THE CLASSIFICATION OF HYPERFINITE BOREL EQUVALENCE RELATIONS

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Let $X$ be a standard Borel space and $E$ a Borel equivalence relation on $X$. We call $E$ hyperfinite if there is a Borel automorphism $T$ of $X$ such that $xEy \Leftrightarrow \exists n \in \mathbb{Z}(T^n x = y)$.

For Borel equivalence relations $E, F$ on $X, Y$ resp. we write

$$E \preceq F \Leftrightarrow \exists f : X \rightarrow Y (f \text{ Borel, injective with } E = f^{-1}[F])$$

$$E \approx F \Leftrightarrow E \preceq F \text{ and } F \preceq E$$

$$E \cong F \Leftrightarrow \exists f : X \rightarrow Y (f \text{ a Borel isomorphism with } E = f^{-1}[F])$$

A Borel equivalence relation $E$ on $X$ is called smooth if there is a Borel map $f : X \rightarrow Y$ ($Y$ some standard Borel space) with $xEy \Leftrightarrow f(x) = f(y)$.

**THEOREM 1.** (Doughe rty-Jackson-Kechris). Let $E, F$ be two non-smooth, hyperfinite Borel equivalence relations. Then $E \approx F$.

A hyperfinite $E$ is called aperiodic if every $E$-equivalence class of $E$ is infinite. Given such an $E$, we denote by $\mathcal{E}(E)$ the space of $E$-ergodic, invariant probability measures. (A
measure is $E$-ergodic if every $E$ invariant Borel set is null or conull and $E$-invariant if it is $T$-invariant for a Borel automorphism $T$ that induces $E$ - this is independent of $T$).

**THEOREM 2.** (Dougherty-Jackson-Kechris). Let $E, F$ be aperiodic, non-smooth hyperfinite Borel equivalence relations. Then

$$E \cong F \iff \text{card } (\mathcal{E}(E)) = \text{card } (\mathcal{E}(F)).$$

This has been conjectured by M.G. Nadkarni, who proved first the case when the above cardinality is countable.

It follows that up to Borel isomorphism the only aperiodic, non-smooth hyperfinite Borel equivalence relations are

- $E_t$ (on $2^\mathbb{N}$, where $x E_t y \iff \exists n \exists m \forall k(x_{n+k} = y_{m+k})$
- $E_0 \times \Delta(n)$ (where $E_0$ on $2^\mathbb{N}$ is given by $x E_0 y \iff \exists n \forall m \geq n(x_m = y_m)$ and $\Delta(n)$ is the equality relation on $n$ elements, for $1 \leq n \leq \aleph_0$)
- $E_5^*$ (where $E_5^*$ is the aperiodic part of the equivalence relation induced by the shift on $2^\mathbb{Z}$).

The above results will appear in a forthcoming paper by the author entitled: The structure of hyperfinite Borel equivalence relations.

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