Π₁¹ Borel Sets
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§0. Introduction. The results in this paper were motivated by the following question of Sacks. Suppose $T$ is a recursive theory with countably many countable models. What can you say about the least ordinal $\alpha$ such that all models of $T$ have Scott rank below $\alpha$? If Martin’s conjecture is true for $T$ then $\alpha \leq \omega \cdot 2$.

Our goal was to look at this problem in a more abstract setting. Let $E$ be a $\Sigma^1_1$ equivalence relation on $\omega$ with countably many classes each of which is Borel. What can you say about the least $\alpha$ such that each equivalence class is $\Pi^0_\alpha$? This problem is closely related to the following question. Suppose $X \subseteq \omega$ is $\Pi^1_1$ and Borel. What can you say about the least $\alpha$ such that $X$ is $\Pi^0_\alpha$?

In §1 we answer these questions in ZFC. In §2 we give more informative answers under the added assumptions $V = L$ or $\Pi^1_1$-determinacy. The final section contains related results on the separation of $\Pi^1_{2n+1}$ sets by Borel sets.

Our notation is standard. The reader may consult Moschovakis [5] for undefined terms.

Some of these results were proved first by Sami and rediscovered by Kechris and Marker.

§1. Borel $\Pi^1_1$-sets.

Definition 1.1. If $X \subseteq \omega$, let $\hat{X} = \{\omega \in \omega \text{O} : |w| \in X\}$. If $\hat{X}$ is $\Pi^1_1$ in the codes, we say that $X$ is $\Pi^1_\alpha$ in the codes. An ordinal $\alpha$ is a basis for subsets of $\omega$ which are $\Pi^1_1$ in the codes iff whenever $X \subseteq \omega$, $X \neq \emptyset$ and $\hat{X}$ is $\Pi^1_1$, there is $\beta \in X$ such that $\beta < \alpha$. We let $\gamma^1_\alpha$ be the least such ordinal.

In [1] Kechris showed that, assuming PD, $\gamma^1_{2n+1} = \delta^1_{2n+1}$ for $n \geq 1$.

Lemma 1.2. If $X \subseteq \omega$ is $\Pi^1_1$ and Borel, then $X$ is $\Pi^1_\beta$ for some $\beta < \gamma^1_2$.

Proof. Let $F$ be a recursive function such that $x \in X$ if and only if $f(x) \in \omega$. For $\eta < \omega_1$ let $X_\eta = \{x \in X : |f(x)| < \eta\}$. Since $X$ is Borel, $f(X)$ is a $\Sigma^1_1$ subset of $\omega$. Thus there is $\eta < \omega_1$ such that $X = X_\eta$. Let $Z = \{w \in \omega \text{O} : \forall x \in X |f(x)| < |w|\}$. Then $Z \in \Pi^1_2$, so there is $w \in Z$ such that $\eta = |w| < \gamma^1_2$. Thus $X$ is Wadge reducible to $\omega_\eta$, but by Stern [9] $\omega_\eta$ is $\Sigma^0_\eta$. Hence since $\gamma^1_2$ is closed under ordinal addition $X$ is $\Pi^1_\beta$ for some $\beta < \gamma^1_2$. 

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We will next see that \( \gamma_1^2 \) is the least such ordinal. Suppose \( X \) is a nonempty bounded initial segment of \( \omega_1 \) and \( X \) is \( \Sigma_4^1 \). There is a tree \( \mathcal{T} \) on \( \omega \times \omega_1 \) such that \( X = \{ x \in \omega: 3f: \omega \rightarrow \omega_1 \langle x, f \rangle \in \mathcal{T} \} \). For \( \alpha < \omega_1 \) and \( x \in \omega \), let \( \mathcal{T}_x^n = \{ \tau: n \rightarrow \alpha: n \in \omega \text{ and } \langle x | n, \tau \rangle \in \mathcal{T} \} \). Then there is a recursive function \( f \) such that if \( \omega \in \mathcal{T} \) and \( x \in \omega \), then \( f(\omega, x) \) is a code for \( \mathcal{T}_x^w \) and \( \mathcal{T}_x^w \) is \( \mathcal{T}_x^w \). Let \( R \in \Pi_1^1 \) and \( R \in \Sigma_1^1 \) be such that for \( u, v \in \mathcal{T} \) and \( x \in \omega \)

\[
R^*(u, v, x) \Leftrightarrow R**(u, v, x) \Leftrightarrow R(|u|, |v|, x).
\]

