Surface-wave group-delay and attenuation kernels

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SUMMARY
We derive both 3-D and 2-D Fréchet sensitivity kernels for surface-wave group-delay and anelastic attenuation measurements. A finite-frequency group-delay exhibits 2-D off-ray sensitivity either to the local phase-velocity perturbation $\delta c/c$ or to its dispersion $\omega(\partial/\partial\omega)(\delta c/c)$ as well as to the local group-velocity perturbation $\delta C/C$. This dual dependence makes the ray-theoretical inversion of measured group delays for 2-D maps of $\delta C/C$ a dubious procedure, unless the lateral variations in group velocity are extremely smooth.

Key words: attenuation, Fréchet derivatives, global seismology, $Q$, sensitivity, surface waves.

1 INTRODUCTION
Increasing theoretical attention has been paid in recent years to the limitations of JWKB ray theory as a basis for inverting surface-wave dispersion measurements. Finite-frequency sensitivity kernels that account for the ability of a mantle Love or Rayleigh wave to ‘feel’ 3-D structure off an unperturbed great-circle ray have been developed by a number of investigators (e.g. Snieder 1986; Snieder & Nolet 1987; Yomogida & Aki 1987; Friederich et al. 1993; Friederich 1999; Spetzler et al. 2000; Selby & Woodhouse 2000, 2002; Gung & Romanowicz 2004). This paper is not self-contained; in the interest of brevity we shall adopt the notation of ZDN04 and make frequent references to equations and figures therein, generally without explanation or comment.

2 NOTATIONAL REVIEW
For simplicity we consider only single-frequency, fundamental-mode measurements made using untapered seismic recordings in the time domain; the effect of applying either a single or multiple tapers in the measurement process can be easily accounted for using the procedures described in Sections 4 and 9 of ZDN04. The perturbations in the frequency-dependent phase $\delta \phi(\omega)$ and logarithmic amplitude $\delta \ln A(\omega)$ of a surface wave are related to the 3-D velocity and density perturbations $\delta a/a$, $\delta \beta/\beta$ and $\delta \rho/\rho$ by eqs (3.8) of ZDN04:

$$\delta \phi = \iint \left[ K^\phi_\alpha(\delta a/a) + K^\phi_\beta(\delta \beta/\beta) + K^\phi_\rho(\delta \rho/\rho) \right] d^3 x,$$

(1)

$$\delta \ln A = \iint \left[ K^\ln A_\alpha(\delta a/a) + K^\ln A_\beta(\delta \beta/\beta) + K^\ln A_\rho(\delta \rho/\rho) \right] d^3 x.$$

(2)

The 3-D phase and amplitude sensitivity kernels $K^\alpha_{\phi, \beta, \rho}(x, \omega)$ and $K^{\ln A}_{\phi, \beta, \rho}(x, \omega)$ are given by ZDN04 eqs (3.9) and (3.10):

$$K^\alpha_{\phi, \beta, \rho} = -\Im \left( \frac{S R \Omega^{\alpha, \beta, \rho} \mathcal{R}^{\alpha} e^{-i[k(\Delta' + \Delta') - (n' + n')/2\pi]} \mathcal{S} \mathcal{R} \sin \Delta}{i \sin \Delta} \right),$$

(3)

$$K^{\ln A}_{\phi, \beta, \rho} = \Re \left( \frac{S R \Omega^{\alpha, \beta, \rho} \mathcal{R}^{\alpha} e^{-i[k(\Delta' + \Delta') - (n' + n')/2\pi]} \mathcal{S} \mathcal{R} \sin \Delta}{i \sin \Delta} \right).$$

(4)
where \( k(\omega) \) is the wavenumber measured in radians per second on the unit sphere, \( \Delta \) is the angular epicentral distance, \( n \) is the number of polar passages, and we have ignored higher-mode coupling for the reasons articulated by ZDN04 and Zhou et al. (2005). The prime and double prime identify quantities associated with the source-to-scatterer and scatterer-to-receiver great-circle paths, of angular arc lengths \( \Delta' \) and \( \Delta'' \) and having \( n' \) and \( n'' \) polar passages, respectively. The quantities \( S \) and \( S' \) account for the surface-wave radiation pattern at the source, whereas \( R \) and \( R' \) account for the polarization of the receiver. The scattering factors \( \Omega_{\alpha, \beta, \rho} \) are a measure of the strength of the self-scattering off a 3-D elastic heterogeneity \( \delta a/\alpha, \delta \beta/\beta \) or \( \delta \rho/\rho \) situated at the point \( x \).

