On the Interpretation of Heat in Relativistic Thermodynamics

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This article investigates the interpretation of the right-hand side of the relativistic second law of thermodynamics

\[ \left( \frac{\phi_0}{dS} \right)_\mu \left( -e^\lambda \right) dx^\mu dx^\alpha dx^\beta dx^\gamma \equiv \frac{dQ_0}{T_0} \]

and shows that the quantity \( dQ_0 \) can be interpreted as the heat—measured by a local observer at rest in the fluid at the point of interest—which flows relative to the fluid into an element of the fluid having the instantaneous proper volume \( dV_0 \) during the proper time \( dt_0 \), where these quantities are chosen so as to satisfy the numerical equality \( dV_0 dt_0 = \left( -e^\lambda \right) dx^\mu dx^\alpha dx^\beta dx^\gamma \) and the quantity \( T_0 \) is taken as the temperature ascribed to this heat by the local observer.

§1. Introduction

The analogue in relativistic thermodynamics of the usual second law of thermodynamics may be conveniently written in either of the two equivalent forms

\[ \left( \frac{\phi_0}{dS} \right)_\mu \left( -e^\lambda \right) dx^\mu dx^\alpha dx^\beta dx^\gamma \equiv \frac{dQ_0}{T_0} \]

or

\[ \frac{\partial}{\partial x^\mu} \left( \phi_0 \right) \left( -e^\lambda \right) dx^\mu dx^\alpha dx^\beta dx^\gamma \equiv \frac{dQ_0}{T_0} \]

where the sign "is greater than" (\( > \)) applies to irreversible processes and the sign "is equal to" (\( = \)) applies to reversible processes.

The quantity \( \phi_0 \) occurring in these expressions is the proper density of entropy as measured by a local observer at rest at the point of interest with respect to the thermodynamic fluid or working substance which is under consideration. The quantities \( dx^\mu/dS \) are the components of the macroscopic "velocity" of this fluid at the point of interest. And the quantity \( dQ_0/T_0 \) may be described as the heat which flows into the element of the fluid and during the time denoted by \( dx^\alpha dx^\beta dx^\gamma dx^\delta \), divided by its temperature—both of these quantities being measured in proper coordinates.

In the applications of relativistic thermodynamics which have so far been made, there has been no flow of heat relative to the fluid under consideration, so that the term \( dQ_0/T_0 \) has actually been zero. These applications have been in the first place to cases of static thermodynamic equilibrium where the heat flow was zero on account of the condition of static equilibrium. In the second place, applications have been made to the reversible expansion and contraction of certain models of the universe in which there has been no flow of heat from one portion of the material filling the model to another owing to uniformity of conditions throughout the models considered. In the third place, applications have also been made to the irreversible expansion and contraction of models of the universe in which, however, there was still no flow of heat from one portion of the material to another, again on account of the assumed homogeneity of the models.

In general, nevertheless, the expression of the relativistic second law given by (1) and (2) should be valid when the heat flow does not happen to be equal to zero. And the purpose of the present note is to make as clear as may be a correct method in the general case for the specification of the quantity \( dQ_0/T_0 \).

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1 Tolman, Proc. Nat. Acad. 14, 268 (1928); ibid. 14, 701 (1928); Phys. Rev. 35, 896 (1930).
4 Tolman, Phys. Rev. 37, 1639 (1931); ibid. 38, 797 (1931); ibid. 38, 1758 (1931).
§2. INVARIANCE OF $dQ_0/T_0$ WITH RESPECT TO COORDINATE TRANSFORMATIONS

Examining the expression for the relativistic second law in the form given by (1), it will be noted that the two factors $(\phi_0 dx^0/ds)_0$ and $(-g)^{1/2} dx^0 dx^1 dx^2 dx^3$, which are multiplied together to give the left-hand side of the expression, are both of them invariant scalars. The first of these factors $(\phi_0 dx^0/ds)_0$ is a scalar since it is the contracted covariant derivative of a vector and it will have a numerical value which is the same in all coordinate systems. The second of the factors $(-g)^{1/2} dx^0 dx^1 dx^2 dx^3$ is the known expression in natural measure for the four-dimensional volume specified by the range $dx^0 dx^1 dx^2 dx^3$, and hence is also a scalar, having a value proportional to the designated range, but otherwise independent of the coordinate system.

