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FORCING WITH Δ PERFECT TREES AND MINIMAL Δ-DEGREES

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This paper is a sequel to [3] and it contains, among other things, proofs of the results announced in the last section of that paper.

In §1, we use the general method of [3] together with reflection arguments to study the properties of forcing with Δ perfect trees, for certain Spector pointclasses Γ, obtaining as a main result the existence of a continuum of minimal Δ-degrees for such Γ's, under determinacy hypotheses. In particular, using PD, we prove the existence of continuum many minimal Δ^1_{2n+1}-degrees, for all n.[2]

Following an idea of Leo Harrington, we extend these results in §2 to show the existence of minimal strict upper bounds for sequences of Δ-degrees which are not too far apart. As a corollary, it is computed that the length of the natural hierarchy of Δ^1_{2n+1}-degrees is equal to ω when $n \geq 1$. (By results of Sacks and Richter the length of the natural hierarchy of Δ^1_1-degrees is known to be equal to the first recursively inaccessible ordinal.)

§0. Preliminaries. We will follow in this paper standard notation and terminology, as in Moschovakis' book [7]. Letters e, i, j, k, l, m, n vary over the set of natural numbers ω, a, b, c over the Cantor space 2^ω and $\alpha, \beta, \gamma, \delta, \dots$ over the set of reals ω^ω. Finally $\xi, \eta, \kappa, \lambda$ always denote ordinals.

For the notion of Spector pointclass and its basic properties, as well as all other standard results of descriptive set theory which we use without explicit reference, we refer also to [7]. By IND(**R**) we denote the pointclass of all absolutely inductive (see [8]) pointsets and by HYP(**R**) = IND(**R**) $\cap$ $\neg$IND(**R**) its ambiguous part.

It is understood that we work in ZF + DC, with all other set theoretical hypotheses stated explicitly throughout this paper.

§1. Minimal Δ-degrees.

1.1. Let Γ be a Spector pointclass. For $\alpha, \beta \in \omega^\omega$ define the *Δ-reducibility*,

$$\alpha \leq_\Delta \beta \Leftrightarrow \alpha \in \Delta(\beta)$$

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[2]Forcing with perfect trees has also been previously studied in a fairly general, but different way from ours, context by Sacks. In particular, in his paper *F-recursiveness*, in ***Logic Colloquium '69*** (R.O. Gandy and C.E.M. Yates, Editors), North-Holland, Amsterdam, 1971, Sacks obtains minimal Δ-degrees for a variety of rigid (see [6] for an explanation of this term) Spector pointclasses as well as some non-Spector ones.

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and put

$$\alpha \equiv_\Delta \beta \Leftrightarrow \alpha \leq_\Delta \beta \wedge \beta \leq_\Delta \alpha.$$

The *Δ-degree* of α, in symbols $[\alpha]_\Delta$, is given by

$$[\alpha]_\Delta = \{\beta : \beta \equiv_\Delta \alpha\}.$$

If $\boldsymbol{\alpha} = [\alpha_\Delta]$, $\boldsymbol{\beta} = [\beta]_\Delta$ are Δ-degrees, let $\boldsymbol{\alpha} \leq \boldsymbol{\beta} \Leftrightarrow \alpha \leq_\Delta \beta$ and $\boldsymbol{\alpha} < \boldsymbol{\beta} \Leftrightarrow \alpha \leq \boldsymbol{\beta} \wedge \boldsymbol{\beta} \nleq \boldsymbol{\alpha}$. Also put $\mathbf{0} = [\lambda t.\, 0]_\Delta =$ the least Δ-degree.

1.2. DEFINITION. A Δ-degree $\boldsymbol{\alpha}$ is *minimal* if

(i) $\mathbf{0} < \boldsymbol{\alpha}$,

(ii) $\neg \boldsymbol{\beta}(0 < \boldsymbol{\beta} < \boldsymbol{\alpha})$.

We will be concerned in this section with the construction of minimal Δ-degrees for various pointclasses Γ.

1.3. The basic technique for achieving this is forcing with perfect binary splitting trees. As is well known, the origins of this idea go back to Spector's construction of minimal Turing degrees and, in a context closer to ours, to the Gandy-Sacks construction of minimal hyperdegrees [2].

To see what is the motivation for the use of this notion of forcing, let us start by analyzing a little closer the concept of the Δ-reducibility $\alpha \leq_\Delta \beta$.

Fix $\mathscr{P} \subseteq \omega \times \omega^\omega$, a universal for the Γ subsets of ω^ω Γ set, and also a Γ-norm $\sigma : \mathscr{P} \twoheadrightarrow \kappa$. If $\alpha \leq_\Delta \beta$, there is a Γ relation R such that

$$\alpha(k) = l \Leftrightarrow R(k, l, \beta),$$

so pick $e \in \omega$ with

$$\alpha(k) = l \Leftrightarrow \mathscr{P}(e, \langle k, l, \beta\rangle).$$

By boundedness, there is some $n \in \omega$ with $(n, \beta) \in \mathscr{P}$ and

$$\alpha(k) = l \Rightarrow \sigma(e, \langle k, l, \beta\rangle) \leq \sigma(n, \beta),$$

so we have

$$\alpha(k) = l \Leftrightarrow \mathscr{P}(e, \langle k, l, \beta\rangle) \Leftrightarrow \sigma(e, \langle k, l, \beta\rangle) \leq \sigma(x),$$

for any $x \in \mathscr{P}$ with $\sigma(x) \geq \sigma(n, \beta)$.

Define now for each $e \in \omega$ and each $\xi < \kappa$ the function $F_e^\xi : \omega^\omega \to \omega^\omega$, given by

$$F_e^\xi(\gamma)(k) = \begin{cases} \text{least } l \text{ such that } \sigma(e, \langle k, l, \gamma\rangle) \leq \xi & \text{if such exists,} \\ 0 & \text{otherwise.} \end{cases}$$

Then we have that $\alpha = F_e^\xi(\beta)$ for any $\xi \geq \sigma(n, \beta)$. Put

$$\lambda_\Gamma^\beta(\mathscr{P}, \sigma) \equiv \lambda_\Gamma^\beta = \sup\{\sigma(n, \beta) : \mathscr{P}(n, \beta)\}.$$

(It is easy to check that also $\lambda_\Gamma^\beta = \sup\{\sigma(x) : x \in \mathscr{P} \wedge x \in \Delta(\beta)\}$.) Then the above imply

1.4. LEMMA. *Let Γ be a Spector pointclass and $\alpha, \beta \in \omega^\omega$. Then the following are equivalent.*

(i) $\alpha \leq_\Delta \beta$.

(ii) *There is* $e \in \omega$ *and* $\eta < \lambda_\Gamma^\beta$ *such that for all* $\xi \geq \eta$, $\alpha = F_e^\xi(\beta)$.

From this it follows that in certain instances the reducibility $\alpha \leq_\Delta \beta$ can take a particularly simple form. This is the case when

$$\lambda_\Gamma^\beta = \lambda_\Gamma^0 = \sup\{\sigma(n, \lambda t.0) \colon \mathscr{P}(n, \lambda t.0)\}.$$

Because then if $\eta < \lambda_\Gamma^\beta$, there is $n \in \omega$ with $(n, \lambda t.0) \in \mathscr{P}$ and $\xi = \sigma(n, \lambda t.0) \geq \eta$. Abbreviating $F_e^n \equiv F_e^{\sigma(n,\lambda t.0)}$, we have that F_e^n is Δ-recursive (i.e. $F_e^n(\gamma)(k) = l$ is in Δ, for each fixed n, e) and moreover, if $\lambda_\Gamma^\beta = \lambda_\Gamma^0$, then for any $\alpha \leq_\Delta \beta$ there is n, e as above with $\alpha = F_e^n(\beta)$. For convenience putting $I = \{i \in \omega \colon ((i)_0, \lambda t.0) \in \mathscr{P}\}$ and for $i \in I$, $F_i = F_{(i)_1}^{(i)_0}$ we have

1.5. LEMMA. *Let* Γ *be a Spector pointclass and* $\alpha, \beta \in \omega^\omega$. *Then the following are equivalent when* $\lambda_\Gamma^\beta = \lambda_\Gamma^0$.