By neglecting the angular deflection \( \eta = \arccos(\hat{k}' \cdot \hat{k}) \) of a surface wave upon scattering, we can reduce the 3-D dependence of \( \delta \phi(\omega) \) and \( \delta \ln A(\omega) \) in eqs (1) and (2) to a 2-D dependence upon the local fractional phase-velocity perturbation:

\[
\delta \phi = \int \int \int K^c_{b}(\delta c/c) d\Omega, \quad \delta \ln A = \int \int \int K^c_{a}(\delta c/c) d\Omega, \tag{5}
\]

where the integration is over the unit sphere \( \Omega = \{ \hat{r} : \hat{r} \cdot \hat{r} = 1 \} \). The 2-D phase-velocity kernels \( K^c_{b}(\hat{r}, \omega) \) and \( K^c_{a}(\hat{r}, \omega) \) are given by eqs (6.3) and (6.4) of ZDN04:

\[
K^c_{b} = -\frac{2k^{3/2} \sin[k(\Delta' + \Delta'') - (n' + n'')\pi/2 + \pi/4]}{\sqrt{8\pi} |\sin \Delta'| |\sin \Delta''| / |\sin \Delta'|}, \tag{6}
\]

\[
K^c_{a} = -\frac{2k^{3/2} \cos[k(\Delta' + \Delta'') - (n' + n'')\pi/2 + \pi/4]}{\sqrt{8\pi} |\sin \Delta'| |\sin \Delta''| / |\sin \Delta'|}, \tag{7}
\]

where we have made a forward-propagating \((S' = S \text{ and } R'' = R)\) as well as a forward-scattering \((\eta = 0)\) approximation for convenience in what follows. As we shall see, the 3-D and 2-D sensitivities of group-delay and anelastic attenuation measurements can all be expressed in terms of the phase-delay and geometrical attenuation kernels \( k^{a, b, \rho}_{\phi}(x, \omega), K^{a}_{\phi}(\hat{r}, \omega) \) and \( K^{b}_{b}(\hat{r}, \omega) \) given in eqs (3), (6) and (7).

### 3 3-D GROUP-DELAY SENSITIVITY KERNEL

The perturbation \( \delta t(\omega) \) in the group delay of a surface wave is related to the perturbation \( \delta \phi(\omega) \) in the phase delay by

\[
\delta t = \frac{d(\delta \phi)}{d \omega}. \tag{8}
\]

The phase delay \( \delta \phi(\omega) \) is measured in radians whereas the group delay \( \delta t(\omega) \) is measured in seconds. Since the model perturbations \( \delta a/\alpha, \delta \beta/\beta \) and \( \delta \rho/\rho \) are independent of the angular frequency \( \omega \), differentiation of eq. (1) immediately yields the desired 3-D group-delay kernel:

\[
\delta t = \int \int \int [K^r_{a}(\delta a/\alpha) + K^r_{b}(\delta \beta/\beta) + K^r_{\phi}(\delta \rho/\rho)] d^3x, \tag{9}
\]

where

\[
K^a_{a, b, \rho} = \frac{\partial K^{a, b, \rho}_{\phi}}{\partial \omega}. \tag{10}
\]

We have made no attempt to evaluate the derivative in eq. (10) analytically; see Gilbert (1976) for a related but simpler problem. However, we have found it straightforward to compute \( K^a_{a, b, \rho}(x, \omega) \) with sufficient accuracy for the purposes of inversion using a simple numerical first-difference method. The lower half of Fig. 1 shows an illustrative example of a group-delay, shear-velocity kernel \( K^a_{\phi}(x, \omega) \) for a 10 mHz Love wave; the corresponding phase-delay kernel \( K^b_{\phi}(x, \omega) \) is plotted above for comparison. The off-great-circle sidebands are more pronounced for \( K^a_{a}(x, \omega) \) than for \( K^b_{b}(x, \omega) \); this enhanced off-path sensitivity of \( \delta t(\omega) \) is consistent with the analytical 2-D Gaussian beam analysis of Nolet & Dahlen (2000). In addition, the group-delay sensitivity is slightly more compressed toward the Earth’s surface than the phase-delay sensitivity; however, this is a minor effect compared to the higher sidebands and difficult to discern on the scale of the AB slice views in Fig. 1.