As a result of this invariant scalar character of the left-hand side of (1), it is evident from the principle of covariance that the right-hand side of the expression must also be a similar invariant scalar, since otherwise the postulated law would not lead to the same results when applied in different coordinate systems. Hence the quantity $dQ_0/T_0$ must itself have a value, which is of course proportional to the four-dimensional volume $(-g)^{1/2} dx^0 dx^1 dx^2 dx^3$, but is otherwise independent of the coordinate system. This result is very useful, since it permits us to obtain a general determination of the quantity $dQ_0/T_0$ by first employing a specially chosen coordinate system in which the interpretation will be quite simple and clear.

§3. DETERMINATION OF $dQ_0/T_0$ BY USING "CO-MOVING" COORDINATES

To obtain a coordinate system which will make the interpretation easy, we may take for the space-like coordinates $x^1, x^2, x^3$ a three-dimensional network of lines permanently connecting adjacent macroscopically identifiable points of the fluid under consideration and moving with the fluid, and take for the time-like coordinate $t^4$ the readings of a set of natural clocks which have been distributed to a sufficient number of different points throughout the fluid and are then allowed to move therewith. The possibility of obtaining such a coordinate system would appear guaranteed by the specifications for setting it up, and such systems which might be called "co-moving coordinates" are often very useful.

With such coordinates, it is evident that the fluid will always be permanently everywhere at rest with respect to the space-like coordinates, so that we can write

$$dx^0/ds = dx^3/ds = dx^4/ds = 0$$  \hspace{1cm} (3)

for the spatial components of its macroscopic "velocity." Furthermore, for the temporal component we can write

$$dx^4/ds = 1$$  \hspace{1cm} (4)

since increments in coordinate time and proper time have been made the same by our specification of $x^4$ with the help of natural clocks moving with the fluid.

Substituting (3) and (4) in the relativistic second law as given in the form (2), we then find that this reduces to the simple expression

$$\frac{\partial}{\partial x^4} (\phi_0(-g)^{1/2}dx^0 dx^1 dx^2 dx^3) dx^4 \equiv \frac{dQ_0}{T_0}$$  \hspace{1cm} (5)

and because of the mutual independence of the coordinates this can be rewritten in the form

$$\frac{\partial}{\partial x^4} (\phi_0(-g)^{1/2}dx^0 dx^1 dx^2 dx^3) dx^4 \equiv \frac{dQ_0}{T_0}.$$  \hspace{1cm} (6)

To see more clearly, moreover, the significance of this result we may now introduce the evidently valid substitutions

$$dt_0 = dx^4; \quad dV_0 = (-g)^{1/2}dx^0 dx^1 dx^2 dx^3,$$

where $dt_0$ is an element of proper time as measured by a local observer at rest in the fluid at the point of interest, and $dV_0$ is the proper spatial volume, associated with the four-dimensional region $dx^0 dx^1 dx^2 dx^3$, as measured by this same local observer. Introducing (7) in (6), we can then write

$$\frac{\partial}{\partial t_0} (\phi_0 dV_0) dt_0 \equiv \frac{dQ_0}{T_0}.$$  \hspace{1cm} (8)