(i) $\alpha \leq_\Delta \beta$.

(ii) $\exists i \in I(F_i(\beta) = \alpha)$.

So for β's with the property that $\lambda_\Gamma^\beta = \lambda_\Gamma^0$, it is easy to understand what it means for a real α to be Δ-reducible to β. It is obtained from β by applying one of the functions in a fixed countable sequence of Δ-recursive functions (the F_i's).

We can see now the motivation for the use of Δ-coded perfect trees, after introducing the following convenient terminology.

1.6. DEFINITION. A Spector pointclass Γ is called *Baire-suitable* if

(i) every $A \subseteq 2^\omega$ in Δ has the property of Baire and if nonmeager, there is a perfect tree $T \subseteq 2^{<\omega}$ in Δ with $[T] \subseteq A$,

(ii) for each $A \subseteq \omega \times 2^\omega$ in Δ, the relation $P(n) \Leftrightarrow \{a \colon A(n, a)\}$ is meager, is in Δ.

We shall give several examples of such Spector pointclasses in the sequel. For the moment we prove the following lemma.

1.7. LEMMA. *Let* Γ *be a Baire-suitable Spector pointclass. If* $F \colon \omega^\omega \to \omega^\omega$ *is a* Δ-*recursive function, then for each perfect tree* $T \in \Delta$ *there is a perfect tree* $T' \in \Delta$ *with* $T' \subseteq T$, *such that either* (i) $F \restriction [T']$ *is* 1-1 *and continuous or* (ii) $F \restriction [T']$ *is constant.*

PROOF. Assume without loss of generality that $T = 2^{<\omega}$. For each finite sequence s from ω, let $N_s = \{\alpha \in \omega^\omega \colon s \subseteq \alpha\}$. Also put $f = F \restriction 2^\omega$. Then $f^{-1}[N_s]$ is in Δ so it has the property of Baire (in 2^ω). Let

$$G_s = \bigcup\{N_t^2 \colon t \in 2^{<\omega} \wedge N_t^2 - f^{-1}[N_s] \text{ is meager}\},$$

where $N_t^2 = \{a \in 2^\omega \colon t \subseteq a\}$. Then if $P_s = f^{-1}[N_s] \triangle G_s$, P_s is meager and if $A = 2^\omega - \bigcup_s P_s$, A is comeager and $f \restriction A$ is continuous. This is because $(f \restriction A)^{-1}[N_s] = G_s \cap A$. Moreover

$$a \notin A \Leftrightarrow \exists s(a \in P_s) \Leftrightarrow \exists s[(f(a) \in N_s \wedge a \notin G_s) \vee (f(a) \notin N_s \wedge a \in G_s)].$$

But since $a \in G_s \Leftrightarrow \exists t(\{b \in N_t^2 \colon f(b) \notin N_s\}$ is meager $\wedge\ a \in N_t^2)$, clearly (by (ii) of 1.6) this relation is in Δ and therefore so is A. By (i) of 1.6 now, find $T^* \subseteq 2^{<\omega}$ perfect with $T^* \in \Delta$ and $[T^*] \subseteq A$. If $f \restriction [T']$ is constant for some $T' \in \Delta$ perfect, $T' \subseteq T^*$, we are done. Otherwise for each $t \in T^*$ there are incompatible extensions

t_0, t_1 of t both in T^* with

$$f[[T^*] \cap N^2_{t_0}] \cap f[[T^*] \cap N^2_{t_1}] = \emptyset.$$

Moreover t_0, t_1 can be found in a Δ-way from t. Indeed, since $f \upharpoonright (N^2_t \cap [T^*])$ is not constant, there are t_0, t_1 incompatible extensions of t both in T^* and s_0, s_1 incompatible finite sequences, such that $f[N^2_{t_0} \cap [T^*]] \subseteq N_{s_0}$ and $f[N^2_{t_1} \cap [T^*]] \subseteq N_{s_1}$. If for each $u \in T$ we denote by $\bar{u}$ the leftmost branch of $[T^*] \cap N^2_u$, we then have

$$\begin{aligned}&\forall t \in T^* \exists t_0, t_1 [t_0, t_1 \in T^* \wedge t_0, t_1 \text{ extend } t \wedge t_0 \text{ is incompatible with } t_1 \wedge\\ &\exists s_0, s_1 (s_0, s_1 \text{ are incompatible} \wedge\\ &\forall u_0 \supseteq t_0 \forall u_1 \supseteq t_1 (u_0, u_1 \in T^* \Rightarrow f(\bar{u}_0) \in N_{s_0} \wedge f(\bar{u}_1) \in N_{s_1}))].\end{aligned}$$

As the expression in square brackets is in Δ, standard results on Spector pointclasses allow us to find a Δ-recursive function g which for every $t \in T^*$ picks t_0, t_1, s_0, s_1, satisfying this expression. If now

$$a \in [T^*] \cap N^2_{t_0}, \qquad b \in [T^*] \cap N^2_{t_1} \quad \text{and} \quad a \upharpoonright k_0 = t_0, b \upharpoonright k_1 = t_1,$$

then for every $N \geq \max\{k_0, k_1\}, f(\overline{a \upharpoonright N}) \in N_{s_0}$ and $f(\overline{b \upharpoonright N}) \in N_{s_1}$. But $\overline{a \upharpoonright N} \to_{N\to\infty} a$ and $\overline{b \upharpoonright N} \to_{N\to\infty} b$, so since $f \upharpoonright [T^*]$ is continuous $f(a) \in N_{s_0}, f(b) \in N_{s_1}$ and we are done.

It is routine now to construct a perfect tree $T' \subseteq T^*$, such that $T' \in \Delta$ and $f \upharpoonright [T']$ is 1-1, by iterating the process $t \to t_0, t_1$. ■

Notice now that if F is Δ-recursive and $F \upharpoonright [T]$ is 1-1 and continuous, where $T \in \Delta$ is a binary splitting perfect tree, then for each $\beta \in [T]$ we have $\beta \equiv_\Delta F(\beta)$, while if $F \upharpoonright [T]$ is constant, then for each $\beta \in [T]$ we have that $F(\beta) \in \Delta$ i.e. $F(\beta) \equiv_\Delta \lambda t.0$. From that it is routine to show that in forcing with Δ perfect binary splitting trees, all sufficiently generic reals β have minimal Δ-degree, *unless they satisfy* $\lambda^\beta_\Gamma \neq \lambda^0_\Gamma$.

More precisely, consider for each $A \subseteq \omega^\omega$ the following game.