### 4 2-D GROUP-DELAY KERNELS

We can likewise determine the 2-D sensitivity of a group-delay measurement \( \delta t(\omega) \) by differentiation of the first of eqs (5). In this case we must be cognizant of the fact that both the 2-D phase-delay kernel \( K^b_{\phi} \) and the perturbation \( \delta c/c \) depend upon the angular frequency \( \omega \):

\[
\delta t = \int \int \int \left[ \frac{\partial K^b_{\phi}}{\partial \omega}(\delta c/c) + K^b_{\phi} \frac{\partial}{\partial \omega}(\delta c/c) \right] d\Omega. \tag{11}
\]

The derivative of the fractional phase-velocity perturbation \( \delta c/c \) is related to the fractional group-velocity perturbation \( \delta c/c \) by eq. (22) of Spetzler et al. (2002):

\[
\frac{\partial}{\partial \omega} \left( \frac{\delta c}{c} \right) = \frac{1}{kC} \left( \frac{\delta C}{C} - \frac{\delta c}{c} \right). \tag{12}
\]

The only frequency dependence of the kernel \( K^b_{\phi}(\hat{r}, \omega) \) is via the wavenumber \( k(\omega) \), which appears in the phase multiplying the angular detour distance \( \Delta' + \Delta'' - \Delta \) and as a \( k^{3/2} \) pre-factor in eq. (6). Noting that \( dk/d\omega = C^{-1} \) and making use of the relation (12) we find—to our initial
consternation and in disagreement with eqs (23)–(24) of Spetzler et al. (2002)—that the group delay $\delta t(\omega)$ depends upon both the fractional group- and phase-velocity perturbations:

$$\delta t = \int_\Omega K^C_x (\delta C / C) d\Omega + \int_\Omega K^C_y (\delta c / c) d\Omega,$$

where

$$K^C_x = \left( \frac{1}{kC} \right) K^C_\phi, \quad K^C_y = \left( \frac{1}{2kC} \right) K^C_\phi + \left( \frac{\Delta^\prime + \Delta^\prime\prime - \Delta}{C} \right) K^C_\phi. \quad (14)$$

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5 REFORMULATION OF THE DUAL DEPENDENCE

In an exemplary review of the originally submitted version of this paper, Mike Ritzwoller has pointed out to us that the strong 2-D dependence of $\delta t(\omega)$ upon the phase velocity can be ameliorated by using eq. (12) to rewrite eqs (13)–(14) in the alternative form

$$\delta t = \int \int K_C^t(\delta C/C) d\Omega + \int \int K_c^t \left[ \frac{\partial}{\partial \omega} (\delta c/c) \right] d\Omega,$$

(15)

where

$$K_C^t = K_C^{e} + K_C^{r} = \left( \frac{3}{2kC} \right) K_\phi + \left( \frac{\Delta' + \Delta'' - \Delta}{C} \right) K_4,$$

(16)

$$K_c^t = -\left( \frac{C}{\omega} \right) K_c^{e} = \left( \frac{1}{2\omega} \right) K_\phi - \left( \frac{\Delta' + \Delta'' - \Delta}{c} \right) K_4.$$

(17)
Eqs (15)–(17) express the dual dependence not directly upon the phase velocity \( \delta c/c \) but rather upon its dimensionless dispersion \( \omega (\partial / \partial \omega) (\delta c/c) \). This has the advantage that the sensitivity to \( \delta C/C \) is greater than that to \( \omega (\partial / \partial \omega) (\delta c/c) \), particularly in the central Fresnel zone, where the amplitude of the reformulated kernel \( \tilde{K}^C_{Ct}(\hat{r}, \omega) \) exceeds that of \( -\tilde{K}^c_{Ct}(\hat{r}, \omega) \) by approximately a factor of three, as illustrated in Fig. 3. Because of this, and because the amplitude of the group-velocity perturbation \( \delta C/C \) exceeds that of the phase-velocity dispersion \( \omega (\partial / \partial \omega) (\delta c/c) \) by roughly the same factor in the current generation of smooth, upper-mantle 3-D models, Barmin et al. (2005) have suggested that it is permissible simply to ignore the dependence upon the phase velocity and approximate the sensitivity of a measured group delay by the first term in eq. (15). Such an approximation may not be unreasonable for the largest-scale lateral variations in the group velocity \( \delta C/C \); however, any attempt to resolve small-scale structure in \( \delta C/C \) is likely to be plagued by the strong sidebands of the reformulated 2-D kernel \( \tilde{K}^C_{Ct}(\hat{r}, \omega) \). We believe that it is preferable to eschew an intermediate 2-D inversion for \( \delta C/C \), and instead to invert measured group delays \( \delta t(\omega) \) directly for 3-D structural variations in the \( S \)-wave velocity \( \delta \beta/\beta \), using eqs (9)–(10).