Let

\[
A(v, w', w) \Leftrightarrow w, w', v \in \mathcal{T} \land |w| = |w'| \land \exists n_0, n_1 R^*(v|n_0, v|n_1, w') \land \forall v'(|v'| < |v| \rightarrow \forall w*\{|w*| = |w| \rightarrow \forall n_0, n_1 \rightarrow R**(v'|n_0, v'|n_1, w*)\}).
\]

Then \( A \) is \( \Pi_1^1 \).

For all \( \alpha \in X \) we can find \( \beta_x = \mu \beta \exists w \in \mathcal{T} |w| = \alpha \land \forall \nu \in \mathcal{T} \). Let \( w_x \) be chosen such that \( \mathcal{T}^w_x \notin \mathcal{T} \). Let \( \gamma_x = \mu \gamma R(\beta_x, \gamma, w_x) \) and let \( \delta_x = \sup \{ \beta_x + 1, \gamma_x + 1 \} \). Then

\[
A = \bigcup_{v \in X} \{(v, w', w) : |w| = \alpha \land |v| = \delta_x \land |w| = |w'| \rightarrow R(\beta_x, \gamma_x, w')\}.
\]

Since \( X \) is bounded, \( A \) is a countable union of Borel set and hence Borel. For all \( \alpha \in X \) there are \( V \) and \( w' \) such that \( \{ w : A(v, w', w) \} = \{ w \in \mathcal{T} : |w| = \alpha \} \). Thus, for all \( \alpha \in X \), \( A \) has Borel rank greater than or equal to \( \alpha \).

**Lemma 1.3** (Stern [9]). Suppose \( \alpha = \omega^\beta \). Then \( \{ x : x \in \mathcal{T} |x < \alpha \} \) is \( \Sigma_2^{\beta+1} \) and \( \{ x \in \mathcal{T} : |x| \leq \alpha \} \) is not \( \Sigma_2^{\beta+1} \). In particular, \( \{ x \in \mathcal{T} : |x| = \alpha \} \) is not \( \Pi_2^{\beta+1} \).

**Theorem 1.4.** \( \gamma_1^2 = \sup \{ \alpha : \exists X \subseteq \omega \alpha \text{ } X \text{ is } \Pi_1^1, \text{ Borel and } \alpha \text{ is the least ordinal such that } X \text{ is } \Pi_1^1 \} \).

**Proof.** In view of Lemma 1.2 we need only show that if \( \delta < \gamma_1^2 \), there is a \( \Pi_1^1 \) Borel set \( A \) which is not \( \Pi_1^0 \).

Let \( X \subseteq \omega_1 \) be nonempty, bounded and \( \Sigma_1^1 \) in the codes such that, for all \( \alpha \leq \delta, \alpha \in X \). Let \( X^* = \{ \beta : \exists x \in X \forall \gamma \leq \alpha \forall y \in X \land \beta \leq \omega^\beta \}. \) There is a recursive function \( f \) such that \( f(x) \in \mathcal{T} \) and \( f(x) = \omega^{|f(x)|} \). Thus \( X^* = \{ \omega \in X \forall n \in \omega \land |n| \leq |f(x)| \}. \) Then \( X^* \) is a proper initial segment of \( \omega_1 \) containing \( \omega^\delta \) which is \( \Sigma_1^1 \) in the codes. By the above construction we can find a \( \Pi_1^1 \) Borel set \( A \) which has \( \{ w \in \mathcal{T} : |w| = \omega^\delta \} \) as a section. By Lemma 1.3, \( A \) is not \( \Pi_1^0 \).

**Corollary 1.5.** For all \( \alpha < \gamma_1^2 \) there is a \( \Sigma_1^1 \) equivalence relation with countably many classes such that each class is Borel but at least one class is not \( \Pi_1^0 \).