I T_0 T_2	I plays a perfect binary splitting tree $T_0 \subseteq \omega^{<\omega}$ in Δ;
$\cdots$	II plays a perfect binary splitting tree $T_1 \subseteq T_0$ in Δ,
II T_1 T_3	of diameter ≤ 1;

I plays a perfect binary splitting tree $T_2 \subseteq T_1$ in Δ, of diameter $\leq \frac{1}{2}$, etc. Let $\lim T_n$ be the unique real in $\bigcap_n [T_n]$. Then II wins iff $\lim T_n \in A$. We abbreviate by

$$\forall T_0 \exists T_1 \subseteq T_0 \forall T_2 \subseteq T_1 \exists T_3 \subseteq T_2 \cdots (\lim T_n \in A),$$

the statement that II has a winning strategy in this game. If we let

$$\mathbf{P}(\Delta) = \{T : T \text{ is a } \Delta \text{ perfect binary splitting tree}\}$$

and for $T, T' \in \mathbf{P}(\Delta)$ we define the partial ordering $T \leq T' \Leftrightarrow T \supseteq T'$, we have a notion of forcing and if (in the notation of [3, 1. 1]) we let $X(\mathbf{P}(\Delta))$ be its associated topological space and we define $f: X(\mathbf{P}(\Delta)) \to \omega^\omega$ by $f(T_0, T_1, \ldots) =$ the leftmost branch of $\bigcap_n [T_n]$, then (in the notation again of [3, 4.2]) we have

$$\forall T_0 \exists T_1 \subseteq T_0 \cdots (\lim T_n \in A) \Leftrightarrow 0 \Vdash_{\overline{\mathbf{P}(\Delta), f}} A$$
$$\Leftrightarrow A \text{ holds for all sufficiently generic relative to } \mathbf{P}(\Delta), f \text{ reals.}$$

From now on we shall drop many times the explicit reference to f and write only $\Vdash_{\mathbf{P}(\Delta)}$ or say "for all sufficiently generic for $\mathbf{P}(\Delta)$ reals", etc.

After these explanations we can now state the following.

1.8. LEMMA. *Assume Γ is a Baire-suitable Spector pointclass. Then for all sufficiently generic for $\mathbf{P}(\Delta)$ reals β, we have $\lambda_\Gamma^\beta \neq \lambda_\Gamma^0 \vee \beta$ has minimal Δ-degree.*

PROOF. Consider the following strategy for player II in the type of game described before.

When I plays $T_{2n} \in \mathbf{P}(\Delta)$, II plays $T_{2n+1} \subseteq T_{2n}$ such that

(i) $T_{2n+1} \in \mathbf{P}(\Delta)$,

(ii) T_{2n+1} has diameter $\leq 1/(2n+1)$,

(iii) $F_n \upharpoonright [T_{2n+1}]$ is either 1-1 and continuous or constant,

(iv) $\beta_n \notin [T_{2n+1}]$ where $\{\beta_0, \beta_1, \ldots\}$ is an enumeration of the Δ reals. Here $\{F_i\}$ comes of course from 1.5.

Let $\beta = \lim T_n$. We can, of course, assume that $\lambda_\Gamma^\beta = \lambda_\Gamma^0$ (otherwise, we are done). By (iv) since $\beta \in \bigcap_n [T_n]$ we have that $\beta \notin \Delta$ i.e. $\boldsymbol{\beta} = [\beta]_\Delta > \mathbf{0}$. Let now $\boldsymbol{\alpha} = [\alpha]_\Delta \leq \boldsymbol{\beta}$. Then $\alpha \leq_\Delta \beta$ so by 1.5 we can find n with $\alpha = F_n(\beta)$. By (i), (ii), (iii) and our preceding remarks, if $F_n \upharpoonright [T_{2n+1}]$ is 1-1 and continuous, then since $\beta \in [T_{2n+1}]$ we have $\alpha \equiv_\Delta \beta$ i.e. $\boldsymbol{\alpha} = \boldsymbol{\beta}$, while if $F_n \upharpoonright [T_{2n+1}]$ is constant we have that $\alpha \in \Delta$ i.e. $\boldsymbol{\alpha} = \mathbf{0}$. So β has minimal Δ-degree and we are done. ∎

1.9. From this lemma it is clear that in order to show that for all sufficiently $\mathbf{P}(\Delta)$-generic reals β, β has minimal Δ-degree, which is our ultimate goal, it is enough to demonstrate that for all sufficiently generic such reals β we have $\lambda_\Gamma^\beta = \lambda_\Gamma^0$.

To establish this generic preservation of the ordinal assignment λ_Γ, for certain Γ's, we use the general ideas of forcing over a pointclass developed in §4 of [3].

1.10. First let us notice that $\mathbf{P}(\Delta)$ is a Γ-coded notion of forcing, i.e. there is $P \subseteq \omega$ in Γ and a surjection $\pi: P \twoheadrightarrow \mathbf{P}(\Delta)$ such that the relation $\pi(n) \leq \pi(m)$ is Δ on P i. e. there are relations A, B in Γ, $\neg\Gamma$ respectively, so that for $m, n \in P$: $\pi(n) \leq \pi(m) \Leftrightarrow A(m, n) \Leftrightarrow B(m, n)$. Moreover, we can arrange so that the relation $s \in T$ is Δ in the codes provided by π i.e. the relation $s \in \pi(n)$ is Δ on P (here $s \in \omega^{<\omega}$). We shall fix such P, π in the sequel.

Then, according to 4.4.1 of [3], verifying that $\lambda_\Gamma^\beta = \lambda_\Gamma^0$ for all sufficiently generic for $\mathbf{P}(\Delta)$ reals β is reduced to computing the following definability estimate for the forcing relation $\Vdash_{\mathbf{P}(\Delta)}$: for each $A \subseteq \omega^\omega \times \omega$ in Γ the relation

$$(*) \qquad T \nVdash_{\mathbf{P}(\Delta)} \neg A(\cdot, n),$$

is in Γ. Here $T \Vdash_{\mathbf{P}(\Delta)} B(\cdot, x)$ stands for $T \Vdash_{\mathbf{P}(\Delta)} \{\beta : B(\beta, x)\}$, where for each $C \subseteq \omega^\omega$ and each $T \in \mathbf{P}(\Delta)$ we define

$$T \Vdash_{\mathbf{P}(\Delta)} C \Leftrightarrow \forall T_0 \subseteq T \exists T_1 \subseteq T_0 \forall T_2 \subseteq T_1 \cdots (\lim T_n \in C).$$

(The expression on the right stands as usual for the statement: player II has a winning strategy in the following game

$$\begin{array}{lcccl} \text{I} & T_0 & & T_2 & \\ & & & & \cdots \\ \text{II} & & T_1 & & T_3 \end{array}$$

I, II alternatively play $T_0, T_1, \ldots$ in $\mathbf{P}(\varDelta)$ with $T_0 \subseteq T$, $T_{i+1} \subseteq T_i$ and diameter of $T_{i+1} \leq 1/(i+1)$; I wins iff $\lim T_n \in C$).

Moreover to say that (*) as above is in Γ means of course that the relation $D(m, n) \Leftrightarrow m \in P \wedge \pi(m) \nVdash_{\mathbf{P}(\varDelta)} \neg A(\cdot, n)$ is in Γ.

A word of caution: According to 4.4.1 of [3] the above reduction holds only when all sets in Γ have the property of Baire relative to $\mathbf{P}(\varDelta), f$ (see [3, 4. 2]).

We can summarize now the three conditions on a Spector pointclass which guarantee that all sufficiently generic for $\mathbf{P}(\varDelta)$ reals have minimal $\varDelta$-degree.

(A) Γ is Baire-suitable.

(B) Every $A \in \Gamma$ has the property of Baire relative to $\mathbf{P}(\varDelta), f$.

(C) The relation $T \nVdash_{\mathbf{P}(\varDelta)} \neg A(\cdot, n)$ is in Γ, for each $A \in \Gamma$, $T \in \mathbf{P}(\varDelta)$.

Using §5 of [3] we can see that there is a wide class of Spector pointclasses Γ for which (A) and (B) are satisfied. Let us introduce the following convenient terminology first.

1.11. DEFINITION. A pointclass Γ' is called *nice* if

(i) Γ' contains all the recursive pointsets and is closed under recursive substitutions and $\wedge$, $\vee$, $\exists^\omega$,

(ii) Γ' is ω-parametrized and scaled,

(iii) $\Gamma' \supseteq \Pi^0_1$, unless $\Gamma' = \Sigma^0_1$.