6 REDUCTION TO RAY THEORY

If the lateral variations in \( \delta C/C \), \( \delta c/c \) and \( \omega (\partial / \partial \omega) (\delta c/c) \) are sufficiently smooth across the great-circle ray path, these factors can be extracted from the cross-path integrals in eqs (13) and (15). The remaining cross-path integrals of the Fréchet kernels \( K^C_{Ct}(\hat{r}, \omega) \), \( K^c_{Ct}(\hat{r}, \omega) \) and \( \tilde{K}^C_{Ct}(\hat{r}, \omega), \tilde{K}^c_{Ct}(\hat{r}, \omega) \) can be performed analytically by making the paraxial approximation

\[
\Delta' + \Delta'' - \Delta \approx \frac{1}{2} \left[ \frac{\sin \Delta}{\sin \Delta \sin(\Delta - x)} \right] y^2,
\]

(18)
where $x$ and $y$ are the along-path and cross-path angular coordinates, and we have restricted attention to a minor-arc wave, as in Section 7 of ZDN04, for simplicity. Upon making use of the Gaussian integral identities in eq. (7.13) of ZDN04 we find that

$$\int_{-\infty}^{\infty} K_c^i \, dy = \int_{-\infty}^{\infty} \tilde{K}_c^i \, dy = -1/C, \quad \int_{-\infty}^{\infty} K_r^i \, dy = \int_{-\infty}^{\infty} \tilde{K}_r^i \, dy = 0,$$

so that the dependencies upon the phase velocity $\delta c/c$ or its dispersion $\omega(\partial/\partial \omega)(\delta c/c)$ vanish, and we recover the 1-D, ray-theoretical dependence upon the along-ray group-velocity variations,

$$\delta t \approx -C^{-2} \int_{0}^{\Delta} \delta c \, dx,$$

in this infinite-frequency limit, as expected. The Fermat path-integral relation (20) has been used as the basis for making 2-D maps of $\delta c/C$ in a number of regional group-velocity investigations. Because of the strong sidebands of the 2-D kernels $K_c^i(\hat{r}, \omega)$ and, especially, $K_r^i(\hat{r}, \omega)$ and $\tilde{K}_r^i(\hat{r}, \omega)$, the cross-path variations of $\delta c/C, \delta c/c$ and $\omega(\partial/\partial \omega)(\delta c/c)$ need to be smooth out to a considerable distance off the great-circle ray path for the ray-theoretical approximation (20) to be valid.

### 7 3-D ANELASTIC ATTENUATION KERNELS

Starting with an unperturbed, perfectly elastic, spherical earth model with bulk and shear moduli $\kappa$ and $\mu$, we next consider a purely imaginary 3-D perturbation of the form

$$\frac{\delta x}{\kappa} = i Q^{-1}_\kappa, \quad \frac{\delta \mu}{\mu} = i Q^{-1}_\mu, \quad \frac{\delta \rho}{\rho} = 0,$$

or, equivalently,

$$\frac{\delta \alpha}{\alpha} = \frac{i}{2} \left( 1 - \frac{4 \beta^2}{3 \alpha^2} \right) Q^{-1}_\alpha + \frac{i}{2} \left( \frac{4 \beta^2}{3 \alpha^2} \right) Q^{-1}_\beta, \quad \frac{\delta \beta}{\beta} = \frac{i}{2} Q^{-1}_\beta, \quad \frac{\delta \rho}{\rho} = 0,$$

where $Q^{-1}_\alpha$ and $Q^{-1}_\beta$ are the spatially variable, inverse bulk and shear quality factors. Upon inserting eqs (22) into eq. (2) and rearranging terms, we obtain the Fréchet derivative relationship expressing the 3-D sensitivity of a measured amplitude perturbation $\delta \ln A(\omega)$ to the inverse quality factors:

$$\delta \ln A = \iint_{\Omega} \left[ K^{Q_\kappa}_d Q^{-1}_\kappa + K^{Q_\mu}_d Q^{-1}_\mu \right] d^3 x,$$

where

$$K^{Q_\kappa}_d = \frac{1}{2} \left( 1 - \frac{4 \beta^2}{3 \alpha^2} \right) K^\kappa, \quad K^{Q_\beta}_d = \frac{1}{2} \left( K^\beta + \frac{4 \beta^2}{3 \alpha^2} K^\rho \right).$$

