tr>
<td></td>
<td>CD-Bonn</td>
<td>23</td>
<td>0.816</td>
<td>0.889</td>
<td>0.857</td>
<td>0.918</td>
<td>0.870</td>
<td>0.914</td>
</tr>
<tr>
<td>$^{130}$Te</td>
<td>Argonne</td>
<td>23</td>
<td>0.845</td>
<td>0.970</td>
<td>0.920</td>
<td>1.000</td>
<td>0.934</td>
<td>0.989</td>
</tr>
<tr>
<td></td>
<td>CD-Bonn</td>
<td>23</td>
<td>0.783</td>
<td>0.891</td>
<td>0.852</td>
<td>0.918</td>
<td>0.861</td>
<td>0.915</td>
</tr>
<tr>
<td>$^{134}$Xe</td>
<td>Argonne</td>
<td>23</td>
<td>0.851</td>
<td>0.912</td>
<td>0.917</td>
<td>0.963</td>
<td>0.911</td>
<td>0.973</td>
</tr>
<tr>
<td></td>
<td>CD-Bonn</td>
<td>23</td>
<td>0.790</td>
<td>0.840</td>
<td>0.850</td>
<td>0.887</td>
<td>0.842</td>
<td>0.903</td>
</tr>
<tr>
<td>$^{136}$Xe</td>
<td>Argonne</td>
<td>23</td>
<td>0.782</td>
<td>–</td>
<td>0.885</td>
<td>0.926</td>
<td>0.864</td>
<td>0.950</td>
</tr>
<tr>
<td></td>
<td>CD-Bonn</td>
<td>23</td>
<td>0.726</td>
<td>–</td>
<td>0.821</td>
<td>0.853</td>
<td>0.800</td>
<td>0.881</td>
</tr>
</tbody>
</table>
and in the tables that follow, the half-life remains unknown.

In the previous section we explained how the parameter $g_{T}^{\nu}$ can be still adjusted such that the half-life of the $2\nu\beta\beta$ decay is correctly reproduced, exactly as done before. In fact, the previously used common value of $g_{pp}$ and the new parameter $g_{pp}^{T=0}$ are essentially the same, as we demonstrate in the next section.

IV. RESULTS AND DISCUSSION

In the previous section we explained how the parameter $g_{pp}^{T=1}$ is determined. The determination of the other renormalization parameter $g_{pp}^{T=0}$ is analogous and follows the suggestion made long time ago in Ref. [2], as already stated. We fit $g_{pp}^{T=0}$ from the requirement that the calculated values of the full $2\nu\beta\beta$ matrix elements $M_{GT}^{2\nu}$ agree with their experimental values. For most nuclei in Table I the half-lives $T_{1/2}^{2\nu}$ have been measured; we use the recommended values in Ref. [14] plus the $^{136}$Xe half-life of Refs. [15,16]. But for several nuclei in that table, and in the tables that follow, the half-life remains unknown. In those cases we proceed as follows: for $^{110}$Pd we use the estimate of Ref. [17] that uses the single-state dominance assumption; for $^{124}$Sn and $^{134}$Xe we use an interval of possible $M_{GT}^{2\nu}$ values: $0 \leq M_{GT}^{2\nu} \leq 0.2 \text{ (0.1) MeV}^{-1}$ for $^{124}$Sn ($^{134}$Xe), respectively. For these two nuclei we show in Tables II–IV the results with $g_{A} = 1.27$ for the lower limit $M_{GT}^{2\nu} = 0$. Our results for these two nuclei therefore reflect our incomplete knowledge of the corresponding $2\nu$ half-life.