Examples of nice Γ's are Σ^0_n for $n \geq 1$, Π^1_1, Σ^1_2 and assuming PD, all Π^1_{2n+1}, Σ^1_{2n+2}. Assuming Determinacy($\mathbf{HYP}(\mathbf{R})$), IND($\mathbf{R}$) is another example.

Now call a Spector pointclass Γ $\mathfrak{D}$*-generated* (where $\mathfrak{D}$ is the game quantifier $\mathfrak{D}\, \alpha P(x, \alpha) \Leftrightarrow \exists \alpha(0)\ \forall \alpha(1)\ \exists \alpha(2)\ \forall \alpha(3) \cdots P(x, \alpha)$) if there is a nice Γ' with $\Gamma = \mathfrak{D}\, \Gamma' = \{\mathfrak{D}\, \alpha P(x, \alpha) : P \in \Gamma'\}$.

All the usual Spector pointclasses in descriptive set theory are $\mathfrak{D}$-generated, since $\Pi^1_1 = \mathfrak{D}\, \Sigma^0_1$, $\Sigma^1_2 = \mathfrak{D}\, \Pi^1_1$ and assuming PD, $\mathfrak{D}\, \Sigma^1_{2n} = \Pi^1_{2n+1}$ and $\mathfrak{D}\, \Pi^1_{2n+1} = \Sigma^1_{2n+2}$. Also IND($\mathbf{R}$) $= \mathfrak{D}$ IND($\mathbf{R}$) is $\mathfrak{D}$-generated. Note that for each nice Γ', $\mathfrak{D}\, \Gamma'$ is a Spector pointclass (see [7]).

1.12. THEOREM. *Let Γ be a $\mathfrak{D}$-generated Spector pointclass, say $\Gamma = \mathfrak{D}\, \Gamma'$ with Γ' nice. Assume* Determinacy($B(\Gamma')$), *where $B(\Gamma')$ is the smallest pointclass containing Γ' and closed under Borel substitutions. Then*

(A) *Γ is Baire-suitable,*

(B) *every $A \in \Gamma$ has the property of Baire relative to* $\mathbf{P}(\varDelta), f$.

PROOF. (A) is immediate from Theorem 5.3.1 of [3].

(B) can be proved as 5.2.2 of [3]. ■

Finally, we discuss conditions under which property (C) (the definability of forcing) is satisfied. We try to use a method analogous to that of §5 in [3], whose key ingredient is the use of the Game Formula 3.3.1 of [3]. The new problem that arises is that the notion of forcing $\mathbf{P}(\varDelta)$ is Γ-coded but not necessarily $\varDelta$-coded. This difficulty is overcome by the use of reflection arguments. These apply only to the so-called reflecting Spector pointclasses, among which we find most of the interesting $\mathfrak{D}$-generated Spector pointclasses, except for a couple of exceptions which sometimes can be handled separately (as for example $\Gamma = \Pi^1_1$ —see below). Let us first define this notion.

1.13. DEFINITION. A Spector pointclass Γ is called *reflecting* if for each real α, each $A \subseteq \omega$ in $\Gamma(\alpha)$ and each $R \subseteq \omega^\omega$ in $\Gamma(\alpha)$ we have

$$R(A) \Rightarrow \exists B \in \Delta(\alpha) R(B).$$

(When we write $R(A)$, where $R \subseteq \omega^\omega$ and $A \subseteq \omega$, we really mean $R(\chi_A)$, where χ_A is the characteristic function of A.)

Examples of reflecting Spector pointclasses are Σ^1_2 and, assuming PD, all Π^1_{2n+1} for $n \geq 1$ and all Σ^1_{2n+2}. Also if Determinacy(**HYP(R)**) holds, IND(**R**) is reflecting (see [4]). In fact, a more recent unpublished result of Martin and Solovay together with a theorem of [6] (16.1(iii)) imply that if $\Gamma' \supseteq \Pi^0_2$ is nice, then $\mathfrak{D}\,\Gamma'$ is reflecting. Thus, essentially all natural $\mathfrak{D}$-generated Spector pointclasses, *except* $\mathfrak{D}\,\Sigma^0_1 = \Pi^1_1$ *and* $\mathfrak{D}\,\Sigma^0_2$, are reflecting.

We have now the main result in this section.

1.14. THEOREM. *Let $\Gamma = \mathfrak{D}\,\Gamma'$ be a $\mathfrak{D}$-generated Spector pointclass, where Γ' is nice. If Γ is reflecting, then, assuming* Determinacy($B(\boldsymbol{\Gamma}')$), *we have that all sufficiently generic for* $\mathbf{P}(\Delta)$ *reals have minimal Δ-degree. In particular there is a continuum of minimal Δ-degrees.*

PROOF. We have only, according to the preceding results, to show that the relation $T \nVdash_{\mathbf{P}(\Delta)} \neg A(\cdot, n)$ is in Γ for each $A \subseteq \omega^\omega \times \omega$ in Γ. Since every set in $\boldsymbol{\Gamma}$ has the property of Baire relative to $\mathbf{P}(\Delta)$, f, this expression is (by the Negation Formula 3.2.1 of [3]) equivalent to

$$\exists T_0 \subseteq T \forall T_1 \subseteq T_0 \cdots A(\lim T_i, n).$$

Since $\Gamma = \mathfrak{D}\,\Gamma'$, let $B \in \Gamma'$ be such that $A(\beta, n) \Leftrightarrow \mathfrak{D}\,\alpha B(\beta, n, \alpha)$. Thus

$$\exists T_0 \subseteq T \forall T_1 \subseteq T_0 \cdots A(\lim T_i, n) \Leftrightarrow \exists T_0 \subseteq T\, \forall T_1 \subseteq T_0 \cdots \mathfrak{D}\,\alpha B(\lim T_i, n, \alpha).$$

Now we apply the Game Formula 3.3.1 of [3], to obtain that this last expression is equivalent to

$$\exists T_0 \subseteq T \exists \alpha(0)\, \forall T_1 \subseteq T_0 \forall \alpha(1)\, \exists T_2 \subseteq T_1 \exists \alpha(2) \cdots B(\lim T_i, n, \alpha),$$

where this abbreviates that player I has a winning strategy in the following game, G:

$$\begin{array}{llll} \mathrm{I} & \alpha(0), T_0 & & \alpha(2), T_2 \\ & & & \cdots \\ \mathrm{II} & & \alpha(1), T_1 & \end{array} \qquad \begin{array}{l} T_i \in \mathbf{P}(\Delta);\ \text{diameter}(T_{i+1}) \leq 1/(1+i); \\ T_0 \supseteq T_1 \supseteq T_2 \supseteq \cdots;\ \mathrm{I}\ \text{wins iff} \\ B(\lim T_i, n, \alpha). \end{array}$$

Notice now that if I has a winning strategy in G, then he has a winning strategy in which he can arrange so that not only the diameter of T_{2i+2} is $\leq 1/(2i+2)$ but also that $\alpha, \beta \in [T_{2i}] \Rightarrow \alpha \restriction i = \beta \restriction i$. But there is a total recursive function g such that if $T_0 \supseteq T_1 \supseteq T_2 \supseteq \cdots$ is a sequence of trees with these properties, then $g(T_0, T_1, \ldots) = \lim T_i$. Since for $k \in P$ the relation $s \in \pi(k)$ is in Δ, it follows that there is a total recursive function h of two arguments such that if $\gamma(0), \gamma(1), \ldots \in P$ and $\pi(\gamma(0)) \supseteq \pi(\gamma(1)) \supseteq \cdots$ and $\alpha, \beta \in [\pi(\gamma(2i))] \Rightarrow \alpha \restriction i = \beta \restriction i$, then $h(\gamma, C) = \lim \pi(\gamma(i))$, where C is some fixed Γ subset of ω. Using now the fact that π: $P \twoheadrightarrow \mathbf{P}(\Delta)$ and these remarks, we have (fixing some l with $\pi(l) = T$),