It is noteworthy that the anelastic attenuation kernels $K^{Q_\kappa}_d(x, \omega)$ and $K^{Q_\beta}_d(x, \omega)$ are linear combinations of the elastic phase-delay kernels $K^\kappa_d(x, \omega)$ and $K^\beta_d(x, \omega)$ rather than the geometrical attenuation kernels $K^\kappa_d(x, \omega)$ and $K^\beta_d(x, \omega)$.

The results in eq. (24) can be rewritten in a form analogous to eq. (3), namely

$$K^{Q_\kappa}_d(\omega) = -\text{Im} \left\{ \frac{\mathcal{S} \mathcal{M} Q_\kappa \mathcal{R} e^{-i[\mathcal{S} \mathcal{M} + \Delta \kappa - \delta \kappa + \kappa U + kr^{-1} W]} - \kappa(U + 2r^{-1} U - kr^{-1} V)^2 \text{ Love waves} \right\}$$

$$K^{Q_\beta}_d(\omega) = -\text{Im} \left\{ \frac{-\mu \left[(W - r^{-1} W) \cos \eta + k^2 r^{-2} W^2 \cos 2\eta \right]}{\sqrt{3\kappa \rho}} \text{ Love waves} \right\}$$

The anelastic self-scattering factors (26)–(27) are analogous to the corresponding elastic scattering factors $\Omega^{\alpha, \beta, \rho}$ tabulated in Appendix A of ZDN04; the quantities $U$, $V$ and $W$ are the Rayleigh and Love eigenfunctions, normalized in accordance with eqs (2.11)–(2.12) of ZDN04, and a dot denotes differentiation with respect to radius $r$.

Love-wave attenuation is independent of the bulk anelasticity $Q^{-1}_\kappa$ whereas Rayleigh waves are attenuated by both $Q^{-1}_\kappa$ and $Q^{-1}_\beta$, but much more strongly by the latter. To illustrate this we show examples of $K^{Q_\kappa}_d(x, \omega)$ for a 10 mHz Love wave and of both $K^{Q_\kappa}_d(x, \omega)$ and $K^{Q_\beta}_d(x, \omega)$ for a 10 mHz Rayleigh wave in Fig. 4. All three sensitivity kernels are negative within the first Fresnel zone, as expected if a physically permissible inverse quality factor, $Q^{-1}_\kappa, Q^{-1}_\beta > 0$, is to lead to a reduction in the wave amplitude. Roughly speaking, a 10 mHz Rayleigh wave is an order of magnitude less sensitive to bulk anelasticity variations $Q^{-1}_\kappa$ than to shear anelasticity variations $Q^{-1}_\beta$.

Finally we note that the above results can be written more succinctly in terms of the inverse $P$-wave and $S$-wave quality factors, defined by eqs (9.59)–(9.60) of Dahlen & Tromp (1998) and commonly used in body-wave seismology:

$$Q^{-1}_u = 1 - \frac{4 \beta^2}{3 \alpha^2} Q^{-1}_\kappa + \frac{4 \beta^2}{3 \alpha^2} Q^{-1}_\beta, \quad Q^{-1}_p = Q^{-1}_\mu.$$