**Proof.** Let \( A \) be \( \Pi_1^1 \) and Borel but not \( \Pi_1^0 \). Let \( \Psi : A \rightarrow \omega_1 \) be a \( \Pi_1^1 \)-norm. Since \( A \) is Borel, there is \( \delta < \omega_1 \) such that \( \forall x \in A \Psi(x) < \delta. \) Define an equivalence relation \( x \equiv y \Leftrightarrow (x \notin A \land y \notin A) \lor \Psi(x) = \Psi(y). \)

If each \( E \) class is \( \Pi_1^0 \), then \( A \) would be \( \Sigma_1^0 \), a contradiction.

If \( E \) is a \( \Sigma_1^1 \) equivalence relation with countably many equivalence classes each of which is Borel, then there is \( \alpha < \gamma_1^2 \) such that all \( E \) classes are \( \Pi_1^0 \). In fact the following stronger theorem is true.
THEOREM 1.6 [6]. If $E$ is a $\Sigma^1_1$ equivalence relation with Borel equivalence classes and there is a bound on the ranks of the classes, then there is $\alpha < \gamma^*_2$ such that every $E$ class is $\Pi^0_\alpha$.

**Proof.** Let $f$ be a recursive function such that $xEy \iff f(x, y) \notin WO$. For $\eta < \omega_1$, say

$$xE_{\eta}y \iff \neg(f(x, y) \in WO \land f(x, y) \leq \eta).$$

Then $E = \bigcap_{\eta < \omega_1} E_{\eta}$. For any $x$, since $\{y : yEx\}$ is Borel, by boundedness we can find a $\gamma_x < \omega_1$ such that $\forall y xE_y \rightarrow xE_{\gamma_x}y$, so $xEy \rightarrow xE_{\gamma_x}y$. If each $E$ class is $\Pi^0_\alpha$, then for each $\beta$ and $x$ we can separate $\{y : xEy\}$ from $\{y : xE_{\beta}y\}$ by a $\Pi^0_\alpha$ set. On the other hand if for all $\beta$ we can separate $\{y : xEy\}$ from $\{y : xE_{\beta}y\}$ by a $\Pi^0_\beta$ set, then since eventually $\{y : xEy\} = \{y : xE_{\beta}y\}$, $\{y : xEy\}$ is $\Pi^0_\alpha$.

Suppose $v, w \in WO$. Since $\{y : yEx\} = \Sigma^1_1(x)$ and $\{y : yE_{|w|x}\}$ is $\Delta^1_1(x, w)$, if they can be separated by a $\Pi^0_1$ set, by Louveau's separation theorem [3] they can be separated by a $\Pi^0_{|w|}$ set with code hyperarithmetic in $\langle v, w, x \rangle$.

Thus if $Z = \{w \in WO : \forall E$ class is $\Pi^0_{|w|}\}$, then

$$w \in Z \iff w \in WO \land \forall x, v (v \in WO \rightarrow \exists z \leq hyp \langle x, v, w \rangle z \text{ is a } \Pi^0_{|w|}	ext{-code}$$

$$\land \forall y((xEy \rightarrow y \in B(z)) \land (xE_{|w|x}y \rightarrow y \notin B(z))),$$

where $B(z)$ is the Borel set coded by $z$. Since the quantifier $\exists z \leq hyp \langle x, v, w \rangle$ is really universal, $Z$ is $\Pi^1_1$. Thus there is $w \in Z$ with $|w| < \gamma^*_2$.

**Question.** Suppose $G$ is a Polish group acting on $^\omega \omega$ with countably many orbits. What can we say about the least $\alpha$ such that every orbit is $\Pi^0_\alpha$? By results of Sami [7], $\alpha < \delta^*_2$.

**§2. Bounds on $\gamma^*_2$.**

**Lemma 2.1.** $\delta^*_2 < \gamma^*_2$.

**Proof.** If $X = \{x : \alpha < \delta^*_1\}$, then $\hat{X} = \{y \in WO : \exists x \in \Delta^1_1 x \in WO \land |x| = |y|\}$. Since $\hat{X}$ is $\Sigma^1_2$, there is $\delta \in \omega_1 - X$ such that $\delta^*_2 \leq \delta < \gamma^*_2$.