Before presenting the results for the $0\nu\beta\beta$ nuclear matrix elements, several comments are in order. Since the main effect considered here is the change in $M_{F}^{2\nu}$ and the associated change in $M_{GT}^{2\nu}$, let us analyze these changes using the radial dependence of $M_{GT}^{2\nu}$ explained in Ref. [5]. In Fig. 2 the functions $C_{F(c)}^{2\nu}(r)$ with the old and new parametrization of $g_{pp}$ are plotted, together with the function $C_{F(c)}^{2\nu}(r)$ (scaled by $1/3$ for clarity). As one can see, with the new $g_{pp}^{T=1}$ the tail of $C_{F(c)}^{2\nu}(r)$ becomes more negative and therefore its integral vanishes, as required. Let us remind ourselves that

$$M_{F(c)}^{2\nu} = \int_{0}^{\infty} C_{F(c)}^{2\nu}(r) dr,$$

and in analogy

$$M_{GT(c)}^{2\nu} = \int_{0}^{\infty} C_{GT(c)}^{2\nu}(r) dr.$$  

Another comment concerns the fact that, as we will see, with the new parametrization $\chi_{F} \approx (0.3–0.4)$, or more precisely $\chi_{F} \approx 1/3$. A somewhat similar conclusion is obtained in the shell model [10], where isospin is conserved by definition. The explanation is based on the fact that the ground states of even-even nuclei consist dominantly of the $J^{\pi} = 0^{+}$, $T = 1$ Cooper pairs that, in turn, are mostly in the $S = 0$, $L = 0$ state. Since such states are eigenstates of the operator $\sigma_{1} \cdot \sigma_{2}$ with eigenvalue $1$, our conclusion simply follows.

In Fig. 3 we show examples of the decomposition of the function $C_{F(c)}^{2\nu}(r)$ into its $S = 0$ and $S = 1$ components. These are rather typical cases. The dominance of the $S = 0$ component in the pure pairing case [Fig. 3(a)] is easily understood. However, that feature is still present in the realistic case with $g_{pp} \neq 0$, hence explaining our finding that, usually, $\chi_{F} \approx 1/3$.

For both modes, $0\nu\beta\beta$ and $2\nu\beta\beta$, we can find relations between the Fermi and Gamow-Teller parts and their $S = 0$ and $S = 1$ components. These relations are exact in the closure approximation and when the higher order weak currents (and thus the tensor part $M_{GT}^{2\nu}$) are neglected and the nucleon form factors have the same cutoff values for the vector and axial vector parts. In addition, since the neutrino potentials indirectly depend on the assumed averaged energy, these $E$ values must be chosen to be the same for the Fermi and Gamow-Teller matrix elements. Using the properties of the $\sigma_{1} \cdot \sigma_{2}$ operator and that $M_{F}^{2\nu} = 0$ with our new parametrization we find that only one of the four components is independent and

$$M_{F(c)}^{2\nu}(S = 0) = -M_{F(c)}^{2\nu}(S = 1) = -M_{GT(c)}^{2\nu}(S = 1) = -M_{GT(c)}^{2\nu}(S = 0)/3.$$  

For the $0\nu$ mode, however, $M_{F}^{0\nu} \neq 0$ and hence the above relations must be modified:

$$M_{F}^{0\nu}(S = 0) = M_{F}^{0\nu} - M_{F}^{2\nu}(S = 1) = M_{F}^{0\nu} - M_{GT}^{0\nu}(S = 1) = -M_{GT}^{0\nu}(S = 0)/3,$$

$$M_{GT}^{0\nu}(S = 0) = M_{F}^{0\nu} - 4M_{GT}^{0\nu}(S = 0).$$
TABLE II. Nuclear matrix elements (NMEs) for both $\beta\beta$ decay modes with the old parametrization ($g^{T=0}_{pp} = g_{pp}^{\nu\nu}$) compared to those with the new one ($g^{T=0}_{pp} \neq g_{pp}^{\nu\nu}$). The adopted values of the parameter $g^{T=1}_{pp}$ are also shown. The results for two values of the axial coupling constant $g_A$ are displayed: the quenched value $g_A = 1.0$ and the standard value $g_A = 1.27$. The $G$-matrix elements of a realistic Argonne V18 nucleon-nucleon potential are considered. The nuclear radius $R = r_0A^{1/3}$ with $r_0 = 1.2$ fm is used.