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Figure 4. Various views of the 3-D anelastic attenuation kernels for 10 mHz Love and Rayleigh waves. (a) Love-wave kernel $K_A^{Q_α}(x,ω)$, expressing the sensitivity of a measured amplitude perturbation $δ\ln A(ω)$ to 3-D variations in the inverse shear quality factor $Q_α^{-1}$. (b) Shear attenuation kernel $K_A^{Q_μ}(x,ω)$ for a 10 mHz Rayleigh wave. (c) Bulk attenuation kernel $K_A^{Q_κ}(x,ω)$ for the same Rayleigh wave. Top three plots are map views plotted at 100 km depth; middle three plots are vertical slices along cross-section AB with 100 km depth indicated by the dotted line; bottom three plots show variation of $K_A^{Q_{α,μ}}(x,ω)$ along profile AB at 100 km depth. Both the Love-wave and Rayleigh-wave sources are 10-km-deep, vertical strike-slip faults (green beachballs), with the latter rotated in strike by $π/4$ with respect to the former, so that the maximum radiation is in both instances in the direction of propagation to the receiver (green triangle). The Love-wave sensitivity kernel is for a transverse-component measurement and the Rayleigh-wave kernel is for a vertical-component measurement, both made on cosine-tapered seismograms of 520 s duration, centred upon the group arrival time in the reference earth model 1066A (Gilbert & Dziewonski 1975).

In this notation the imaginary velocity perturbations in eq. (22) are simply $δα/α = (i/2)Q_α^{-1}$, $δβ/β = (i/2)Q_β^{-1}$ so that the 3-D Fréchet kernel relationship (23)–(24) reduces to

$$δ\ln A = \int\int\int_{Ω} \left[ K_A^{Q_α} Q_α^{-1} + K_A^{Q_μ} Q_μ^{-1} \right] d^3x \quad \text{where} \quad K_A^{Q_{α,μ}} = \frac{1}{2} K_φ^{α,μ}. \tag{29}$$

8 2-D ATTENUATION KERNEL

A 2-D anelastic sensitivity kernel can be derived by making a forward-scattering ($η = 0$) approximation in eqs (26)–(27) and evaluating the resulting integral over depth; the anelastic analogue of eq. (6.2) of ZDN04 is

$$\int_{0}^{a} \left[ Ω_{n=0}^{Q_α} Q_α^{-1} + Ω_{n=0}^{Q_μ} Q_μ^{-1} \right] r^2 dr = -k^2 \left( \frac{c}{CQ} \right), \tag{30}$$

where $Q$ is the local quality factor of a Love or Rayleigh wave, given by eqs (16.148)–(16.153) of Dahlen & Tromp (1998). Alternatively, it is possible to start with the 2-D amplitude kernel $K_A^φ(\hat{r},ω)$ in the second of eqs (5) and note that anelasticity corresponds to an imaginary perturbation in the local fractional phase velocity of the form

$$\frac{δc}{c} = \frac{i}{2} \left( \frac{c}{CQ} \right). \tag{31}$$
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2-D attenuation kernels \( K^Q_A \)

(a) Love-wave kernel \( K^Q_A(\hat{r}, \omega) \) expressing the sensitivity of \( \delta \ln A(\omega) \) to 2-D variations in the inverse quality factor \( Q^{-1} \) of a 10 mHz Love wave. (b) Same but for a 10 mHz Rayleigh wave. Top plots show map views and bottom plots show variation along the perpendicular cross section AB, midway between the source and receiver; the amplitude measurements are presumed to be made on cosine-tapered seismograms of 520 s duration, as in Figs 1–4.

Either method yields the same 2-D sensitivity to the inverse surface-wave quality factor \( Q^{-1} \), namely

\[
\delta \ln A = \int \int_{\Omega} K^Q_A Q^{-1} d\Omega \quad \text{where} \quad K^Q_A = \left( \frac{c}{2C} \right) K^\phi.
\]

Just as the 3-D kernels \( K^{Q, \kappa}(x, \omega) \) and \( K^{Q, \mu}(x, \omega) \) are linear combinations of \( K^{\alpha, \beta}(x, \omega) \), the 2-D kernel \( K^Q_A(\hat{r}, \omega) \) is simply a frequency-dependent constant \( c/(2C) \) times the 2-D phase-delay kernel \( K^\phi(\hat{r}, \omega) \). In Fig. 5 we show illustrative examples of the 2-D anelastic attenuation kernels \( K^Q_A(\hat{r}, \omega) \) along the same source–receiver path as in Figs 1–4. The 2-D sensitivity of a 10 mHz Rayleigh wave is slightly higher than that of a 10 mHz Love wave because it has a slightly larger wavenumber: \( (k_R/k_L)^{3/2} \approx 1.3 \).