The interpretation of $dQ_0/T_0$ with the help of this result is then quite simple. Since $\phi_0$ is the density of entropy as measured by a local ob-
server at rest in the fluid, and \(dV_0\) is the spatial
volume as measured by this observer of a small
region through the boundaries of which no fluid
is passing, the left-hand side of the expression
gives the change which the local observer finds in
the time \(dt_0\) in the entropy of a definite small ele-
ment of the fluid. In accordance with the principle
of equivalence, however, this change which a
local observer finds in the entropy of a system
small enough so that gravitational curvature can
be neglected must be related to the heat that
flows into the system in the way given by the
ordinary second law of thermodynamics. Hence
we can now conclude that \(dQ_0\) is the heat as
measured by the local observer that flows in the
time \(dt_0\) into the element of fluid instantaneously
contained in the proper volume \(dV_0\), and \(T_0\) is
the temperature associated with this heat.
Furthermore, as a result of our previous discussion
in §2 of the invariance of \(dQ_0/T_0\) to coordinate
transformations, we are thus furnished with a
correct value to use for this quantity in any co-
ordinate system, provided of course that we allow
for the proportionality between the magnitude of
\(dQ_0/T_0\) and the magnitude \((-g)dx^1dx^2dx^3dx^4\)
of the four-dimensional volume associated with the
specified coordinate range.

Hence, returning to our general expression for
the relativistic second law, which is valid in any
coordinate system

\[
\left( \frac{dx^\mu}{d\phi_0} \right)_\mu (-g)^{1/2}dx^1dx^2dx^3dx^4 \equiv \frac{dQ_0}{T_0}
\]

we may now state that the quantity \(dQ_0\) occurring
in this expression can be taken as the heat—
measured by a local observer at rest in the fluid
at the point of interest—which flows into an
element of the fluid having the instantaneous
proper volume \(dV_0\) during the proper time \(dt_0\),
where these quantities are chosen so as to satisfy
the numerical equality

\[
dV_0dt_0 = (-g)^{1/2}dx^1dx^2dx^3dx^4
\]

and the quantity \(T_0\) is taken as the temperature
ascribed to this heat by the local observer. We
are thus provided with a perfectly definite inter-
pretation of all the quantities occurring in the
relativistic second law which will satisfy the
principle of covariance by leading to the same
results no matter what coordinate system is used,
and which satisfies the principle of equivalence by
reducing, as we have seen with the help of a
specially convenient coordinate system, to the
ordinary second law for an infinitesimal portion
of the fluid.

§4. Determination of \(dQ_0/T_0\) by Using
Natural Coordinates

In accordance with the above specifications
for determining a correct value to use in the rel-
ativistic second law for the flow of heat \(dQ_0\) into
the element and during the time denoted by
\(dx^1dx^2dx^3dx^4\), it will be noted that this quantity is
to be obtained from measurements by a local
observer of the rate at which heat flows into a
specified element of the fluid. Hence this quantity
is to be regarded as corresponding to a flow of
heat measured relative to boundaries at rest in
the fluid, rather than in some way relative to
boundaries at rest with respect to the spatial
coordinates that are actually being employed, as
might at first sight have seemed plausible. Since
this conclusion was obtained, however, with the
help of a coordinate system specially chosen so
that boundaries at rest with respect to the fluid
were also at rest with respect to the spatial co-
dinates, it will be illuminating to show that
we should also be led to the same interpretation
for \(dQ_0/T_0\) using a system of coordinates with
respect to which the fluid is not at rest.

To obtain such a demonstration, we shall now
use a set of coordinates \(x, y, z, t\) with respect to
which the fluid is not at rest, but which are so
chosen as to be natural coordinates for the point
of interest. In accordance with the principle of
equivalence such natural coordinates for any de-
sired spacetime point can always be found, and
with respect to them the principles of special
relativity as expressed in their usual form may be
taken as valid in the immediate neighborhood
of the point in question. Hence, by choosing these
coordinates, we shall be able to employ the
principles of thermodynamics as developed for
the special theory of relativity by Planck and
Einstein in arriving at the desired interpretation.

