MAGMA SOLITONS

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Abstract. Motivated to understand the process of melt migration in the earth’s mantle, we have studied a generalised form of Darcy’s law that describes porous flow in a matrix that can deform by creep. We find a remarkable richness of phenomena, including a new class of solitons. These consist of shape-preserving waves of high liquid fraction which buoyantly ascend through a stationary matrix. This may have important implications for the morphology and geochemistry of primary igneous processes, and applicability to other porous flow problems.

Formulation

Porous flow processes [Bear, 1972] occur in many natural systems. They allow, for example, the movement of water or hydrocarbons through sedimentary rocks and the movement of water through glacial ice; our interest lies in the migration of melt through partially molten rocks in a planetary interior [Turcotte, 1982]. The governing equation, known as Darcy’s law, states that the flux of liquid through a permeable medium is proportional to the pressure gradient in the liquid. This equation is usually stated in simple algebraic form but must be generalised to a non-linear partial differential equation if the matrix can deform by creep. This deformation takes the form of compaction or distension and is a response to spatial variations of liquid content.

Although the phenomena described here are novel, the basic equations have been obtained independently by several workers [Sleep, 1974; Stevenson, 1980; Fowler, 1984; McKenzie, 1984], with minor differences. A conceptual development is given here. Consider a partially molten medium consisting of a solid matrix phase and a liquid phase, both fully connected and incompressible, in a uniform vertical gravitational field. For clarity in this description, we neglect phase transitions and allow only vertical motions. Conservation of mass and energy in an elemental layer can be stated as:

\[ \frac{\partial}{\partial z} (f \nu_l) = -\frac{\partial}{\partial z} [(1-f) \nu_m] = -\frac{\partial}{\partial t} \Phi \]

\[ \frac{\partial}{\partial z} [u(\sigma_l-\sigma_m)] + g \Delta p u = \frac{\eta_l}{k} u^2 + (\sigma_l-\sigma_m) \frac{\partial u}{\partial z} \]

where \( z \) is the vertical coordinate, \( t \) is time, \( f \) is the mean volume fraction of liquid, \( \nu_l \) and \( \nu_m \) are the mean liquid and matrix velocities, \( u = f \nu_l - (1-f) \nu_m \) is the mean liquid flux in the barocentric frame, \( \sigma_l \) and \( \sigma_m \) are the mean vertical nor-

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Fig. 1. A simple example illustrating the control exerted by matrix compaction on liquid ascent. The vertical scale is non-dimensional height, the horizontal scale is liquid fraction in percent. The plot shows the profile of liquid fraction, initially and after 100 time units. The initial state is a uniform liquid fraction of 1%. The bottom boundary is impermeable ($u=0$), so the matrix must compact in the lower region to allow liquid to escape. Simple analysis predicts the non-dimensional height of the compacting boundary layer to be $f_{(n-m)/2}$, where $n$ and $m$ are the permeability and effective viscosity parameters in equation (4). Here $n=3$ and $m=1$ so the height is 0.01 units. The layer develops in a non-dimensional time of $t=(n+m)/2$, here equal to 100 units. Liquid escapes freely through the top boundary, so the deviation from Darcy's law in the upper part of the column is small.

The effective viscosity $\eta^*$ includes a bulk (or 'second') viscosity that has never been measured for a partially molten medium. However, recent experiments [Cooper and Kohlstedt, 1984] on an olivine-basalt assemblage indicate that the shear viscosity in the partially molten state is not greatly reduced from that in the completely solid state. We therefore feel justified in using a typical asthenospheric estimate, $\eta^*=10^{20}$ Poise. With a grain size of 1 mm, the unit of length is then $=50$ km and the unit of time is $=10,000$ yr.

We have studied the system of equations (3) and (4) using numerical and analytic techniques, concentrating on the one-dimensional case and on understanding qualitatively the processes in igneous systems. A range of power laws have been used for the matrix permeability and effective viscosity parameters in equation (4). Here $n=3$ and $m=1$ so the height is 0.01 units. The layer develops in a non-dimensional time of $t=(n+m)/2$, here equal to 100 units. Liquid escapes freely through the top boundary, so the deviation from Darcy's law in the upper part of the column is small.