**Theorem 2.2** [5]. If $V = L$, then $\gamma^*_1 = \delta^*_1$.

**Proof.** If $V = L$, then $\Delta^1_3$ is a basis for $\Sigma^1_3$. Thus every nonempty $\Pi^1_1$ set contains a $\Delta^1_3$ member. So $\gamma^*_1 \leq \delta^*_3$.

Suppose $y \in WO$ is $\Delta^1_3$. Say $y(n) = m \iff \exists r A(r, \langle n, m \rangle)$ and $y(n) \neq m \iff \exists r B(r, \langle n, m \rangle)$, where $A$ and $B$ are $\Pi^1_1$. Then

$$x = y \iff \exists r \forall n, m((x(n) = m \rightarrow A(r_n, \langle n, m \rangle)) \land (x(n) \neq m \rightarrow B(r_m, \langle n, m \rangle))).$$

Thus $x = y \iff \exists r C(r, x)$, where $C$ is $\Pi^1_1$ and $x$ is recursive in every element of $C$.

Let $Z = \{z \in WO : L_{|z|} \models KP \land \exists r, x \in L_{|z|}(r, x) \in C\}$. Since $V = L$, $Z$ is non-empty. Thus

$$Z' = \{z \in WO : \forall x \in L_{|z|} (x \in WO \rightarrow |x| < |z|) \land \exists r, x \in L_{|z|}(r, x) \in C\} \models Z \neq \emptyset.$$

But “$x \in L_{|z|}$” is $\Delta^1_1$ and if $r, x \in L_{|z|}$, then $r, x \leq hyp z$, so

$$z \in Z' \iff \forall x((x \in L_{|z|} \land x \in WO) \rightarrow |x| < |z|) \land \exists r, x \leq hyp z (r, x) \in C.$$

So $Z'$ is $\Pi^1_1$. Thus there is $z \in Z'$ such that $|z| < \gamma^*_2$ and $y \in L_{|z|}$. Thus $|y| < |z| < \gamma^*_2$. $\square$
We will see that under the assumption of $H_1$-determinacy $\gamma_2$ is quite large in $L$ but much smaller than $\delta_1$.

**Theorem 2.3.** Suppose, for all $\xi < \gamma_2$, $\mathcal{N}_{\xi}^L < \aleph_1$. Then, for all $\xi < \gamma_2$, $\mathcal{N}_{\xi}^L < \gamma_2$.

**Proof.** Pick $\xi < \gamma_2$. Let $X \subseteq \omega_1$ be a bounded initial segment of $\omega_1$ containing $\xi$ which is $\Sigma^1_2$ in the codes.

Suppose $(\omega, E)$ is a transitive, well founded model of $KP + V = L$. If $w \in \omega_1$ and $f: \text{dom}(w) \rightarrow \omega$ we say that $f$ is a $\omega$-chain in $(\omega, E)$ if and only if for all $n \in \text{dom}(w)$ if $f(n) = m$, then $(\omega, E) \models \"m = \mathcal{N}_{[w[n]]}\".$

Claim. For all $\alpha < \gamma_2$, $\mathcal{N}_\alpha^L = \sup \{\mathcal{N}_{(\omega, E)}^L: (\omega, E) \text{ is a transitive, well founded model of } KP + V = L, \alpha < \text{On}(\omega, E) \text{ and } (\omega, E) \models \mathcal{N}_\alpha \text{ exists}\}$.

$(\geq)$ Clear since, for some $\beta$, $(\omega, E) \models (\mathcal{E}^{\omega+1}_\beta)$; then $\mathcal{N}_{(\omega, E)}^L = \mathcal{N}_\beta^L$. This is possible since $\alpha + 1 < \gamma_2$, so $\omega_{\alpha+1} < \aleph_1$.