Making use of these natural coordinates \(x, y,
z, t\), we shall evidently have

\[(-g)^{1/2} = 1\]
into the relativistic second law of thermodynamics in the form given by (2), we can now write the left-hand side of that expression in our present coordinates in the form

$$\left[ \frac{\partial}{\partial x} \left( \frac{dx}{ds} \right) + \frac{\partial}{\partial y} \left( \frac{dy}{ds} \right) + \frac{\partial}{\partial z} \left( \frac{dz}{ds} \right) + \frac{\partial}{\partial t} \left( \frac{dt}{ds} \right) \right] dsdydzdt$$

(12)

where $dx/ds$, $dy/ds$, and $dz/ds$ are the components of the macroscopic velocity of the fluid at the point of interest with respect to proper time, and by a simple substitution we can rewrite this in the form

$$\left[ \frac{\partial}{\partial x} \left( \frac{dt}{ds} \frac{dx}{dt} \right) + \frac{\partial}{\partial y} \left( \frac{dt}{ds} \frac{dy}{dt} \right) + \frac{\partial}{\partial z} \left( \frac{dt}{ds} \frac{dz}{dt} \right) + \frac{\partial}{\partial t} \left( \frac{dt}{ds} \right) \right] dsdydzdt,$$

(13)

where $dx/dt$, $dy/dt$, and $dz/dt$ give the components of the velocity of the fluid as ordinarily expressed in terms of the coordinate time $t$.

In accordance with the special theory of relativity, however, entropy is an invariant for the Lorentz transformation, and hence entropy density will be affected by the Lorentz factor of contraction $ds/dt$ in such a way that we can substitute

$$\phi = \phi_0 (dt/ds),$$

(14)

where $\phi$ is the density of entropy at the point of interest taken with respect to our present system of coordinates $x$, $y$, $z$, $t$.

Substituting (14) into (13), and writing for simplicity $u$, $v$, and $w$ for the three components of velocity $dx/dt$, $dy/dt$, and $dz/dt$, we then obtain

$$\left[ \frac{\partial}{\partial x} \left( \frac{dt}{ds} \frac{dx}{dt} \right) + \frac{\partial}{\partial y} \left( \frac{dt}{ds} \frac{dy}{dt} \right) + \frac{\partial}{\partial z} \left( \frac{dt}{ds} \frac{dz}{dt} \right) + \frac{\partial}{\partial t} \left( \frac{dt}{ds} \right) \right] dsdydzdt,$$

(15)

and by performing the indicated differentiations and rearranging the order, this can be rewritten in the form

$$\left[ \frac{\partial \phi}{\partial t} + \frac{\partial \phi}{\partial x} u + \frac{\partial \phi}{\partial y} v + \frac{\partial \phi}{\partial z} w \right] dx dy dz dt.$$

(16)

Finally, noting the significance of the various terms in (16), we may now rewrite this expression for the left-hand side of the relativistic second law in the simple form

$$(d\phi/\partial t) dV dt + \phi (d/dt) (dV) dt,$$

(17)

where we have written $dV$ for the volume of the element of fluid, instantaneously contained in the region $dx dy dz$, and have written $d\phi/\partial t$ for the total rate of change in entropy density as we follow the moving fluid. Hence in natural coordinates the left-hand side of the relativistic second law is seen to be equal to the change that takes place in time $dt$ in the entropy of the element of fluid instantaneously contained in the region $dx dy dz$, and we can write for these coordinates

$$\left( \frac{dx^\mu}{ds} \right) (dV) dV = \left( \frac{dV}{dt} \right).$$

(18)

In accordance, however, with special relativistic thermodynamic theory we can relate this change in the entropy, of the little thermodynamic system consisting of this element of fluid, to heat flow and temperature by the expression

$$d/\partial t (dV)/dt = dQ/T,$$

(19)

where $dQ$ is the heat absorbed by the element at temperature $T$ and in the time $dt$ referred to our present coordinates. Furthermore, since the ratio of heat to temperature is an invariant for the Lorentz transformation, we can also take

$$dQ/T = dQ_0/T_0,$$

(20)

where $dQ_0$ is the absorbed heat as measured in
proper coordinates and $T_\theta$ is the proper temperature of the element. Moreover, in accordance with the Lorentz contraction for volume elements and Lorentz dilation for time intervals we can write

$$dVdt = dV_\phi dt_\phi$$

where $dV_\phi$ is the volume of this element as measured in proper coordinates and $dt_\phi$ is the proper time during which the heat absorption takes place.