Solitons

We can demonstrate solitary wave solutions which propagate over a constant, low background liquid fraction $f_0$. In the usual way [Drazin, 1983], these solutions are sought by substitution of $\psi=\psi(y), \; y=z-ct$, in equations (3) and (4); $c$ is the ascent velocity of the wave. Integration of (4) gives $u=u_0+c\psi$, where $u_0$ is a constant of integration. As $y\to\pm\infty$,

in the background region, we demand $\phi=f_0, \; \phi_y=0$; hence $f_0^n-u_0+ct\phi_0$ and $u_0$ can be eliminated (again, the subscript $y$ denotes differentiation). The following ordinary differential equation in $\phi$ is obtained from equation (4):

$$\frac{[c(\phi-f_0)-(\phi^n-f_0^n)]}{\phi^n} = c(\phi^{-m}\phi_y)_y$$

which can be integrated once using the identity

$$(\phi^{-m}\phi_y)_y \equiv \frac{1}{2m}\frac{d}{d\phi}[\phi^{-2m}\phi_y^2].$$

In the case $m=0$, we obtain (for $n\neq 2$),

$$\phi_y = \pm \left\{ \frac{2f_0}{c} \right\}^{1/2} \left( \frac{c^*}{(n-1)(n-2)} - \psi + \frac{n}{(n-1)-1} \right)^{1/2}$$

where $c^*=c/f_0^{n-1}$, and $\psi=\psi/f_0$. $c^*$ and $\psi$ are the ascent velocity and profile of the wave scaled to the ascent velocity of the background liquid and the background liquid fraction. In the special case of $n=2$, the expression inside the square bracket is replaced by

$$c^* \left\{ \ln \left( \psi - 1 + \frac{1}{\psi} \right) - \left( \psi - 2 + \frac{1}{\psi} \right) \right\}$$

Solitary wave solutions exist provided $\phi_y^2$ is positive for some finite range of $\psi$ and goes to zero at $\psi=\psi\neq 1$ as well as at $\psi=1$. $\psi$ is then the scaled wave amplitude. These requirements are satisfied for $\psi>1$ ('positive' solitary waves) provided $n>1$. The dispersion relation, obtained by setting $\phi_y=0$ at $\psi=\Psi$, is, for $n\neq 2$;

$$c^* = \left( n-1)(n-2) \right) \left[ \frac{\psi-n}{(n-1)n-1} \right]^{1/2}$$

and for $n=2$;

$$c^* = \left( n-1 \right) \left[ \frac{\psi-2+\frac{1}{\psi}}{\ln \psi - 1 + \frac{1}{\psi}} \right]$$

In the limit $\Psi>>1$, equation (6) can be integrated to give

$\psi=\Psi-\gamma/2c_0, \; \Psi>>1$, implying an approximately parabolic solitary wave.

In a similar way, the case $m=1$ leads to solitary waves for $n>1$, with approximately Gaussian form. The corresponding dispersion relation is:

$$c^* = n(n-1) \left[ \frac{\ln \psi - 1 + \frac{1}{\psi}}{n\psi^m} \right]$$

$$\left[ 1 - \frac{n}{\psi^m} + \frac{n(n-1)}{\psi^m-1} \right]$$

In the limit $\Psi>>1$, equation (8) can be integrated to give

$\psi=\Psi-\gamma/2c_0, \; \Psi>>1$, implying an approximately parabolic solitary wave.
Fig. 2. Dispersion relations between the scaled ascent velocity $c^*$ and scaled amplitude $\Psi$ of solitary waves (see text). The solid lines are the theoretical predictions for two choices of $n$ and $m$; the crosses are values measured in numerical experiments. These experiments used $f_0=0.1\%$, so values of $\Psi$ greater than about 300 (corresponding to a liquid fraction greater than 30%) are physically unreasonable since the solid phase would no longer be connected (Arzi, 1978).