$$\exists T_0 \subseteq T \exists \alpha(0)\ \forall T_1 \subseteq T_0 \forall \alpha(1)\ \exists T_2 \subseteq T_1 \exists \alpha(2) \cdots B(\lim T_i, n, \alpha)$$

$$\Leftrightarrow \exists \gamma(0)\ \exists \alpha(0) \forall \gamma(1)\ \forall \alpha(1)\ \exists \gamma(2)\ \exists \alpha(2) \cdots \Big\{ [\gamma(0) \in P \wedge \pi(\gamma(0)) \subseteq \pi(1)] \wedge$$

$$\Big[\exists t (\gamma(0), \ldots, \gamma(2t) \in P \wedge \forall i < 2t \,(\text{diameter}(\pi(\gamma(i+1)) \leq \frac{1}{i+1} \wedge$$

$$\forall i \leq t \forall u, v \in \pi(\gamma(2i))\ (\text{length}(u), \text{length}(v) = i \Rightarrow u = v) \wedge$$

$$(\gamma(2t+1) \notin P \vee \pi(\gamma(2t+1)) \nsubseteq \pi(\gamma(2t)) \vee \text{diameter}(\gamma(2t+1)) \not\geq \frac{1}{2t+1})) \Big] \vee$$

$$\Big[\forall t (\gamma(t) \in P \wedge \pi(\gamma(t)) \supseteq \pi(\gamma(t+1)) \wedge \text{diameter}(\pi(\gamma(t+1)) \leq \frac{1}{t+1} \wedge$$

$$\forall u, v \in \pi(\gamma(2t))(\text{length}(u), \text{length}(v) \geq t \Rightarrow u \restriction t = v \restriction t)) \wedge$$

$$B(g(\gamma, C), n, \alpha) \Big] \Big\}.$$

The expression in $\{\ \}$ is clearly in $\Gamma'(D)$, where D is some fixed Γ subset of ω. Thus by the fact that Γ' has the scale property, we have by Moschovakis [7, Chapter 6] that if player I wins this game, he has a winning strategy in $\Delta(D)$ and therefore there is a winning strategy in $\Delta(D)$ for player I in the game G. By the proof of the Game Formula 3.3.1 of [3], it follows that I has a $\Delta(D)$ winning strategy also in the original game, G':

$$\begin{array}{llll} \text{I} & T_0 & & T_2 \\ & & & \cdots \\ \text{II} & & T_1 & \end{array} \qquad \begin{array}{l} T \supseteq T_0 \supseteq T_1 \supseteq \cdots;\ \text{diameter}(T_{i+1}) \leq 1/(i+1); \\ \\ \text{I wins iff } A(\lim T_i, n). \end{array}$$

From this and a routine splitting argument we have finally

$$\exists T_0 \subseteq T \forall T_1 \subseteq T_0 \cdots A(\lim T_i, n) \Rightarrow \exists S \subseteq T (S \in \Delta(D) \wedge [S] \subseteq \{\beta \colon A(\beta, n)\}).$$

For each perfect binary splitting tree S, let $h_S \colon 2^\omega \twoheadrightarrow [S]$ be the canonical homeomorphism. Then we have that if

$$(1) \Leftrightarrow \exists T_0 \subseteq T \forall T_1 \subseteq T_0 \cdots A(\lim T_i, n)$$

holds then

$\exists S \in \Delta(D)(S \subseteq T \wedge S$ is perfect binary splitting $\wedge \{a \in 2^\omega \colon A(h_S(a), n)\}$ is comeager).

By Theorem 5.3.1(ii) of [3] this last expression is equivalent to $R(n, D)$ for some relation R in Γ. So by reflection, if (1) holds, there is $E \in \Delta$ such that $R(n, E)$ i.e.

$\exists S \in \Delta(E)(S \subseteq T \wedge S$ is perfect binary splitting $\wedge \{a \in 2^\omega \colon A(h_S(a), n)\}$ is comeager),

therefore if (1) holds then (2) holds, where

$$(2) \Leftrightarrow \exists S \in \mathbf{P}(\Delta)(S \subseteq T \wedge \{a \in 2^\omega \colon A(h_S(a), n)\} \text{ is comeager}).$$

Now applying an (obvious) strengthening of Theorem 5.3.1(vi) of [3], i.e. that every comeager Γ set contains a Δ coded perfect subset, we see that in turn, if (2) holds there is $T_0 \subseteq T$ in $\mathbf{P}(\Delta)$ with $[T_0] \subseteq \{\beta \colon A(\beta, n)\}$, in which case clearly

(1) holds. Thus $\exists T_0 \subseteq T \forall T_1 \subseteq T_0 \cdots A(\lim T_i, n) \Leftrightarrow \exists\, S \in \Delta(S$ is a perfect binary tree $\wedge\, S \subseteq T \wedge \{a \in 2^\omega\colon A(h_S(a), n)\}$ is comeager) and by [3, Theorem 5.3.1(ii)] this last expression is in Γ so we are done. ■

We have now the following immediate corollary.

1.15. COROLLARY. (i) *Let* $n \geq 2$ *and assume* Determinacy(Σ^1_{n-1}). *Then there is a continuum of minimal* Δ^1_n*-degrees.*

(ii) *If* Determinacy(**IND**(**R**)) *holds, then there is a continuum of minimal* HYP(**R**)*-degrees.*

For even n, 1.15(i) can be substantially improved, see [1] and [5]. Theorem 1.14 clearly does not apply to the case $\Gamma = \mathfrak{D}\Sigma^0_1 = \Pi^1_1$ or $\Gamma = \mathfrak{D}\Sigma^0_2$. The result is of course true for Π^1_1 by Gandy and Sacks [2]. We show below how this case can be handled in our framework, by giving an inductive definability (instead of a reflection) argument for computing the complexity of the relation $T \not\Vdash_{\mathbf{P}(\Delta^1_1)} \neg A(\cdot, n)$ in this case. This seems to provide a new way of showing that all sufficiently generic for $\mathbf{P}(\Delta^1_1)$ reals have minimal Δ^1_1-degree. We do not know how to give a similar special argument to cover the case of $\mathfrak{D}\,\Sigma^0_2$, although we believe that this can also be done.

1.16. Let us deal now with the case $\Gamma = \Pi^1_1$. We want to show again that the relation $\exists T_0 \subseteq T \forall T_1 \subseteq T_0 \cdots A(\lim T_i, k)$ is Π^1_1 for each $A \in \Pi^1_1$. Let $U \in \Sigma^0_1$ be such that $A(\beta, n) \Leftrightarrow \forall \alpha U(\beta, k, \alpha)$, so that by the Game Formula

$$\exists T_0 \subseteq T \forall T_1 \subseteq T_0 \cdots A(\lim T_i, k) \Leftrightarrow \exists T_0 \subseteq T \forall T_1 \subseteq T_0 \forall \alpha(0) \exists T_2 \subseteq T_1 \forall T_3 \subseteq T_2 \forall \alpha(1) \cdots U(\lim T_i, k, \alpha).$$

Our argument is motivated by some ideas of Solovay in [10], where he uses an inductive analysis similar to the one below, to obtain an effective version of the open set Ramsey Theorem.

Since k is carried through as a parameter, we will not indicate it explicitly from now on. Also let t, u vary below over finite sequences from ω and S over perfect binary splitting trees, not necessarily in Δ^1_1. For such an S and every $t \in S$ we let $S_t = \{u \in S\colon u$ is compatible with $t\}$.

