

Path integrals, density matrices, and information flow with closed timelike curves

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Two formulations of quantum mechanics, inequivalent in the presence of closed timelike curves, are studied in the context of a solvable system. It illustrates how quantum field nonlinearities lead to a breakdown of unitarity, causality, and superposition using a path integral. Deutsch's density matrix approach is causal but typically destroys coherence. For each of these formulations I demonstrate that there are yet further alternatives in prescribing the handling of information flow (inequivalent to previous analyses) that have implications for any system in which unitarity or coherence is not preserved.

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Quantum mechanics has been suggested [1,2] as a possible means of resolving some of the classic paradoxes of time travel. The basic idea is to require some sort of consistency around closed timelike curves (CTC's). As it is not yet known whether or not a compact, bounded region of CTC's can arise [3] (due to gravitational interactions of quantum mechanical matter), one possible line of inquiry is the following: given such a spacetime, can it support any consistent, nontrivial mechanics? Two distinct generalizations of quantum mechanics have been proposed. One is defined by a coherent, action-weighted sum over all single-valued histories defined on the spacetime [1]. The other is a linear time evolution of a density matrix subject to certain periodic boundary conditions around the CTC's. The resulting mechanics of the two schemes are quite different, certainly peculiar, but in no apparent way self-inconsistent.

One purpose of this paper is to analyze in some detail a system that is simple enough to be solved exactly in both formulations but rich enough to exhibit various striking phenomena. In fact, intermediate steps in the calculations involve nothing more complicated than the interaction of two half-integer spins. However, I will indicate the parallels between this system and any arbitrary, interacting, quantum field theory.

A second goal is to illuminate issues regarding the flow of information in circumstances where unitarity or coherence are not preserved by the dynamics.¹ After establishing how pure initial states evolve according to the two proposed generalizations of quantum mechanics, I will show that the handling of initial mixed states is not unambiguous in either formulation. In particular, I will argue for an implementation of the density matrix mechanics (motivated by fairly classical notions of ensembles and probability) that it is inequivalent to the formulation proposed in Ref. [2]. In the path integral mechanics I show that the density matrix does not encode all that can be known about a mixed state; however,

the dynamics preserves what is known about the system, although traditional entropy is not a good measure of information.

In the context of the path integral approach, one new result presented here is that the nonunitarity of evolution from before to after a compact epoch of CTC's, hitherto identified in perturbation theory [4–7], persists in the exact solution of a nonlinear theory. (Previously, the only interacting models solved exactly exhibited no nonunitarity [7].) Although this phenomenon has been identified earlier, it may be of interest to see how, in a very simple context, CTC's lead to nonunitary amplitudes using an action that would otherwise have preserved unitarity in the absence of CTC's. It may also be helpful to have an explicit example in which to implement Hartle's general analysis of nonunitary evolution [8], which amounts to renormalizations to preserve a probabilistic interpretation. I present, in passing, an additional argument to those given by Hartle as to the necessity of following histories to the future of all CTC's. A remarkable consequence of nonunitarity and the consideration of full histories is that experiments completed before the CTC epoch are sensitive to the existence of CTC's in their future. In particular, such experiments violate the quantum superposition principle.

Deutsch's density matrix mechanics was originally presented in an even more abstract context than the model analyzed here. So certain particulars, which he did not address, need to be established. The hallmark of the density matrix approach is that pure states can evolve into mixed states after traversing a compact CTC epoch. Physics before that epoch is the same whether or not the CTC's come into existence, thus evading the acausality that arises in the path integral formulation.

Comparing the two approaches we will see that there are interactions which lead to nonunitary evolution from the path integral but preserve the coherence of any initial pure state under density matrix evolution. Conversely, there are interactions that do not preserve coherence using density matrices but are totally unitary in the path integral sense. More generally, these examples (which involve only four-by-four matrix Hamiltonians) offer some insight into possible distinctions between complex and simple quantum systems.

¹Black holes are thought by some to be more plausible examples of such systems than time machines.

THE MODEL SYSTEM

I will describe the model system (which is really only two coupled spins) as an abbreviated quantum field theory. The correspondences thus established should make clear both the motivation for the particular rules of the CTC mechanics and the appropriate generalization to more realistic systems.

In the absence of even an existence proof, there is certainly no “realistic” candidate for a spacetime. Hence, I choose one of a variety of possible generic structures [9]. Begin with a flat spacetime. Identify a compact region of space at time $t=0$ with a compact region of space at $t=T$ such that world lines entering the identified region from $t \leq 0$ connect smoothly and immediately to the identified point at $t=T$ and continue with $t > T$; world lines entering the identified region from $t \leq T$ connect smoothly and immediately to the identified point at $t=0$ and continue with $t > 0$. If the identified regions are the same, then the obvious point-by-point identification leaves the spacetime flat except for conical singularities. This construction is illustrated in the accompanying figure for one-plus-one dimensions (Fig. 1). There are four classes of straight world lines: (1) lines unaffected by the identification (these may either miss the time machine altogether or pass between the identified regions so quickly as to be unaltered); (2) lines that jump from $t=0$ to $t=T$ without ever passing in between; (3) lines that first encounter the time machine at $t=T$ and wind around some number of times before exiting to $t \rightarrow \infty$; and (4) CTC's that have no existence for $t < 0$ or $t > T$.