<table>
<thead>
<tr>
<th>Nucleus</th>
<th>$g_A$</th>
<th>$g_{pp}^{T=0}$</th>
<th>Parameter</th>
<th>$2\nu\beta\beta$-decay NMEs</th>
<th>$0\nu\beta\beta$-decay NMEs</th>
<th>$\chi_F$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$M^{2\nu}_{F}$</td>
<td>$M^{2\nu}_{GT}$</td>
<td>$M^{0\nu}_{F}$</td>
</tr>
<tr>
<td>$^{40}$Ca</td>
<td>1.00</td>
<td>0.771</td>
<td>old</td>
<td>0.331</td>
<td>0.0736</td>
<td>-0.794</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.770</td>
<td>new</td>
<td>0.00</td>
</tr>
<tr>
<td>$^{76}$Ge</td>
<td>1.00</td>
<td>0.728</td>
<td>old</td>
<td>0.240</td>
<td>0.220</td>
<td>-2.688</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.728</td>
<td>new</td>
<td>0.00</td>
</tr>
<tr>
<td>$^{82}$Se</td>
<td>1.00</td>
<td>0.751</td>
<td>old</td>
<td>0.231</td>
<td>0.137</td>
<td>-2.632</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.750</td>
<td>new</td>
<td>0.00</td>
</tr>
<tr>
<td>$^{96}$Zr</td>
<td>1.00</td>
<td>0.806</td>
<td>old</td>
<td>0.180</td>
<td>0.153</td>
<td>-2.394</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.817</td>
<td>new</td>
<td>0.00</td>
</tr>
<tr>
<td>$^{100}$Mo</td>
<td>1.00</td>
<td>0.841</td>
<td>old</td>
<td>0.100</td>
<td>0.373</td>
<td>-2.757</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.840</td>
<td>new</td>
<td>0.00</td>
</tr>
<tr>
<td>$^{110}$Pd</td>
<td>1.00</td>
<td>0.785</td>
<td>old</td>
<td>0.081</td>
<td>0.423</td>
<td>-2.666</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.783</td>
<td>new</td>
<td>0.00</td>
</tr>
<tr>
<td>$^{116}$Cd</td>
<td>1.00</td>
<td>0.870</td>
<td>old</td>
<td>0.008</td>
<td>0.206</td>
<td>-1.633</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.870</td>
<td>new</td>
<td>0.00</td>
</tr>
<tr>
<td>$^{124}$Sn</td>
<td>1.00</td>
<td>0.628</td>
<td>old</td>
<td>0.132</td>
<td>0.20</td>
<td>-1.779</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.626</td>
<td>new</td>
<td>0.00</td>
</tr>
<tr>
<td>$^{128}$Te</td>
<td>1.00</td>
<td>0.785</td>
<td>old</td>
<td>0.086</td>
<td>0.00</td>
<td>-1.473</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.785</td>
<td>new</td>
<td>0.00</td>
</tr>
<tr>
<td>$^{130}$Te</td>
<td>1.00</td>
<td>0.770</td>
<td>old</td>
<td>0.133</td>
<td>0.0776</td>
<td>-2.540</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.769</td>
<td>new</td>
<td>0.00</td>
</tr>
<tr>
<td>$^{134}$Xe</td>
<td>1.00</td>
<td>0.739</td>
<td>old</td>
<td>0.103</td>
<td>0.0545</td>
<td>-2.232</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.774</td>
<td>new</td>
<td>0.00</td>
</tr>
<tr>
<td>$^{136}$Xe</td>
<td>1.00</td>
<td>0.730</td>
<td>old</td>
<td>0.0652</td>
<td>0.0313</td>
<td>-1.228</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.730</td>
<td>new</td>
<td>0.00</td>
</tr>
</tbody>
</table>

Two components are independent in this case. In realistic cases these relations are not exact, but they are still valid in a reasonable approximation.