We define now a monotone operator Φ on the set of all pairs (u, S) as follows

$$(u, S) \in \Phi(X) \Leftrightarrow N_u \times [S] \subseteq U \vee \forall n \exists m \forall t[t \in S \wedge \operatorname{length}(t) \geq m \Rightarrow (u^\frown n, S_t) \in X].$$

Let, as usual, Φ^ξ be the ξth iterate of Φ and $\Phi^\infty = \bigcup_\xi \Phi^\xi$ its least fixed point. If $(u, S) \in \Phi^\infty$ we let $|u, S| =$ least ξ such that $(u, S) \in \Phi^\xi$, while if $(u, S) \notin \Phi^\infty$ we let $|u, S| = \infty$, so that $(u, S) \in \Phi^\infty \Leftrightarrow |u, S| < \infty$. The following two properties are now easy to verify.

(i) $(u, S) \in \Phi^\xi \wedge S' \subseteq S \Rightarrow (u, S') \in \Phi^\xi$, thus $S' \subseteq S \Rightarrow |u, S'| \leq |u, S|$.

(ii) For $T \in \mathbf{P}(\Delta^1_1)$, the relation $|u, T| < \infty$ is Π^1_1.

The following lemma is the key to the proof. We let T vary below over trees in $\mathbf{P}(\Delta^1_1)$.

1.17. LEMMA. *If* $\forall T_0 \subseteq T(|u, T_0| = \infty)$, *then* $\forall T_0 \subseteq T \exists n \exists T_1 \subseteq T_0 \forall T_2 \subseteq T_1(|u^\frown n, T_2| = \infty)$.

PROOF. Fix $T_0 \subseteq T$ and assume towards a contradiction that

$$\forall n \forall T_1 \subseteq T_0 \exists T_2 \subseteq T_1(|u\hat{\ }n, T_2| < \infty).$$

Then by the usual Π_1^1 dependent choice argument, we can define a binary system $\{u_t\}_{t \in 2^{<\omega}}$ of finite sequences from ω and a binary system $\{T_t\}_{t \in 2^{<\omega}}$ of elements of $\mathbf{P}(\Delta_1^1)$ such that the mappings $t \mapsto u_t$, $t \mapsto T_t$ are both Δ_1^1 and

(i) $u_{t\frown 0}$, $u_{t\frown 1}$ are incompatible extensions of u_t,

(ii) $u_{t\frown t'} \in T_t \subseteq T_0$, for all $t, t' \in 2^{<\omega}$,

(iii) $T_{t\frown 0} \in (T_t)_{u_t\frown 0}$, $T_{t\frown 1} \in (T_t)_{u_t\frown 1}$,

(iv) $|u\frown\text{length}(t), T_t| < \infty$.

Let $T' = \{s \in \omega^{<\omega}: s$ is subsequence of some $u_t\}$. Then $T' \subseteq T_0 \subseteq T$, so $|u, T'| = \infty$. But on the other hand we claim that $|u, T'| < \infty$, therefore establishing the desired contradiction. Indeed, given n, take $m = \max\{\text{length}(u_t): t \in 2^n\}$. Then if $s \in T$ and $\text{length}(s) \geq m$, s extends some u_t with $\text{length}(t) = n$, so $T'_s \subseteq T'_{u_t} \subseteq T_t$, therefore $|u\hat{\ }n, T'_s| \leq |u\hat{\ }n, T_t| = |u\frown\text{length}(t), T_t| < \infty$ and we are done.

To complete the proof notice now that if $\exists T_0 \subseteq T(|\varnothing, T_0| < \infty)$, then $\exists T_0 \subseteq T \forall T_1 \subseteq T_0 \forall \alpha(0) \exists T_2 \subseteq T_1 \forall T_3 \subseteq T_2 \forall \alpha(1) \cdots U(\lim T_i, \alpha)$ while, by the lemma, if $\forall T_0 \subseteq T(|\varnothing, T_0| = \infty)$, then $\neg \exists T_0 \subseteq T \forall T_1 \subseteq T_0 \forall \alpha(0) \exists T_2 \subseteq T_1 \forall T_3 \subseteq T_2 \forall \alpha(1) \cdots U(\lim T_i, \alpha)$ therefore $\exists T_0 \subseteq T \forall T_1 \subseteq T_0 \forall \alpha(0) \exists T_2 \subseteq T_1 \forall T_3 \subseteq T_2 \forall \alpha(1) \cdots U(\lim T_i, \alpha) \Leftrightarrow \exists T_0 \subseteq T(|\varnothing, T_0| < \infty)$ and this last expression is clearly Π_1^1, so we are done. ∎

§2. Minimal strict upper bounds for sequences of Δ-degrees.

2.1. DEFINITION. Let Γ be a Spector pointclass and $d_0 \leq d_1 \leq \cdots$ an ascending sequence of Δ-degrees. A Δ-degree d is a *minimal strict upper bound* of $\{d_i\}$ if

(i) $\forall i(d_i < d)$,

(ii) $d' \leq d \wedge \forall i(d_i < d') \Rightarrow d' = d$.

Sacks [9] has proved that most natural sequences of Δ_1^1-degrees have minimal strict upper bounds, but it is not yet known if this is true for *all* such sequences. On the other hand Friedman [1] proved that all ascending sequences of Δ_2^1-degrees have minimal strict upper bounds and this was extended to all Δ_{2n}^1 in [5] from PD. This is done by using a construction with Δ_{2n}^1-pointed perfect binary splitting trees in $I(\{d_i\}) = \{\alpha: \exists i([\alpha]_{\Delta_{2n}^1} \leq d_i)\}$ where $d_0 \leq d_1 \leq \cdots$ is the given ascending sequence of Δ_{2n}^1-degrees. After seeing a preliminary version of the results in §1, Leo Harrington pointed out to us that the technique used there could be also applied to forcing with Δ_{2n+1}^1-pointed perfect binary splitting trees in $I(\{d_i\})$, for certain sequences $\{d_i\}$, to show the existence of minimal strict upper bounds for them. We give below a somewhat more general result, which shows that one can find (a continuum of) minimal strict upper bounds for sequences of Δ-degrees which are not too "far apart", when Γ is $\mathfrak{D}$-generated and reflecting, thereby strengthening Theorem 1.14.

2.2. DEFINITION. Let Γ be a Spector pointclass. Let $d_0 \leq d_1 \leq \cdots$ be an ascending sequence of Δ-degrees. We call $\{d_i\}$ *short* if there is a real γ with the following properties.

(i) $I(\{d_i\}) \equiv \{\alpha: \exists i([\alpha]_\Delta \leq d_i)\} = \{(\gamma)_j: j \in \omega\}$.

(ii) For each $i \in \omega$, $\alpha \in d_i$ and each R in $\Gamma(\alpha)$, $\exists x \in \Delta(\gamma)R(x) \Rightarrow \exists x \in \Delta(\alpha)R(x)$.

To see some examples, take $\Gamma = \Pi^1_{2n+1}$, with $n \geq 1$ and $\{d_i\}$ any sequence of Δ^1_{2n+1}-degrees such that for some $\delta \in Q_{2n+1}(\alpha_0)$, where $[\alpha_0]_{\Delta^1_{2n+1}} = d_0$, we have $d_i \leq [\delta]_{\Delta^1_{2n+1}}$ for all i. (For the definition of Q_{2n+1} see [4].) Then there is $\gamma \in Q_{2n+1}(\alpha_0)$ satisfying (i) and so if $\alpha \in d_i$ for some i, we have $\gamma \in Q_{2n+1}(\alpha_i)$ and $\exists x \in \Delta(\gamma)R(x) \Rightarrow \exists x \in \Delta(\alpha)R(x)$ for $R \in \Pi^1_{2n+1}(\alpha)$ by 3A of [4]. On the other hand if $\Gamma = \Sigma^1_{2n}$, *every* ascending sequence of Δ^1_2-degrees is short, by the basis theorem, and the same is true for $\Gamma = \mathrm{IND}(\mathbf{R})$. We grant PD (or Determinacy(**HYP(R)**) for the last example) above. Finally, notice that if Γ is reflecting, then $\{d_i\}$ is short, where $d_0 = d_1 = \cdots$.