Let $Z = \{v \in \omega_1: \exists E, f, w, g(\omega, E) \text{ is a transitive well founded model of } KP + V = L, w \in \vec{X}, f: \text{dom}(w) \rightarrow \omega \text{ is a } \omega \text{-chain for } (\omega, E) \text{ and } g: \text{dom}(v) \rightarrow \{m \in \omega: \exists n \in \text{dom}(w) (\omega, E) \models \("m \text{ is an ordinal and } m \leq f(n)\") \text{ is order preserving}\} \}$. Then $Z = \Sigma^1_2$ and $Z = \{v \in \omega_1: \exists \delta \in \omega \mid |v| < \mathcal{N}_{\delta}^L\}$. Let $z \in \omega_1 - Z$ be such that $|z| < \gamma_2$. Then $\mathcal{N}_{\gamma_2}^L < \gamma_2^L$.

**Corollary 2.4.** $(\forall x \mathcal{N}_{\mathcal{L}^{\omega}(x)}^L < \aleph_1) \mathcal{N}_{\gamma_2}^L = \gamma_2$.

**Proof.** $\aleph_1$ is inaccessible in $L$. Thus $\aleph_1 = \mathcal{N}_{\aleph_1}^L$. So $\mathcal{N}_{\gamma_2}^L < \aleph_1$.

On the other hand, $\gamma_2$ will always behave reasonably well in $L$.

**Theorem 2.4.** $\gamma_2$ is definable in $L$ and $\text{cf}_L(\gamma_2) = \omega$.

**Proof.** In [2] Kechris and Moschovakis show that every subset of $\omega_1$ which is $H_2$ in the codes is constructible. In fact if $U \subseteq \omega \times R$ is an $\omega$-universal $\Pi^1_2$ set and $\vec{U} = \{(n, x): x \in \omega_1 \land \forall y \in \omega_1 \mid |y| = |x| \rightarrow (n, y) \in U\}$, then $Y = \{(n, x) \in \omega \times \omega_1: \exists x \in \omega_1 \land (n, x) \in U\}$ is constructible. Thus in $L$ we can define $\langle \alpha_n: n \in \omega \rangle \in L$, where

$$\alpha_n = \begin{cases} 0 & \forall \alpha < \omega_1 (n, \alpha) \notin Y, \\ \text{least } (n, \alpha) \in Y & \text{otherwise}. \end{cases}$$

Then $\gamma_2 = \sup_n \alpha_n$. [We thank the referee for pointing out this simple argument.]

**Corollary 2.5 ($\Pi^1_2$-AD).** $\gamma_2$ is less than the first Silver indiscernible, so $\gamma_2 < \delta_1$.

§3. A separation theorem for $\Pi^1_{2n+1}$-sets. We assume projective determinacy. Let $U$ be an $\omega$-universal $\Pi^1_{2n+1}$ set. Let $\varphi: U \rightarrow \delta_{2n+1}$ be a $\Pi^1_{2n+1}$-norm. Let $A$ and $B$ be disjoint $\Pi^1_{2n+1}$ sets, and let $e_0, e_1 \in \omega$ be such that $A = \{x: (e_0, x) \in U\}$ and $B = \{x: (e_1, x) \in U\}$. For $\eta < \delta_{2n+1}$, let $A_\eta = \{x: \varphi(e_0, x) < \eta\}$ and $B_\eta = \{x: \varphi(e_1, x) < \eta\}$.

The following is a generalization of a weak version of a theorem of Stern [10].

**Theorem 3.1.** Suppose, for all $\eta < \delta_{2n+1}$, $A_\eta$ and $B_\eta$ are $\Pi^0_2$-separable. Then $A$ and $B$ are $\Pi^0_2$-separable.

The proof uses the analysis of certain Wadge-like games from [8]. If $X_0 \cap X_1 = Y_0 \cap Y_1 = \emptyset$, consider the game $G(\langle X_0, X_1 \rangle, \langle Y_0, Y_1 \rangle)$ where if I plays $\alpha \in \omega \omega$ and II plays $\beta \in \omega \omega$, then II wins if and only if $\alpha \in X_0 \rightarrow \beta \in Y_0$ and $\alpha \in X_1 \rightarrow \beta \in Y_1$. We write $\langle X_0, X_1 \rangle \leq \langle Y_0, Y_1 \rangle$ if and only if II has a winning strategy.
LEMMA 3.2. Assume all $X_i$ and $Y_i$ are projective. If $\langle X_0, X_1 \rangle \not\leq \langle Y_0, Y_1 \rangle$, then $\langle Y_0, Y_1 \rangle \not\leq \langle X_1, X_0 \rangle$.

