# Wave-number selection by soft boundaries near threshold

M. C. Cross
*Bell Laboratories, Murray Hill, New Jersey 07974*
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An explanation is proposed for the increase in the width of the band of allowed states near threshold found in recent experiments by Cannell, Dominguez-Lerma, and Ahlers on wave-number selection in a Taylor-Couette system. A simple parametrization of the forcing of the motion by the nonuniform gap between the cylinders used to yield the selection produces many features of the data.

Cannell, Dominguez-Lerma, and Ahlers[1] have recently reported an elegant experimental realization (using Taylor-Couette flow) of a wave-number selection mechanism proposed[2] for the spatially periodic state typical of many nonequilibrium systems. The scheme consists of imposing a spatially varying control parameter $R$ that interpolates slowly, for $-l \leq z \leq 0$, between the bulk superthreshold value $R_b$ for $0 \leq z \leq L$, where the periodic state is investigated, and a subthreshold value for $z \leq -l$, where the periodic state is not a solution. In Ref. 2 it was shown quite generally that for a slow enough spatial variation (ramp) of the control parameter, a single wave number is selected in the homogeneous superthreshold region. The value of the wave number was shown to be independent of the shape of the ramp, but could differ if different physical quantities appearing in the expression for the control parameter were ramped.

The results reported by Cannell *et al.*[1] seem, when extrapolated to arbitrarily slow ramp rates, consistent with the theory of Ref. 2. However, they observed that the deviations at the nonzero ramp rates actually used did not have the same form as in the numerical simulation of simple model equations,[1,2] or as predicted by an amplitude equation that they suggested should give an accurate representation of the flow near threshold. In particular, the width $\delta\lambda$ of the band of possible wavelengths increased as $R_b$ approached the threshold value $R_c$. By contrast, the numerical experiments found a band shrinking linearly to zero towards threshold.

I suggest that these observations may be understood as a consequence of the inhomogeneous forcing of the fluid motion induced by the spatially varying gap between the cylinders used to vary $R$. Such forcing was not included in the equations studied in Ref. 1. A simple parameterization of the forcing consistent with naive expectations leads to the correct trends in the width of the band, and the oscillatory dependence observed for the wavelength as the length of the uniform region $L$ is changed. The one feature of the data not reproduced is the marked asymmetry of the band of wavelengths about the threshold value $\lambda_c$: $\lambda \geq \lambda_c$ was always observed, whereas I predict a symmetric band. It is interesting to remark that a similar, though less pronounced, discrepancy exists between theory and experiment for the width of the band in the case of a uniform control parameter (Fig. 3 of Ref. 1).

The equation I use is the same amplitude equation of Ref. 1, Eq. (1), but supplemented with a forcing term $f(z)$:

$$\tau_0 \dot{A} = \xi_0^2 A'' + A[\epsilon(z) - |A|^2] + f(z) \quad . \tag{1}$$

The notation of Ref. 1 is followed. Thus $A(z)$ is the complex amplitude giving the slow modulation of the onset disturbance, with the fluid flow varying with the axial coordinate $z$ as $\mathrm{Re}[A \exp(ik_c z)]$ with $k_c = 2\pi/\lambda_c$ and $\lambda_c \simeq 2.0$ measured in units of the gap. The dot denotes a time derivative and the prime a derivative with respect to $z$, and $\tau_0$ and $\xi_0$ are known constants[1] setting the time and length scales. Also, $\epsilon(z)$ is $[R(z) - R_c]/R_c << 1$. For $f(z) = 0$, Eq. (1) is consistent with zero axially dependent flow. On the other hand, the full Navier-Stokes equations do not permit such a solution with a nonuniform gap. The forcing $f(z)$, independent of $A$, is introduced to correct this contradiction, and describes the effect of the induced flow on the onset mode.

A calculation of $f(z)$ is very complicated, although similar calculations have been done for convection.[3] However, physical arguments lead to a simple parametrization. I would expect the intrinsic flow induced by the ramp well below threshold to be a weak, large roll spanning the region $-l \leq z \leq 0$. The projection along the critical mode $\exp(ik_c z)$ is then large only at the up and down flow points $-l$ and 0. Since $z = -l$ is strongly subthreshold, I write the important forcing localized at $z = 0$ as

$$f(z) = h\xi_0 \delta(z) \quad , \tag{2}$$

with $h$ proportional to the ramp rate $\alpha = d\epsilon/dz$, and taken real by a choice of origins. Note that Eqs. (1)–(2) predict the small but measurable subthreshold flow

$$A(z) \simeq \tfrac{1}{2} h |\epsilon|^{-1/2} \exp(-|\epsilon^{1/2} z|/\xi_0), \quad \epsilon < 0 \quad . \tag{3}$$