We can now state the main theorem in this section.

2.3. THEOREM. *Let $\Gamma = \mathfrak{D}\,\Gamma'$ be a $\mathfrak{D}$-generated Spector pointclass, where Γ' is nice. If Γ is reflecting, then, assuming* Determinacy($B(\Gamma')$), *we have that all short ascending sequences of Δ-degrees have a continuum of minimal strict upper bounds.*

PROOF. Let $\{d_i\}$ be a short ascending sequence of Δ-degrees. Consider the following notion of forcing.

$$\mathbf{C} = \{T\colon T \text{ is a } \Delta\text{-pointed perfect binary splitting tree in } I(\{d_i\})\},$$
$$T_1 \leq T_2 \Leftrightarrow T_2 \supseteq T_1.$$

(Recall that T is Δ-pointed iff $\forall \alpha \in [T](T \leq_\Delta \alpha)$.) We shall prove that all sufficiently generic for $\mathbf{C}$ reals β have Δ-degree a minimal strict upper bound of $\{d_i\}$.

Let λ^α_Γ be the ordinal defined in §1. We claim that it is enough to show that for all sufficiently generic for $\mathbf{C}$ reals β, we have

$$(*) \qquad \lambda^\beta_\Gamma = \sup_i \lambda^{\alpha_i}_\Gamma,$$

where $[\alpha_i]_\Delta = d_i$. Indeed, pointedness guarantees easily that for all sufficiently generic for $\mathbf{C}$ β's, $d_i < [\beta]_\Delta$, since for each Δ-pointed T and each α with $T \leq_\Delta \alpha$ there is $T^* \subseteq T$, T^* Δ-pointed with $T^* \equiv_\Delta \alpha$ (see [9]). Also granting $(*)$, we have that for all sufficiently generic for $\mathbf{C}$ β's, if $\alpha \leq_\Delta \beta$ then $\alpha = F^\xi_e(\beta)$, where F^ξ_e is as in 1.3 and $\xi < \lambda^\beta_\Gamma = \sup\{\lambda^{\alpha_i}_\Gamma : i \in \omega\}$, therefore $\xi < \lambda^{\alpha_i}_\Gamma$ for some i. Then as in 1.7 and the remarks following it, we can make sure that for all sufficiently generic for $\mathbf{C}$ β's, if $\alpha \leq_\Delta \beta$ then for some i, either $\alpha \leq_\Delta \alpha_i$ or $\beta \leq_\Delta \langle \alpha_i, \alpha \rangle$, which clearly guarantees that $[\beta]_\Delta$ is a strict minimal upper bound for $\{d_i\}$.

So it is enough to verify $(*)$ for all sufficiently generic β's. For that it is, of course, sufficient to show that

$$0 \Vdash_{\mathbf{C}} \lambda^\beta_\Gamma \leq \sup\{\lambda^{\alpha_i}_\Gamma : i \in \omega\}.$$

(Strictly speaking we should write

$$0 \Vdash_{\mathbf{C},f} \lambda^\beta_\Gamma \leq \sup\{\lambda^{\alpha_i}_\Gamma : i \in \omega\},$$

where f is as in 1.7.) If this fails, towards a contradiction, we can find $T \in \mathbf{C}$ with

$$T \Vdash_{\mathbf{C}} \lambda^\beta_\Gamma > \sup\{\lambda^{\alpha_i}_\Gamma : i \in \omega\}.$$

If $\mathscr{P}$, σ are as in 1.3, then we have

$$T \Vdash_{\mathbf{C}} \exists k(\mathscr{P}(k, \beta) \wedge \sigma(k, \beta) \geq \sup\{\lambda^{\alpha_i}_\Gamma : i \in \omega\}),$$

so by the disjunction formula of [3, 3.2.2],

$$\forall T_0 \subseteq T \exists k \exists T_1 \subseteq T_0 (T_1 \Vdash_{\mathbf{C}} \mathscr{P}(k, \beta) \wedge \sigma(k, \beta) \geq \sup\{\lambda_{\Gamma}^{\alpha_i} : i \in \omega\}).$$

So fix $T^* \in \mathbf{C}$ and $k \in \omega$ with

$$T^* \Vdash_{\mathbf{C}} \mathscr{P}(k, \beta) \wedge \sigma(k, \beta) \geq \sup\{\lambda_{\Gamma}^{\alpha_i} : i \in \omega\}.$$

Pick i_0 with $T^* \leq_{\Delta} \alpha_{i_0}$. Then, in particular, we have

$$T^* \Vdash_{\mathbf{C}} \mathscr{P}(k, \beta) \wedge \sigma(k, \beta) \geq \lambda_{\Gamma}^{\alpha_{i_0}}.$$

Claim.

$$(**) \qquad \neg \mathscr{P}(l, \alpha_{i_0}) \Leftrightarrow \exists T' \subseteq T^*(T' \Vdash_{\mathbf{C}} (k, \beta) \leq_{\sigma}^{*} (l, \alpha_{i_0})),$$

where $x \leq_{\sigma}^{*} y \Leftrightarrow \mathscr{P}(x) \wedge (\neg\mathscr{P}(y) \vee \sigma(x) \leq \sigma(y))$.

Indeed, if $\neg \mathscr{P}(l, \alpha_{i_0})$, then since $T^* \Vdash_{\mathbf{C}} \mathscr{P}(k, \beta)$, we have $T^* \Vdash_{\mathbf{C}} (k, \beta) \leq_{\sigma}^{*} (l, \alpha_{i_0})$. Conversely, if $T' \subseteq T^*$ and $T' \Vdash_{\mathbf{C}} (k, \beta) \leq_{\sigma}^{*} (l, \alpha_{i_0})$, while $(l, \alpha_{i_0}) \in \mathscr{P}$, then $T' \Vdash_{\mathbf{C}} \sigma(k, \beta) \leq \sigma(l, \alpha_{i_0})$. But also $T' \Vdash_{\mathbf{C}} \sigma(k, \beta) \geq \lambda_{\Gamma}^{\alpha_{i_0}} > \sigma(l, \alpha_{i_0})$, a contradiction.

So it is enough to show that the expression on the right of (**) is in $\Gamma(\alpha_{i_0})$. (Clearly $\{l: \mathscr{P}(l, \alpha_{i_0})\}$ is a complete $\Gamma(\alpha_{i_0})$ set of integers, so we have immediately a contradiction.)

First consider the notion of forcing $\mathbf{C}' = \{T \in I(\{d_i\}): T$ is a perfect binary splitting tree and T is almost Δ-pointed$\}, T_1 \leq T_2 \Leftrightarrow T_2 \subseteq T_1$, where S is *almost Δ-pointed* iff $\{a \in 2^{\omega}: S \leq_{\Delta} h_S(a)\}$ is comeager, with $h_S: 2^{\omega} \twoheadrightarrow S$ the canonical homeomorphism. Note now that $\mathbf{C}$ is dense in $\mathbf{C}'$. This is because Γ is $\mathfrak{D}$-generated, so that each $\Gamma(x)$ comeager set of reals contains a $\Delta(x)$ perfect set (see [3, 5.3.1]). Thus

$$\exists T' \subseteq T^*(T' \Vdash_{\mathbf{C}} (k, \beta) \leq_{\sigma}^{*} (l, \alpha_{i_0})) \Leftrightarrow \exists T' \subseteq T^*(T' \Vdash_{\mathbf{C}'} (k, \beta) \leq_{\sigma}^{*} (l, \alpha_{i_0}))$$

and we shall work with $\mathbf{C}'$ from now on. Our aim will be to show that

$$\begin{aligned} &\exists T' \subseteq T^*(T' \Vdash_{\mathbf{C}'} (k, \beta) \leq_{\sigma}^{*} (l, \alpha_{i_0})) \Rightarrow \\ &\qquad \exists T' \subseteq T^*(T' \text{ is almost } \Delta\text{-pointed} \wedge T' \leq_{\Delta} \alpha_{i_0} \wedge \\ &\qquad\qquad \{a \in 2^{\omega}: (k, h_{T'}(a)) \leq_{\sigma}^{*} (l, \alpha_{i_0})\} \text{ is comeager}). \end{aligned}$$

Since the converse is clearly true (as in 1.14) and the second expression is in $\Gamma(\alpha_{i_0})$ this will complete the proof. (We have used $\mathbf{C}'$ instead of $\mathbf{C}$ precisely because almost Δ-pointedness is a Γ property, but this is not necessarily true for Δ pointedness.)