We will return to the discussion of the $\chi_F$ values obtained by different approximate methods in the next section.
In Tables II and III we compare the resulting matrix elements $M_F^{0\nu}$, $M_G^{0\nu}$, and $M_{GT}^{0\nu}$ with their components evaluated using the old parametrization ($g_{pp}^{T=1} = g_{pp}^{T=0} \approx g_{pp}$) with the new results, where $g_{pp}^{T=1}$ is fitted to the known experimental values of $M_{GT}^{0\nu}$. The calculations were performed for the unquenched value $g_A = 1.27$ as well as for $g_A = 1.0$. The quantities $M_{GT}^{0\nu} = M_{GT}^{0\nu} \times (g_A/1.27)^2$, as well as $\chi_F$, are also shown. Calculations in both tables were performed within the standard QRPA with all the usual ingredients, i.e., including the higher order weak currents, nucleon form factors, and the short-range correlation treatment of Ref. [13].
constant
Since we adjust the isoscalar particle-particle renormalization
account with our method (and for more details see Ref. [5]).
the old parametrization, with a single
should not be affected by the axial current quenching. With
is associated with the weak vector current, and as such it
value of the axial current coupling constant
the 2
parametrization and, for comparison, the function
M
nu
ν
β
β
AND 2νββ NUCLEAR MATRIX . . .
Let us explain briefly again how the quenching is taken into
account with our method (and for more details see Ref. [5]).
Since we adjust the isoscalar particle-particle renormalization
constant \( g_{pp}^{0} \) in such a way that the experimental half-life of
the 2νββ is correctly reproduced, by changing the effective
value of the axial current coupling constant \( g_{A} \) we are forced
to change also the parameter \( g_{pp}^{T=0} \), albeit only slightly. Those
changes are visible in the third columns of Tables II and III.
Since with smaller \( g_{A} \) the parameter \( g_{pp}^{T=0} \) slightly decreases,
the corresponding \( M_{GT}^{0} \) matrix element increases. However,
the 0νββ decay rate, proportional to the \( (M_{GT}^{0})^{2} \), naturally,
decreases.

In that context it is worthwhile to point out another feature
of the new parametrization. The Fermi matrix element \( M_{F}^{0} \)
is associated with the weak vector current, and as such it
should not be affected by the axial current quenching. With
the old parametrization, with a single \( g_{pp} \), that was not quite true, as seen in Tables II and III. However, with the new
parametrization where isospin symmetry is partially restored,
the \( M_{GT}^{0} \) becomes independent of the effective value of \( g_{A} \), as it
should be. (The tiny changes in Tables II and III are round-off
errors.)

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Since we adjust the isoscalar particle-particle renormalization
constant \( g_{pp}^{0} \) in such a way that the experimental half-life of
the 2νββ is correctly reproduced, by changing the effective
value of the axial current coupling constant \( g_{A} \) we are forced
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changes are visible in the third columns of Tables II and III.
Since with smaller \( g_{A} \) the parameter \( g_{pp}^{T=0} \) slightly decreases,
the corresponding \( M_{GT}^{0} \) matrix element increases. However,
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decreases.

In that context it is worthwhile to point out another feature
of the new parametrization. The Fermi matrix element \( M_{F}^{0} \)
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should not be affected by the axial current quenching. With
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parametrization where isospin symmetry is partially restored,
the \( M_{GT}^{0} \) becomes independent of the effective value of \( g_{A} \), as it
should be. (The tiny changes in Tables II and III are round-off
errors.)

![Figure 2](image2.png)

**FIG. 2.** (Color online) Functions \( C_{F}(r) \) with old and new parametrization and, for comparison, the function \( C_{GT}(r) \) (scaled by 1/3 for clarity) is also shown. This is the case of \(^{76}\text{Ge}\).

**TABLE IV.** Ratio \( \chi_{F} = M_{F}^{0}/M_{GT}^{0} \) [see our definition of \( \chi_{F} \) in Eq. (2)] in ISM [10], QRPA-A, and QRPA-B (present work, \( g_{A} = 1.00 \) and \( g_{A} = 1.27 \) side by side). QRPA-A results are with the Argonne V18 potential; QRPA-B results are with the CD-Bonn potential, IBM-2 [20], and QRPA-JyLa [21].