PROOF. By PD, I has a winning strategy $\sigma$ in $G(\langle X_0, X_1 \rangle, \langle Y_0, Y_1 \rangle)$. Suppose II plays $G(\langle Y_0, Y_1 \rangle, \langle X_1, X_0 \rangle)$ using $\sigma$ (and ignoring I's last move). If I plays $\alpha$ and II plays $\beta$, then I wins $G(\langle X_0, X_1 \rangle, \langle Y_0, Y_1 \rangle)$ on the play $\beta, \alpha$. So either $\beta \in X_0$ and $\alpha \notin Y_0$ or $\beta \in X_1$ and $\alpha \notin Y_1$. Thus $\alpha \in Y_0 \rightarrow \beta \in X_1$ and $\alpha \in Y_1 \rightarrow \beta \in X_0$. So this is a winning strategy for II in $G(\langle Y_0, Y_1 \rangle, \langle X_1, X_0 \rangle)$.

LEMMA 3.3. Let $\mathcal{C}$ be any Wadge class. If $\langle X_0, X_1 \rangle \not\leq \langle Y_0, Y_1 \rangle$ and $Y_0, Y_1$ can be separated by some $D \in \mathcal{C}$, then $X_0$ and $X_1$ can be separated by some $\hat{D} \in \mathcal{C}$.

PROOF. Let $\hat{D}$ be the inverse image of $D$ under the winning strategy.

LEMMA 3.4. If $X_0$ and $X_1$ are projective and $C$ is complete $\Pi_0^0$, then $\langle X_0, X_1 \rangle \not\leq \langle C, \neg C \rangle$ if and only if $X_0$ and $X_1$ are $\Pi_0^0$-separable.

PROOF. (\Rightarrow) Clear by 3.3.

(\Leftarrow) Let $D \in \Pi_0^0$ separate $X_0$ and $X_1$. Then $D$ is Lipschitz reducible to $C$. II can win $G(\langle X_0, X_1 \rangle, \langle C, \neg C \rangle)$ by playing the winning strategy in the Lipschitz game.

PROOF OF 3.1. Suppose $A$ and $B$ are not $\Pi_0^0$-separable. Then $\langle A, B \rangle \not\leq \langle C, \neg C \rangle$, where $C$ is complete $\Pi_0^0$. Thus, as 3.2, $\langle \neg C, C \rangle \not\leq \langle A, B \rangle$. Let $\sigma$ be II's strategy in $G(\langle \neg C, C \rangle, \langle A, B \rangle)$ and let $f_\sigma$ be the continuous function it determines. Then $f(\neg C) \subseteq A$ and $f(C) \subseteq B$. Since $f(\neg C)$ and $f(C)$ are $\Sigma_1^1$ sets, by boundedness there is $\eta < \delta_{2n+1}$ such that $f(\neg C) \subseteq A_\eta$ and $f(C) \subseteq B_\eta$. Thus, using $\sigma$, II also wins $G(\langle \neg C, C \rangle, \langle A_\eta, B_\eta \rangle)$. So $\langle \neg C, C \rangle \not\leq \langle A_\eta, B_\eta \rangle$. Since $A_\eta$ and $B_\eta$ are $\Pi_0^0$-separable, $\langle A_\eta, B_\eta \rangle \leq \langle C, \neg C \rangle$. But then $\langle \neg C, C \rangle \leq \langle C, \neg C \rangle$. But then $\neg C$ is Lipschitz reducible to $C$, a contradiction.

This proof also works if we replace $\Pi_0^0$ by a Wadge class of $\Lambda_{2n+1}$ sets containing a complete set.