Hence combining (18), (19) and (20), we have now obtained, also using our present coordinates, the result

$$\frac{\partial}{\partial s} \left[ \phi_\theta \frac{dx^\mu}{ds} \right] (-g)^{\lambda j} dx^\lambda dx^j dx^3 dx^4 = \frac{dQ_\phi}{T_\theta},$$

where, in accordance with (21), $dQ_\phi$ is to be taken as the heat—measured by a local observer at rest in the fluid at the point of interest—which flows into an element of the fluid having the instantaneous proper volume $dV_\phi$ during the proper time $dt_\phi$, these quantities being chosen so as to satisfy the numerical equality

$$dV_\phi dt_\phi = dV dt = (-g)^{\lambda j} dx^\lambda dx^j dx^3 dx^4.$$  

Thus, using natural coordinates, we have satisfactorily obtained the same interpretation of the quantities occurring in the relativistic second law as was obtained in the preceding section using co-moving coordinates. And the treatment gives the desired illustration of the fact that our interpretation of $dQ_\phi$ as heat flowing through a boundary stationary in the fluid was not a result of our original special choice of coordinates in which the fluid was at rest.

§5. Conclusion

Before concluding, two further remarks may be made concerning the interpretation of the relativistic second law, which may be illuminating.

In accordance with the treatment given in the last section it is to be noted that the quantities $\phi$, $\phi u$, $\phi \varphi$ and $\phi \omega$ must be regarded as total densities of entropy and entropy flux at the point of interest, since otherwise, for example, the left-hand side of expression (19) could not have been interpreted as the change in the total entropy of the specified element of fluid and correlated as was done with the influx of heat. This implies that we are to treat our thermodynamic fluid from a macroscopic point of view and take $\phi$ as the total entropy associated and moving along with unit volume of that material, rather than to try to employ a microscopic point of view and take $\phi$ as entropy belonging in some way solely to the molecules of the fluid with an additional quantity belonging to the radiation in the space between them. Similar remarks apply to $\phi_\theta$ and $\phi_\theta dx^\lambda /ds$. It will be noted that the correct procedure is in entire agreement with ordinary thermodynamic practice and with the spirit of thermodynamics as a macroscopic phenomenological science.

In laying down a correct method for determining the magnitude of the heat flux divided by temperature $dQ_\phi / T_\theta$ which corresponds to the infinitesimal element of fluid and time denoted by $dx^\lambda dx^j dx^3 dx^4$, it will have been noted that we have merely specified that the local observer should measure the proper heat $dQ_\phi$ entering an element of the fluid of instantaneous proper volume $dV_\phi$ during the proper time $dt_\phi$, where $dV_\phi$ and $dt_\phi$ are chosen so as to satisfy the numerical equality

$$dV_\phi dt_\phi = (-g)^{\lambda j} dx^\lambda dx^j dx^3 dx^4$$

and have not introduced any specifications as to the shape of the four-dimensional region $dV_\phi dt_\phi$. This, however, will be seen to be legitimate since the heat flux is directly proportional to the product of volume and time, independent of shape except for higher order differentials which can be neglected on shrinking the four-dimensional region down to the point of interest.

It is hoped that the treatment which we have given in this article will help to make the interpretation of the relativistic second law clear. The considerations presented are in agreement with the results obtained in previous applications of relativistic thermodynamics where there was no flow of heat relative to the fluid under consideration, and should be specially helpful when further applications are made to systems in which a flow of heat relative to the fluid does take place.