Supplementary Appendix

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Proof of Proposition 5

As with the proof of Proposition 4, we condition on \( z = \bar{z} \). Setting \( \delta = x - \bar{x} \) and setting \( \hat{\delta}_n(C) = \hat{x}_n(C)|_{z=\bar{z}} - \bar{x} \), we can rewrite the estimation problem [2] from the main paper as follows:

\[
\hat{\delta}_n(C) = \arg \min_{\delta \in \mathbb{R}^p} \frac{1}{2} \left\| (x^* - \bar{x}) + \frac{\sigma}{\sqrt{n}} \bar{z} - \delta \right\|^2_{\ell_2} \quad \text{s.t.} \quad \delta \in C - \bar{x}.
\]

Letting \( R_1 \) and \( R_2 \) denote orthogonal subspaces that contain \( Q_1 \) and \( Q_2 \), i.e., \( Q_1 \subseteq R_1 \) and \( Q_2 \subseteq R_2 \), and letting \( \delta^{(1)} = \mathcal{P}_{R_1}(\delta), \delta^{(2)} = \mathcal{P}_{R_2}(\delta), \hat{\delta}_n^{(1)}(C) = \mathcal{P}_{R_1}(\hat{\delta}_n(C)), \hat{\delta}_n^{(2)}(C) = \mathcal{P}_{R_2}(\hat{\delta}_n(C)) \) denote the projections of \( \delta, \hat{\delta}_n(C) \) onto \( R_1, R_2 \), we can rewrite the above reformulated optimization problem as:

\[
\left[ \hat{\delta}_n^{(1)}(C), \hat{\delta}_n^{(2)}(C) \right] = \arg \min_{\delta^{(1)} \in Q_1, \delta^{(2)} \in Q_2} \frac{1}{2} \left\| \mathcal{P}_{R_1} \left[ (x^* - \bar{x}) + \frac{\sigma}{\sqrt{n}} \bar{z} \right] - \delta^{(1)} \right\|^2_{\ell_2} + \frac{1}{2} \left\| \mathcal{P}_{R_2} \left[ (x^* - \bar{x}) + \frac{\sigma}{\sqrt{n}} \bar{z} \right] - \delta^{(2)} \right\|^2_{\ell_2}.
\]

As the sets \( Q_1, Q_2 \) live in orthogonal subspaces, the two variables \( \delta^{(1)}, \delta^{(2)} \) in this problem can be optimized separately. Consequently, we have that \( \| \hat{\delta}_n^{(2)}(C) \|_{\ell_2} \leq \alpha \) and that

\[
\| \hat{\delta}_n^{(1)}(C) \|_{\ell_2} \leq \sup_{\delta \in \text{cone}(Q_1) \cap B_{\ell_2}^p} \{ \delta, \frac{\sigma}{\sqrt{n}} \bar{z} + (x^* - \bar{x}) \}.
\]

This bound can be established following the same sequence of steps as in the proof of Proposition 4. Combining the two bounds on \( \hat{\delta}_n^{(1)}(C) \) and \( \hat{\delta}_n^{(2)}(C) \), one can then check that

\[
\| \hat{\delta}_n^{(1)}(C) \|_{\ell_2} + \| \hat{\delta}_n^{(2)}(C) \|_{\ell_2} \leq 2 \left[ \frac{\sigma^2}{n} g(\text{cone}(Q_1) \cap B_{\ell_2}^p) + \| x^* - \bar{x} \|_{\ell_2} \right] + \alpha^2.
\]

To obtain a bound on \( \| \hat{x}_n(C)|_{z=\bar{z}} - x^* \|_{\ell_2} \) we note that

\[
\| \hat{x}_n(C)|_{z=\bar{z}} - x^* \|_{\ell_2} \leq 2 \left[ \| \hat{x}_n(C)|_{z=\bar{z}} - \bar{x} \|_{\ell_2} + \| x^* - \bar{x} \|_{\ell_2} \right] \leq 2\| \hat{\delta}_n^{(1)}(C) \|_{\ell_2} + 2\| \hat{\delta}_n^{(2)}(C) \|_{\ell_2} + 2\| x^* - \bar{x} \|_{\ell_2}.
\]