For all $m$, $c^*-n$ as $m\to 1$ (small amplitude waves), in agreement with the dispersion relation obtained by linearising equation (4). In analogy with the Korteweg-de Vries equation, it is found that the linearised frequency-wavenumber relation predicts dispersion but the non-linearity causes wave-steepening.

In numerical experiments, using an iterative finite-difference scheme, these solitary waves are found to be stable and arise whenever a substantially liquid region underlies a column of low liquid fraction. The liquid preferentially forms solitary waves which ascend through the column with an unchanging profile, at a constant velocity. Figure 2 shows the exact dispersion relations derived above, for two choices of $n$ and $m$, and points from corresponding numerical experiments in which solitary wave amplitudes and velocities were measured. The agreement lends us confidence in the numerical technique. The sizes and shapes of the numerically created solitary waves are also in agreement with theoretical predictions.

In many non-linear systems exhibiting solitary waves [Fornberg and Whitham, 1978], the form and amplitude of the individual waves are conserved in collisions, either exactly or approximately. In these cases, the waves may be termed solitons. This type of interaction is shown in figure 3, demonstrating that the solitary wave solutions to our equation are solitons. Figure 4 shows how the episodic ascent process can arise from a steady-state melting process, that might occur in a rising mantle plume. Referring to figures 3 and 4, we again note the qualitative similarities but quantitative differences that arise from different choices of $n$ and $m$.

Discussion

When discussing consequences of magma solitons in the geological context, we propose the abbreviation 'magmon'. A resolution of the quantitative uncertainties described above is needed because the significance of magmons in igneous processes will depend on their physical size and velocity. This may come through experimental determination of the appropriate matrix permeability and effective viscosity. Plausible estimates of the various parameters lead to magmon heights of a few km and velocities of a few tens of cm/yr, in which case the geodynamic and geochemical consequences

Fig. 3. The interaction of two solitons. Vertical profiles of liquid fraction are shown at successive points in non-dimensional time, as indicated by the upper horizontal axis. In this calculation, we use $n=2$ in the matrix permeability and $m=1$ in the effective matrix viscosity. The initial state consists of two solitons (approximately Gaussian when $m=1$) in a column containing a uniform background liquid fraction of 0.1%. The flux at the upper and lower boundaries is prescribed to maintain this background value. If alone, the individual solitons would ascend through the column at a constant velocity with unchanging form. Here, the larger soliton ascends faster due to its greater total buoyancy, and overtakes the smaller. Both emerge essentially unchanged by the interaction, apart from the phase shifts seen in the plot. The background liquid fraction is also unchanged.

Fig. 4. Creation of solitons by a melting event. Layout of the plot is as in figure 3. This calculation uses an $n=3$ permeability (giving narrower solitons) and a constant ($m=0$) effective matrix viscosity (giving the approximately parabolic soliton profiles). Initially the column contains only a background liquid fraction of 0.1%. Melting then occurs in the lowest tenth of the column, below the dashed horizontal line, and continues at a constant rate. Based on more complete calculations that include melting and thermal effects, this is a reasonable simulation of the pressure-release melting of a rock that consists of a low-melting fraction and a refractory residue. Solitons rise individually, separated by necks where the liquid fraction returns to its background value. Note that the first soliton has a slightly lower amplitude and velocity than the second. The latter catches up and in the last few profiles the type of interaction shown in figure 3 is occurring.
may be resolvable in observations. For example, the episodic ascent process shown in figure 4 would lead to temporally, and perhaps spatially, periodic eruptions. Within a magmon, the liquid is in intimate contact with the matrix, and will acquire some chemical imprint as it ascends. The existence of magmons is also conditional on their stability in two and three spatial dimensions, which is the subject of continuing study. Even if instability occurs (for example, self-focussing [Kelley, 1965; Kadomtsev and Petviashvili, 1970]) occurs, it may have interesting consequences such as streamers or plumes of rising liquid. The possibility of solitary waves in other porous flow problems also deserves attention.

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