<table>
<thead>
<tr>
<th>Nucleus</th>
<th>ISM</th>
<th>QRPA-A</th>
<th>QRPA-B</th>
<th>IBM</th>
<th>QRPA-JyLa</th>
</tr>
</thead>
<tbody>
<tr>
<td>(^{48}\text{Ca})</td>
<td>-0.22</td>
<td>-0.42, -0.51</td>
<td>-0.42, -0.51</td>
<td>-0.68</td>
<td>-0.90</td>
</tr>
<tr>
<td>(^{76}\text{Ge})</td>
<td>-0.16</td>
<td>-0.31, -0.34</td>
<td>-0.30, -0.34</td>
<td>-0.61</td>
<td>-0.35</td>
</tr>
<tr>
<td>(^{82}\text{Se})</td>
<td>-0.16</td>
<td>-0.33, -0.36</td>
<td>-0.33, -0.36</td>
<td>-0.68</td>
<td>-0.45</td>
</tr>
<tr>
<td>(^{96}\text{Zr})</td>
<td>-</td>
<td>-0.45, -0.52</td>
<td>-0.45, -0.51</td>
<td>-0.10</td>
<td>-0.69</td>
</tr>
<tr>
<td>(^{110}\text{Mo})</td>
<td>-0.25</td>
<td>-0.39, -0.44</td>
<td>-0.38, -0.46</td>
<td>-0.05</td>
<td>-0.61</td>
</tr>
<tr>
<td>(^{116}\text{Cd})</td>
<td>-0.30</td>
<td>-0.43, -0.48</td>
<td>-0.42, -0.46</td>
<td>-0.10</td>
<td>-0.45</td>
</tr>
<tr>
<td>(^{124}\text{Sn})</td>
<td>-0.20</td>
<td>-0.39, -0.43</td>
<td>-0.38, -0.41</td>
<td>-0.56</td>
<td>-0.68</td>
</tr>
<tr>
<td>(^{128}\text{Te})</td>
<td>-0.20</td>
<td>-0.39, -0.43</td>
<td>-0.38, -0.41</td>
<td>-0.55</td>
<td>-0.60</td>
</tr>
<tr>
<td>(^{130}\text{Te})</td>
<td>-0.20</td>
<td>-0.41, -0.44</td>
<td>-0.39, -0.42</td>
<td>-0.55</td>
<td>-0.60</td>
</tr>
<tr>
<td>(^{136}\text{Xe})</td>
<td>-0.20</td>
<td>-0.38, -0.41</td>
<td>-0.36, -0.39</td>
<td>-0.55</td>
<td>-0.60</td>
</tr>
</tbody>
</table>

*Reference [22].

![Figure 3](image3.png)

**FIG. 3.** (Color online) (a) The function \( C_{F}(r) \) for the pure pairing case, i.e., \( g_{pp}^{T=0} = g_{pp}^{T=1} = g_{pp} = 0.0 \), separated into \( S = 0 \) and \( S = 1 \) components. (b) The function \( C_{GT}(r) \) for \( g_{pp}^{T=1} = 1.038 \) and \( g_{pp}^{T=0} = 0.750 \) again separated into its \( S = 0 \) and \( S = 1 \) parts. The sum function is also displayed. The dominance of the \( S = 0 \) component is clearly visible in (a). In (b) the two components when integrated over \( r \) are, naturally, equal and opposite. The \( S = 0 \) part, however, clearly is considerably larger in absolute value than the \( S = 1 \) part, at all \( r \) values. This is the case of \(^{76}\text{Ge}\).
relatively small effects for the other multipoles. This is the case of
compared with the old method (with the new parametrization developed in this work (filled squares)
M
\text{element}
leading to
compared. Note the dominant effect for the 0
\nu
87
\text{model treatment. We are, naturally, well aware of the fact that
and hence the Ikeda sum rule is fulfilled in the nuclear shell
only in the case of48Ca is the full oscillator
potential and 0.59 (0.77) with the CD-Bonn potential, while
are treated as arising from pairing, both with the Argonne V18
and 0.71 in the variant where the even-odd mass differences
the Argonne V18 (CD-Bonn) potential in Ref. [19].) Note that
scheme, as in TableI, evaluation using the Argonne V18 potential.
Finally, in order to better visualize the effect of the new
parametrisation of the particle-particle interaction, we show
in Fig. 4 an example of the multipole decomposition of the matrix element M
0\nu
. One can see there that the contribution of
the intermediate multipole 0
\nu
is drastically reduced with our
choice of g
\nu
=F=F=1
, while all the other multipoles are affected only
slightly or not at all. This is, in some sense, analogous to the
situation with M
0\nu
\text{GT}, where the parameter g
\nu
=pp
affects mostly
the intermediate 1
\nu
states, while all the other multipoles
are affected much less.
We compare in Fig. 5 the M
0\nu
matrix elements for all
considered nuclei evaluated with the old and new parametrizations
of g
\nu
=pp
. The smaller values of M
0\nu
in 48Ca, 166Cd, 124Sn,
136Xe, and to some extent also in 96Zr are related to the magic
or semimagic nucleon number in these nuclei, and thus to the
reduced pairing correlations in them.