Taking expectations concludes the proof. \( \square \)

Proof of Proposition 9

The main steps of this proof follow the steps of a similar result in [1], with the principal difference being that we wish to bound Gaussian squared-complexity rather than Gaussian complexity. A central theme in this proof is the appeal to Gaussian isoperimetry. Let \( \mathbb{S}^{p-1} \) denote the sphere in \( p \) dimensions. Then in bounding the expected squared-distance to the dual cone \( \mathcal{K}^* \) with \( \mathcal{K}^* \cap \mathbb{S}^{p-1} \) having a volume of \( \mu \), we need only consider the extremal case of a spherical cap in \( \mathbb{S}^{p-1} \) having a volume of \( \mu \). The manner in which this is made precise will become clear in the proof. Before proceeding with the main proof, we state and derive a result on the solid angle subtended by a spherical cap in \( \mathbb{S}^{p-1} \) to which we will need to appeal repeatedly:
Lemma 2 Let \( \psi(\mu) \) denote the solid angle subtended by a spherical cap in \( \mathbb{S}^{p-1} \) with volume \( \mu \in \left( \frac{1}{4} \exp\left\{-\frac{p}{20}\right\}, \frac{1}{4}\right) \). Then
\[
\psi(\mu) \geq \frac{\pi}{2} \left( 1 - \sqrt\frac{2\log\left(\frac{1}{4}\right)}{p-1} \right).
\]

Proof of Lemma 2: Consider the following definition of a spherical cap, parametrized by height \( h \):
\[
J = \{ \mathbf{a} \in \mathbb{S}^{p-1} \mid a_1 \geq h \}.
\]
Here \( a_1 \) denotes the first coordinate of \( \mathbf{a} \in \mathbb{R}^p \). Given a spherical cap of height \( h \in [0, 1] \), the solid angle is given by:
\[
\psi = \frac{\pi}{2} - \sin^{-1}(h). \tag{10}
\]
We can thus obtain bounds on the solid angle of a spherical cap via bounds on its height. The following result from [2] relates the volume of a spherical cap to its height:

Lemma 3 [2] For \( \frac{2}{\sqrt{p}} \leq h \leq 1 \) the volume \( \tilde{\mu}(p, h) \) of a spherical cap of height \( h \) in \( \mathbb{S}^{p-1} \) is bounded as
\[
\frac{1}{10h\sqrt{p}}(1 - h^2)^{\frac{p-1}{2}} \leq \tilde{\mu}(p, h) \leq \frac{1}{2h\sqrt{p}}(1 - h^2)^{\frac{p-1}{2}}.
\]

Continuing with the proof of Lemma 2, note that for \( \frac{2}{\sqrt{p}} \leq h \leq 1 \)
\[
\frac{1}{2h\sqrt{p}}(1 - h^2)^{\frac{p-1}{2}} \leq \frac{1}{4}(1 - h^2)^{\frac{p-1}{2}} \leq \frac{1}{4} \exp\left(-\frac{p-1}{2}h^2\right).
\]

Choosing \( h = \sqrt{\frac{2\log\left(\frac{1}{4}\right)}{p-1}} \) we have \( \frac{2}{\sqrt{p}} \leq h \leq 1 \) based on the assumption \( \mu \in \left( \frac{1}{4} \exp\left\{-p/20\right\}, \frac{1}{4}\right) \). Consequently, we can apply Lemma 3 with this value of \( h \) combined with (10) to conclude that
\[
\tilde{\mu}\left( p, \sqrt{\frac{2\log\left(\frac{1}{4}\right)}{p-1}} \right) \leq \mu.
\]
Hence the solid angle \( \psi\left( \tilde{\mu}\left( p, \sqrt{\frac{2\log\left(\frac{1}{4}\right)}{p-1}} \right) \right) \) is less than the solid angle \( \psi(\mu) \). Consequently, we use (10) to conclude that
\[
\psi(\mu) \geq \frac{\pi}{2} - \sin^{-1}\left( \sqrt{\frac{2\log\left(\frac{1}{4}\right)}{p-1}} \right).
\]
Using the bound \( \sin^{-1}(h) \leq \frac{\pi}{2}h \), we obtain the desired bound. □