Question. Suppose $X$ and $Y$ are disjoint $\Sigma_{2n+2}$ sets, $A$ and $B$ are $\Pi_{2n+1}$ sets such that $\pi(A) = X$ and $\pi(B) = Y$ and, for $\eta < \delta_{2n+1}$, $X_\eta = \pi(A_\eta)$ and $Y_\eta = \pi(B_\eta)$. Suppose, for all $\eta$, $X_\eta$ and $Y_\eta$ can be separated by a $\Pi_0^0$ set. Can $X$ and $Y$ be separated by a $\Pi_0^0$-set? Stern [10] showed the answer is yes if $n = 0$ (using the weaker assumption $\forall \eta \forall \xi_\eta^{\Pi_0^0} < \eta$).

COROLLARY 3.5. If $A$ and $B$ are Borel separable $\Pi_{2n+1}$ sets, then $A$ and $B$ can be separated by a $\Pi_0^0$ set for some $\alpha < \gamma_{2n+2}$.

PROOF. Let $Z = \{w \in W0: A$ and $B$ are $\Pi_0^0$-separable}. Then

$$w \in Z \iff \forall \eta < \delta_{2n+1} (\exists z(z \text{ is a } \Pi_0^0\text{-code} \land \forall x(x \in A_\eta \rightarrow x \in B(z))) \land \forall y(y \in B_\eta \rightarrow y \notin B(z))),$$

where $B(z)$ denotes the Borel set coded by $z$. By Louveau and Saint Raymond ([3] and [4]), $A_\eta$ and $B_\eta$ are $\Pi_0^0$-separable if and only if for any $(e, s) \in U$ with $\varphi(e, s) = \eta$, there is a $\Pi_0^0$ set separating $A_\eta$ and $B_\eta$ with code in $\Lambda_{2n+1}(w, s)$. Thus

$$w \in Z \iff \forall e, s(e, s) \in U \rightarrow \exists z \in \Lambda_{2n+1}(w, s)(z \text{ is a } \Pi_0^0\text{-code} \land \forall x(\varphi(e_0, x) < \varphi(e, s) \rightarrow x \in B(z)) \land \forall y(\varphi(e_1, y) < \varphi(e, s) \rightarrow y \notin B(z))).$$

Thus $Z$ is $\Pi_{2n+2}^1$. The next result shows this is best possible.
PROPOSITION 3.6. For each \( \alpha < \gamma \frac{1}{2n+2} \) there are \( \Pi^1_{2n+1} \) sets \( A \) and \( B \) which are Borel separable but not separable by a \( \Pi^0_\omega \) set.

PROOF. As in 1.4 we can find a bounded initial segment \( X \) of \( \mathbb{N}_1 \), containing \( \omega^x \) such that \( \hat{X} \) is \( \Sigma^1_{2n+2} \) and if \( U \) is a universal \( \Pi^1_{2n+1} \) set and \( \Psi: U \to \delta^1_{2n+1} \) is a \( \Pi^1_{2n+1} \) norm, we can find a \( \Pi^1_{2n+1} \) \( A \subseteq U \times \hat{X} \times \hat{X} \) such that the following conditions hold:

(i) If \( A(v, w', w) \), then \( |w| = |w'| \).

(ii) For all \( w \in \hat{X} \) there is a unique \( v \) such that for some \( w', w'' \)

\[ |w'| = |w''| = |w| \wedge A(v, w', w''). \]

(iii) If \( A(v, w', w) \) and \( |w''| = |w| \), then \( A(v, w', w') \).

Let \( B(v, w', w) \iff A(v, w', w') \wedge |w| \neq |w'| \). Then \( B \) is \( \Pi^1_{2n+1} \) and \( A \cap B = \emptyset \). Let \( C = \{(v, w', w); |w'| = |w| \in X\} \). Then since \( X \) is bounded, \( C \) is Borel. Clearly \( A \subseteq C \) and \( B \cap C = \emptyset \), so \( A \) and \( B \) are Borel separable. But if \( D \) separates \( A \) and \( B \), then \( \{z \in \mathcal{W}_\omega; |z| = \omega^x\} \) is a section of \( D \). Thus \( D \) is not \( \Pi^0_\omega \).

\[ \square \]

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