So assume $\exists T' \subseteq T^*[T' \Vdash_{\mathbf{C}'} (k, \beta) \leq_{\sigma}^{*} (l, \alpha_{i_0})]$.

By the shortness of $\{d_i\}$ find γ as in 2.2. Then $\{j \in \omega: (\gamma)_j \in \mathbf{C}'\}$ is in $\Gamma(\gamma)$ (since $\mathbf{C}' \in \Gamma$), so by the fact that Γ is reflecting, there is δ such that

(i') $\{(\delta)_j : j \in \omega\} = \mathbf{C}'$,

(ii') $\forall i \in \omega\, \forall \alpha \in d_i \forall R \in \Gamma(\alpha)[\exists x \in \Delta(\delta) R(x) \Rightarrow \exists x \in \Delta(\alpha) R(x)]$.

We have now that

$$\exists T' \subseteq T^*[T' \Vdash_{\overline{\mathbf{C}'}} (k, \beta) \leq_{\sigma}^{*} (l, \alpha_{i_0})],$$

but also that $T \Vdash_{\mathbf{C}'} \alpha_{i_0} \leq_{\Delta} \beta$ for all $T \in \mathbf{C}'$, so that

$$\exists T' \subseteq T^*[T' \Vdash_{\overline{\mathbf{C}'}} \alpha_{i_0} \leq_\Delta \beta \wedge (k, \beta) \leq^*_\sigma (l, \alpha_{i_0})].$$

Abbreviate,

$$U(\beta) \Leftrightarrow \alpha_{i_0} \leq_\Delta \beta \wedge (k, \beta) \leq^*_\sigma (l, \alpha_{i_0}),$$

so that $U \in \Gamma(\alpha_{i_0})$. Then we have

$$\exists T' \subseteq T^* \forall T_0 \subseteq T' \exists T_1 \subseteq T_0 \cdots U(\lim T_i).$$

Since the T's here vary over $\mathbf{C}'$ which is directly enumerated by δ, this expression is a $\Gamma(\alpha_{i_0})$ property of δ and so is the fact "$\forall j((\delta)_j$ is an almost Δ-pointed perfect binary splitting tree". Thus, by (ii′) above we can find a $\gamma' \leq_\Delta \alpha_{i_0}$ such that $\exists S' \subseteq T^* \forall S_0 \subseteq S' \exists S_1 \subseteq S_0 \cdots U(\lim S_i)$, where the S's vary over $\mathbf{C}'' = \{(\gamma')_i : i \in \omega\} \subseteq \mathbf{C}'$ and $T^* \in \mathbf{C}''$. Moreover, the player who wins this game has a $\Delta(\gamma')$ winning strategy τ. By a routine splitting argument we use τ to produce a $T' \leq_\Delta \alpha_{i_0}$ with $T' \subseteq T^*$ and such that every $\beta \in [T']$ is of the form $\lim S_i$, where $S', S_0, S_1, \ldots$ is a run of the above game, with II following his winning strategy τ, therefore $\alpha_{i_0} \leq_\Delta \beta$ and $(k, \beta) \leq^*_\sigma (l, \alpha_{i_0})$. Thus $T' \leq_\Delta \beta$, for all $\beta \in [T']$ i.e. T' is Δ-pointed and also $[T'] \subseteq \{\beta : (k, \beta) \leq^*_\sigma (l, \alpha_{i_0})\}$, so we are done. ■

2.4. COROLLARY. (i) *Let $n \geq 2$ be odd and assume* Determinacy($\boldsymbol{\Delta}^1_{n-1}$). *Then every short ascending sequence of Δ^1_n-degrees has a continuum of minimal strict upper bounds.*

(ii) ([1] *and* [5]) *Let $n \geq 2$ be even and assume* Determinacy($\boldsymbol{\Sigma}^1_{n-1}$). *Then every ascending sequence of Δ^1_n-degrees has a continuum of minimal strict upper bounds.*

(iii) *Assume* Determinacy(**IND**(**R**)). *Then every ascending sequence of* HYP(**R**)-*degrees has a continuum of minimal strict upper bounds.*

As another immediate application we have a computation of the length of the natural hierarchy of Δ-degrees for certain Γ. Let us recall the definition first.

2.5. For each Spector pointclass Γ we define the *natural hierarchy of Δ-degrees* $\{d^\xi_\Delta\}_{\xi<\rho_\Delta}$ as follows

$$d^0_\Delta = [\lambda t.0]_\Delta,$$

$$d^{\xi+1}_\Delta = \text{the } \Delta\text{-jump of } d^\xi_\Delta \quad \text{where the } \Delta\text{-jump of}$$

$$d = [\alpha]_\Delta \text{ is } d' = [W^\alpha]_\Delta \text{ with } W^\alpha \text{ a complete } \Gamma(\alpha) \text{ subset of } \omega,$$

$$d^\lambda_\Delta = \operatorname{lub}\{d^\xi_\Delta : \xi < \lambda\} \quad \text{if } \lambda \text{ is limit,}$$

$$\rho_\Delta = \text{least } \lambda \text{ such that } d^\lambda_\Delta \text{ does not exist.}$$

Also for convenience put $\rho_n = \rho_{\Delta^1_n}$.

It is well known that the length of the hierarchy of Δ^1_1-degrees is $\rho_1 =$ the first recursively inaccessible ordinal (Richter, Sacks). Friedman [1] has shown that if $\exists\alpha(\alpha \notin L)$ then $\rho_2 = \omega$, and this was extended in [5] to $\rho_{2n} = \omega$ for all n, using PD. We have now

2.6. COROLLARY. (i) *Assume* Determinacy($\boldsymbol{\Delta}^1_{2n}$) *and $n \geq 1$. Then $\rho_{2n+1} = \omega$.*

(ii) *Assume* Determinacy(**IND**(**R**)). *Then $\rho_{\mathrm{HYP}(\mathbf{R})} = \omega$.*

PROOF. Notice that if $d_0 < d_1 < \cdots$ is a strictly ascending sequence with at least two distinct minimal strict upper bounds, say d', d'', then $d = \operatorname{lub}\{d_i : i \in \omega\}$ does not exist. Because otherwise, $d \leq d'$ and $d \leq d''$, so that $d = d' = d''$, a contradiction.

Thus (ii) is immediate from 2.3 and the remarks following 2.2, while (i) follows from the same remarks since the sequence $\{d'_{\Delta^1_{2n+1}}\}$ satisfies the condition stated there, as it is shown in 3A of [4]. ■

2.7. We conclude with some open problems related to the results in this paper.

(i) Can the conclusion of Theorem 1.14 be extended to Spector classes that do not satisfy the hypothesis of this theorem, in particular to rigid Spector classes (see [6] for an explanation of this term)?

(ii) What is the structure of initial segments of Δ^1_{2n+1}-degrees for $n > 0$?

(iii) Is there an ascending sequence of Δ^1_{2n+1}-degrees with no minimal strict upper bound for $n > 0$? (We have the feeling that this may be easier to settle than the same problem about Δ^1_1-degrees.)

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