V. COMPARISON OF THE \( \chi_F \) VALUES EVALUATED
BY DIFFERENT METHODS
As we argued in this work, the result of the new parametrisation
of the particle-particle interaction, which partially restores
isospin symmetry and leads to the correct M
0\nu
=F=F=0
value, is the reduction of the Fermi part M
0\nu
=F=F,pp
uclear
matrix element. At the same time, the largest component of
that matrix element, M
0\nu
\text{GT}, remains essentially unaffected. One
can see that most clearly by considering the quantity \( \chi_F \), the
ratio M
0\nu
=F=F/M
0\nu
\text{GT}.
In Table IV we compare the \( \chi_F \) values obtained with
different methods. [An analogous table, naturally without our
new results, appears in Ref. [20] in their Table VII. However,
as we already mentioned, their definition of \( \chi_F \) contains an
extra factor \( (g_V/g_A)^2 \).] One can see in Table IV that in the
nuclear shell model, and in our QRPA calculation with the
new parametrization of g
\nu
=pp, the \( \chi_F \) values are substantially
smaller than in the previous approaches. (In IBM-2 the \( \chi_F \)
are very small when neutrons and protons are in different shells.
That is an artifact of the model where only one shell in each
system is included.)
In the shell model, and in our new QRPA calculations, the
\( \chi_F \) values are relatively close to \(-1/3\), the value one would
obtain in pure \( S = 0 \) states. However, in the shell model the
\( \chi_F \) values are systematically smaller than in our version
of the QRPA. Why this is so remains to be understood. (To be
really precise, \( \chi_F = -1/3 \) would arise for pure \( S = 0 \) when
the higher order terms in the weak current are absent, when
in the nucleon form factor the cutoff parameters for the vector
and axial vector currents are the same, and the average energies
\( E \) are chosen to be the same in both neutrino potentials.) As
we pointed out before, while the \( S = 0 \) component is large,
the other parts, in particular \( S = 1 \), are clearly present.
We may notice that the QRPA values of \( \chi_F \) are always
smaller with the quenched value \( g_A = 1.0 \) compared to the
unquenched value \( g_A = 1.27 \). That trend continues when the
amount of quenching is increased, e.g., to \( g_A = 0.8 \) where \( \chi_F \)
values are really quite close to \(-1/3\). However, the question

From the tables one can see that the new parametrization,
leading to M
0\nu
=F=F=0
= 0, leads to a substantial reduction of the
M
0\nu
component of M
0\nu
and an overall \(-\{10\%–20\%\) reduction of the
final M
0\nu
nuclear matrix elements. It is encouraging
that both variants of the M
0\nu
matrix elements for 48Ca are now
rather close to the results of nuclear shell model evaluation.
(With \( g_A = 1.27 \) our M
0\nu
values are 0.54 in the listed case
and 0.71 in the variant where the even-odd mass differences
are treated as arising from pairing, both with the Argonne V18
potential and 0.59 (0.77) with the CD-Bonn potential, while
the shell model values are 0.59 in Ref. [18] and 0.82 (0.90) for
the Argonne V18 (CD-Bonn) potential in Ref. [19].) Note that
only in the case of 48Ca is the full oscillator \( pf \) shell included
and hence the Ikeda sum rule is fulfilled in the nuclear shell
model treatment. We are, naturally, well aware of the fact that
applying the QRPA in the case of 28Ca is questionable; our
results should be treated with that in mind.