With $A = |A| \exp(i\phi)$ the static form of Eqs. (1)–(2) may be integrated explicitly to give

$$\delta k = k(z) - k_c = \frac{d\phi}{dz} = \frac{c}{|A|^2} + \frac{h|A(0)|\sin\phi(0)}{\xi_0 |A|^2} \Theta(z) \quad , \tag{4}$$

with $\Theta(z)$ the unit step function. The constant of integration $c$ is set to zero by considering the subthreshold region $|A|^2 \to 0$. (Then for $h = 0$ a zero width band $\delta k = 0$ is found in agreement with the numerical work of Ref. 1.) Thus in the uniform region $z > 0$,

$$\xi_0 \delta k \simeq \gamma h \epsilon^{-1/2} \sin\phi(0) \quad , \tag{5}$$

where $\gamma = |A(0)|/|A(z > 0)|$. For $\epsilon$ not too small, $\gamma \simeq 1$, and I predict a band of wave numbers of width $2\delta k_+ = 2h\epsilon^{-1/2}$ as $\sin\phi(0)$ varies between $\pm 1$ as $L$ is varied. The divergence of $2\delta k_+$ towards threshold is presumably ultimately controlled by a decrease in $\gamma$ for suf-

 

ficiently small $\epsilon$ although in practice the Eckhaus instability in the bulk may intervene first.

The dependence of the wavelength $\lambda = 2\pi(k_c + \delta k)^{-1}$ on the length $L$ of the homogeneous region is given by (5) and the relationship

$$\phi(0) = \phi_0 - (k_c + \delta k)L \quad , \tag{6}$$

with $\phi(L) = \phi_0$ assumed to be a constant fixed by the endwall boundary condition. Choosing $h = 7 \times 10^{-3}\xi_0$ to give the observed width[1] $2\delta\lambda \simeq 0.06$ at $\epsilon = 0.02$ for the ramp angle $\alpha = 0.016$ and keeping $\gamma = 1$ leads to Fig. 1 showing the variation of the wavelength as the aspect ratio $L$ or control parameter $\epsilon = (R_b - R_c)/R_c$ is changed, to be compared with Fig. 2 of Ref. 1. Note the common features of the period $\sim 2$ in aspect ratio, and the increasing amplitude of the variation, with increasingly sharp drops, as $\epsilon$ decreases. (In fact, the curves become hysteretic for $\delta k_+ L \geqslant 1$, corresponding to $\epsilon \leqslant 0.014$ for the value of $h$ used.) Again, however, note the symmetric form about $\lambda = \lambda_c \simeq 2.0$ calculated, compared with the asymmetry of the data. At this stage it is not clear if this discrepancy is due to the simple forcing and boundary condition $\phi(L)$ assumed, other effects that may further truncate or shift the band, or is a symptom of a serious flaw in the mechanism proposed here.

In summary, I have shown that the addition of the forcing term to the amplitude equation correctly accounts for an increasing band of stable wavelengths near threshold. From a formal point of view the result is not really surprising: The ramp rate $\alpha = d\epsilon/dz$ must be slow on the length scale relevant to the system which, close to threshold, diverges as $\epsilon^{-1/2}$, leading to the criterion for the validity of the theory of Ref. 2:

$$\alpha\epsilon^{-1/2} << 1 \quad . \tag{7}$$

The result Eq. (5), due to the pinning, with strength $h$ proportional to $\alpha$, competing with the intrinsic elasticity of the periodic pattern decreasing as $\epsilon^{1/2}$, is a physical derivation of the origin of this restriction. Note that the inclusion instead of higher-order (in $\epsilon$) terms in the amplitude equation such as developed for convection[4] will not account for the data: Typical further terms involve higher powers of $\epsilon^{1/2}d/dz \sim \epsilon^{1/2}\alpha$ and become *less* important near threshold. Such terms would, however, account for the width increasing *away* from threshold found for the model equations.[1]

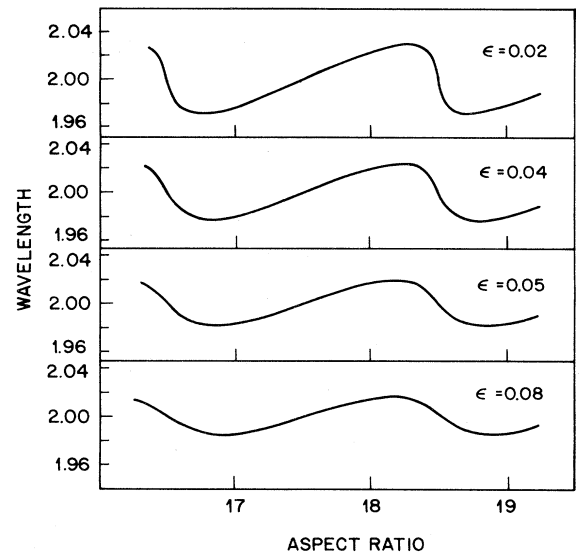


FIG. 1. Predicted variation of the wavelength with aspect ratio from Eqs. (5) and (6) for various values of $\epsilon$ with $\gamma = 1$ and fixed $h$ fitted to the $\epsilon = 0.02$ data of Ref. 1 for ramp rate $\alpha = 0.016$.

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