9 REDUCTION TO RAY THEORY REDUX

In the limit of infinite frequency, \( \omega \to \infty \), the local inverse quality factor \( Q^{-1} \) can be extracted from the cross-path integral in eq. (32), and the remaining cross-path integral over the 2-D kernel \( K^Q_A(\hat{r}, \omega) \) can be evaluated by making the paraxial approximation (18), with the result

\[
\int_{-\infty}^{\infty} K^Q_A d\gamma = -\frac{\omega}{2C}.
\]

In this limit we recover the anelastic ray-theoretical result,

\[
\delta \ln A \approx -\frac{\omega}{2C} \int_{\Omega} \frac{dx}{Q},
\]

as required for the finite-frequency theory to be consistent. A factor of \( C \) rather than \( c \) appears in the denominators of eqs (31) and (34) because the energy of a dispersive wave propagates with the group velocity.
10 GEOMETRICAL PLUS ANELASTIC ATTENUATION

On a realistic earth model with 3-D variations in $\delta \alpha /\alpha$, $\delta \beta /\beta$ and $\delta \rho /\rho$ as well as $Q^{-1}_s$ and $Q^{-1}_\mu$, the amplitude of a surface wave will be perturbed by elastic focusing and defocusing effects as well as by anelastic attenuation. The total first-order amplitude perturbation is the sum of both effects:

$$\delta \ln A = \delta \ln A_\alpha + \delta \ln A_\beta + \delta \ln A_\rho + \delta \ln A_{Q^{-1}_s} + \delta \ln A_{Q^{-1}_\mu},$$

where $\delta \ln A_\alpha(\omega)$ is given by eqs (2) and (4), whereas $\delta \ln A_\beta(\omega)$ is given by eqs (23)–(27). In the 2-D, forward-scattering, forward-propagating approximation, the amplitude depends only upon the local surface-wave phase-velocity and inverse quality factor:

$$\delta \ln A = \int_0^\Delta K_\alpha^0(\delta c/c) \, d\Omega + \int_0^\Delta K_\beta^0(\delta c/c) \, d\Omega.$$  

Finally, in the ray-theoretical limit, $\omega \to \infty$, the amplitude is the sum of two 1-D integrals along the unperturbed great-circle ray path:

$$\delta \ln A = -\frac{1}{2c \sin \Delta} \int_0^\Delta \sin x \sin(\Delta - x) \delta c \, dx - \frac{\omega}{2C} \int_0^\Delta \frac{dx}{Q}.$$  

The spherical-earth, elastic focusing-defocusing term, eq. (7.12) of ZDN04, was first derived using a strictly ray-theoretical argument by Woodhouse & Wong (1986).

11 CONCLUSION

In this paper we have derived Fréchet kernels expressing the linearized sensitivity of a surface-wave group-delay measurement $\delta t(\omega)$ to 3-D elastic velocity and density variations $\delta \alpha /\alpha$, $\delta \beta /\beta$ and $\delta \rho /\rho$ and the sensitivity of an amplitude measurement $\delta \ln A(\omega)$ to 3-D bulk and shear anelasticity variations $Q^{-1}_s$ and $Q^{-1}_\mu$. By making a forward-scattering ($t = 0$) and a forward-propagating ($S' = S$ and $R' = R$) approximation and evaluating the resulting integrals over depth, we have implemented a reduction from 3-D to 2-D, obtaining kernels that express the sensitivity of $\delta t(\omega)$ to the local fractional group-velocity and phase-velocity perturbations $\delta C/C, \delta c/c$ and $\omega(\delta t/\delta x)(\delta c/c)$, and the sensitivity of $\delta \ln A(\omega)$ to the local surface-wave inverse quality factor $Q^{-1}$. In the ray-theoretical limit, $\omega \to \infty$, these 2-D relationships reduce in turn to the expected 1-D, along-ray integrations. The strong sensitivity of a measured group delay $\delta t(\omega)$ to short-scale, off-path variations in the phase velocity $\delta c/c$ could potentially give rise to serious artefacts in 2-D maps of $\delta C/C$ as derived by ray-theoretical, group-delay inversion. In our opinion it is strongly preferable to invert both group-delay measurements $\delta t(\omega)$ and anelastic attenuation measurements $\delta \ln A(\omega)$ using the full 3-D, finite-frequency sensitivity kernels $K_\alpha^0(\omega, t)(x, \omega)$ and $K_\beta^0(\omega, t)(x, \omega)$.

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REFERENCES


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