Proof of Proposition 9: We bound the Gaussian squared-complexity of \( \mathcal{K} \) by bounding the expected squared-distance to the polar cone \( \mathcal{K}^* \). Let \( \tilde{\mu}(U; t) \) for \( U \subseteq \mathbb{S}^{p-1} \) and \( t > 0 \) denote the volume of the set of points in \( \mathbb{S}^{p-1} \) that are within a Euclidean distance of at most \( t \) from \( U \) (recall that the volume of this set is equivalent to the measure of the set with respect to the normalized Haar measure on \( \mathbb{S}^{p-1} \)). We have the
following sequence of relations by appealing to the independence of the direction \( g/\|g\|_{\ell_2} \) and of the length \( \|g\|_{\ell_2} \) of a standard normal vector \( g \):

\[
E[\text{dist}(g, K^*)^2] = E[\|g\|_{\ell_2}^2 \text{dist}(g/\|g\|_{\ell_2}, K^*)^2] = p E[\text{dist}(g/\|g\|_{\ell_2}, K^*)^2] \\
\leq p E[\text{dist}(g/\|g\|_{\ell_2}, K^* \cap S^{p-1})^2] \\
= p \int_0^\infty P[\text{dist}(g/\|g\|_{\ell_2}, K^* \cap S^{p-1})^2 > t]dt \\
= p \int_0^\infty P[\text{dist}(g/\|g\|_{\ell_2}, K^* \cap S^{p-1}) > \sqrt{t}]dt \\
= 2p \int_0^\infty sP[\text{dist}(g/\|g\|_{\ell_2}, K^* \cap S^{p-1}) > s]ds \\
= 2p \int_0^\infty s[1 - \bar{\mu}(K^* \cap S^{p-1}; s)]ds.
\]

Here the third equality follows based on the integral version of the expected value. Let \( V \subseteq S^{p-1} \) denote a spherical cap with the same volume as \( K^* \cap S^{p-1} \). Then we have by spherical isoperimetry that \( \bar{\mu}(V; s) \geq \bar{\mu}(K^* \cap S^{p-1}; s) \) for all \( s \geq 0 \) [3]. Thus

\[
E[\text{dist}(g, K^*)^2] \leq 2p \int_0^\infty s[1 - \bar{\mu}(V; s)]ds. \tag{11}
\]

From here onward, we focus exclusively on bounding the integral.

Let \( \tau(\psi) \) denote the volume of a spherical cap subtending a solid angle of \( \psi \) radians. Recall that \( \psi \) is a quantity between 0 and \( \pi \). As in Lemma 2 let \( \psi(\mu) \) denote the solid angle of a spherical cone subtending a solid angle of \( \mu \). Since the Euclidean distance between points on a sphere is always smaller than the geodesic distance, we have that \( \bar{\mu}(V; s) \geq \bar{\mu}(K^* \cap S^{p-1}; s) \). Further, we have the following explicit formula for \( \tau(\psi) \) [4]:

\[
\tau(\psi) = \omega_p^{-1} \int_0^\psi \sin^{p-1}(v)dv,
\]

where \( \omega_p = \int_0^\pi \sin^{p-1}(v)dv \) is the normalization constant. Combining these latter two observations, we can bound the integral in (11) as:

\[
\int_0^\infty s[1 - \bar{\mu}(V; s)]ds \leq \int_0^\infty s[1 - \tau(\psi(\mu) + s)]ds \\
= \int_0^{\pi - \psi(\mu)} s[1 - \tau(\psi(\mu) + s)]ds \\
= \frac{(\pi - \psi(\mu))^2}{2} - \int_0^{\pi - \psi(\mu)} s\tau(\psi(\mu) + s)ds \\
= \frac{(\pi - \psi(\mu))^2}{2} - \omega_p^{-1} \int_0^{\pi - \psi(\mu)} \int_0^{\psi(\mu) + s} s\sin^{p-1}(v)dvds.
\]
Next we change the order of integration to obtain:

\[
\int_0^\infty s[1 - \bar{\mu}(V; s)]ds \leq \frac{(\pi - \psi(\mu))^2}{2} - \omega_p^{-1} \int_0^\pi \int_{\max\{v - \psi(\mu), 0\}}^{\pi - \psi(\mu)} \sin^{p-1}(v)sdsdv
\]

\[
= \frac{(\pi - \psi(\mu))^2}{2} - \omega_p^{-1} \int_0^\pi \left[ (\pi - \psi(\mu))^2 - (\max\{v - \psi(\mu), 0\})^2 \right] \sin^{p-1}(v)dv
\]

\[
= \omega_p^{-1} \int_0^\pi (\max\{v - \psi(\mu), 0\})^2 \sin^{p-1}(v)dv
\]

\[
= \omega_p^{-1} \int_\psi(\mu) (v - \psi(\mu))^2 \sin^{p-1}(v)dv.
\]

We now appeal to the inequalities \(\omega_p^{-1} \leq \sqrt{p-1}/2\) and \(\sin(x) \leq \exp(-(x - \frac{\pi}{2})^2/2)\) for \(x \in [0, \pi]\) to obtain

\[
\int_0^\infty s[1 - \bar{\mu}(V; s)]ds \leq \frac{\sqrt{p-1}}{2} \int_\psi(\mu) (v - \psi(\mu))^2 \exp \left[-\frac{(v - \frac{\pi}{2})^2}{2}\right] dv.
\]

Performing a change of variables with \(a = \sqrt{p-1}(v - \frac{\pi}{2})\), we have

\[
\int_0^\infty s[1 - \bar{\mu}(V; s)]ds \leq \frac{1}{2} \int_{\sqrt{p-1}(\psi(\mu) - \pi/2)}^{\sqrt{p-1}(\psi(\mu) - \pi/2)} \left( \frac{a^2}{p-1} + (\frac{a}{2} - \psi(\mu))^2 \right) \exp[-\frac{a^2}{2}]da
\]

\[
= \frac{1}{2} \left[ \int_{\sqrt{p-1}(\psi(\mu) - \pi/2)}^{\sqrt{p-1}(\psi(\mu) - \pi/2)} \left( \frac{a^2}{p-1} + (\frac{a}{2} - \psi(\mu))^2 \right) \exp[-\frac{a^2}{2}] \right] \exp[-\frac{a^2}{2}]da
\]

\[
= \frac{1}{2} \left[ \int_{-\infty}^{\sqrt{p-1}(\psi(\mu) - \pi/2)} \exp[-\frac{a^2}{2}]da + \int_{-\infty}^{\sqrt{p-1}(\psi(\mu) - \pi/2)} (\frac{a}{2} - \psi(\mu))^2 \exp[-\frac{a^2}{2}]da + \int_0^{\frac{2a}{\sqrt{p-1}}} (\frac{a}{2} - \psi(\mu)) \cdot \exp[-\frac{a^2}{2}]da \right]
\]

\[
= \frac{1}{2} \left[ \frac{\sqrt{p-1}}{p-1} + \sqrt{2\pi}\left(\frac{\psi(\mu)}{2}\right)^2 + \frac{2a}{\sqrt{p-1}} \left(\frac{a}{2} - \psi(\mu)\right) \cdot \exp[-\frac{a^2}{2}] \right]
\]

Here the inequality was obtained by suitably changing the limits of integration. We now employ Lemma 2 to obtain the final bound:

\[
g(\mathcal{K} \cap B_{\ell_2}^p) \leq p \left[ \frac{\sqrt{2\pi}}{p-1} + \sqrt{2\pi} \left(\frac{2\log\left(\frac{\psi(\mu)}{2}\right)}{p-1}\right)^2 + \frac{2}{\sqrt{p-1}} \left(\frac{2\log\left(\frac{\psi(\mu)}{2}\right)}{p-1}\right) \right]
\]

\[
= \frac{p\sqrt{2\pi}}{p-1} \left[ 1 + \pi \log \left(\frac{1}{10}\right) + \sqrt{\pi} \sqrt{\log \left(\frac{1}{10}\right)} \right]
\]

\[
\leq 20 \log \left(\frac{1}{10}\right).
\]

Here the final bound holds because \(\mu < 1/4e^2\) and \(p \geq 12\). \(\Box\)

References