V. COMPARISON OF THE \( \chi_F \) VALUES EVALUATED
BY DIFFERENT METHODS
As we argued in this work, the result of the new parametrisation
of the particle-particle interaction, which partially restores
isospin symmetry and leads to the correct M
0\nu
=F=F=0
value, is the reduction of the Fermi part M
0\nu
=F=F,pp
uclear
matrix element. At the same time, the largest component of
that matrix element, M
0\nu
\text{GT}, remains essentially unaffected. One
can see that most clearly by considering the quantity \( \chi_F \), the
ratio M
0\nu
=F=F/M
0\nu
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In Table IV we compare the \( \chi_F \) values obtained with
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\nu
=pp, the \( \chi_F \) values are substantially
smaller than in the previous approaches. (In IBM-2 the \( \chi_F \)
are very small when neutrons and protons are in different shells.
That is an artifact of the model where only one shell in each
system is included.)
In the shell model, and in our new QRPA calculations, the
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obtain in pure \( S = 0 \) states. However, in the shell model the
\( \chi_F \) values are systematically smaller than in our version
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the higher order terms in the weak current are absent, when
in the nucleon form factor the cutoff parameters for the vector
and axial vector currents are the same, and the average energies
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We may notice that the QRPA values of \( \chi_F \) are always
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unquenched value \( g_A = 1.27 \). That trend continues when the
amount of quenching is increased, e.g., to \( g_A = 0.8 \) where \( \chi_F \)
values are really quite close to \(-1/3\). However, the question

![FIG. 4.](image1.png)

![FIG. 5.](image2.png)
of quenching of the $0\nu\beta\beta$ matrix elements remains open, and in particular how to treat it properly in the QRPA goes beyond the scope of the present paper.

VI. CONCLUSIONS

By separating the particle-particle neutron-proton interaction into its isovector and isoscalar parts, and renormalizing them each separately with its own fitted parameters $g_{T=1}^{pp}$ and $g_{T=0}^{pp}$, we have achieved partial restoration of isospin symmetry and fulfillment of the requirement that $M_2^{0\nu} = 0.0$. This has been done essentially without introducing new parameters, since $g_{T=1}^{pp} \approx g_{pair}$ as required by the isospin symmetry of the particle-particle force. At the same time the isoscalar parameter $g_{T=0}^{pp}$ is fitted from the requirement that the calculated $2\nu\beta\beta$ half-life is the same as its experimental value. The resulting $g_{T=0}^{pp}$ is then almost the same one as with the old parametrization with the single $g_{pp}$ value.

When the new parametrization of the particle-particle renormalization constants is used in the QRPA evaluation of the $0\nu\beta\beta$ nuclear matrix elements, a substantial reduction of the Fermi part, $M_1^{0\nu}$, is observed, while the Gamow-Teller and tensor parts remain essentially unaffected. The full matrix elements $M_i^{0\nu}$ are reduced by $\sim 10\% - 20\%$, as seen in Fig. 5. We believe that such reduction, which also brings the ratio $\chi_F$ closer to $\approx -1/3$, nearer to its value in the isospin-conserving nuclear shell model values, is realistic and should be used in the future application of the QRPA and its generalizations.

ACKNOWLEDGMENTS

Useful discussions with Kazuo Muto are appreciated. The work of P.V. was partially supported by US Department of Energy Grant No. DE-FG02-92ER40701. F.Š. acknowledges support by the VEGA Grant Agency of the Slovak Republic under Contract No. 1/0876/12 and by the Ministry of Education, Youth and Sports of the Czech Republic under Contract No. LM2011027. F.Š. and A.F. thank the Deutsche Forschungsgemeinschaft for support under Contract No. 436SLK113/14/0-1.

[21] J. Suohon and O. Civitarese (as quoted in Table VII of Ref. [20]).