The first drastic simplification is to reduce the above-described space to two points. One point z_1 is outside the time machine, with a single continuous time history on the interval $-\infty < t < \infty$. The other point z_2 corresponds to the identified region of space. It has two disjoint segments to its history. One is a continuous line from $t = -\infty$ to $t = \infty$ excluding $0 < t < T$. The other is the CTC formed by identifying the spacetime point at $t=T$ with the point at $t=0$.

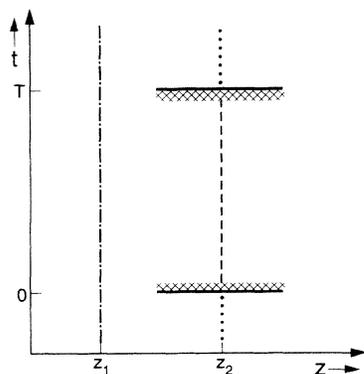


FIG. 1. A class of CTC spacetimes is defined by identifying the like-shaded time boundaries of two spatial regions at times $t=0$ and T . Further simplification comes from restricting the space to two points, z_1 and z_2 , whose respective world histories (broken lines) are connected and disconnected.

The second simplification concerns the nature of the quantum field defined on the space. In second quantization, bosons are described by an oscillator degree of freedom at each point in space. Even for a single point, this corresponds to an infinite dimensional Hilbert space. To make matters simpler I choose to consider a single fermion field. At each spatial point a single fermionic degree of freedom corresponds to a two-dimensional Hilbert space, occupied or unoccupied in the language of second-quantized fermions or, equivalently, spin up or down. With only two sites, the total Hilbert space has only four dimensions.

To make the connection of the thus defined two spin system to quantum field theory and to make explicit the measuring of a sum over histories for the spins, I briefly review fermion path integrals [10], in the absence of CTC's.

Fermions can be described by a Grassmann-number-valued field $\psi(z,t)$. In the present case the positions z take only two values, z_1 and z_2 . So there really are only two Grassmann (fully anticommuting) coordinates:

$$x(t) \equiv \psi(z_1, t), \quad y(t) \equiv \psi(z_2, t).$$

We will also need the independent, conjugate coordinates \bar{x} and \bar{y} . An alternative description is given by the set of operators $a, a^\dagger, b,$ and b^\dagger , corresponding, respectively, to $x, \bar{x}, y,$ and \bar{y} . These anticommute with themselves and each other except for the canonical anticommutators.

$$[a, a^\dagger] = 1, \quad [b, b^\dagger] = 1.$$

The most general wave function is

$$\phi(x, y) = c_0 + c_1 x + c_2 y + c_3 xy,$$

with complex coefficients c_i , defining a vector space equivalent to that of two half-integer spins.

The most general Hamiltonian $H(a, a^\dagger, b, b^\dagger)$ contains terms zero through fourth degree in the operators and is parametrized by 16 real numbers, corresponding to the general four-by-four Hermitian matrix Hamiltonian for the two-spin system.

The path integral that generates quantum amplitudes and, in particular, the evolution kernel, is an integral over the Grassmann fields $\psi(z,t)$ and $\bar{\psi}(z,t)$ or, in the present case, Grassmann paths $x(t), \bar{x}(t), y(t),$ and $\bar{y}(t)$, with suitable boundary conditions [10], of $\exp(iS)$, where the action

$$S = \int dt [-\bar{x}i\partial_t x - \bar{y}i\partial_t y - H(x, \bar{x}, y, \bar{y})].$$

The field theoretic perturbative expansion expresses all integrals in terms of Gaussians. Having only two sites reduces the problem drastically: the model is soluble in as much as one can diagonalize a four-by-four matrix.

For illustrative purposes it will suffice to consider only a few particular examples within the class of possible Hamiltonians. In particular, I will analyze special cases that can be diagonalized by inspection of

$$H = \omega_0(a^\dagger a + b^\dagger b) + \omega_1(a^\dagger b + b^\dagger a) + \gamma_1 a^\dagger a (b + b^\dagger) + \gamma_2 (a + a^\dagger) b^\dagger b + \lambda a^\dagger a b^\dagger b, \quad (1)$$

where $\omega_{0,1}$, $\gamma_{1,2}$, and λ are real parameters. Define a basis set of states for the two spins

$$\{|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle\} \quad (2)$$

by

$$\begin{aligned} a|\downarrow i\rangle &= 0, & a^\dagger|\downarrow i\rangle &= |\uparrow i\rangle, \\ b|\downarrow i\rangle &= 0, & b^\dagger|\downarrow i\rangle &= |\uparrow i\rangle, \end{aligned}$$

etc., where $i = \uparrow$ or \downarrow . In this basis, the Hamiltonian in Eq. (1) is represented by

$$H = \begin{pmatrix} 2\omega_0 + \lambda & \gamma_1 & \gamma_2 & 0 \\ \gamma_1 & \omega_0 & \omega_1 & 0 \\ \gamma_2 & \omega_1 & \omega_0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (3)$$

In the language of second quantized fields, ω_0 corresponds to the particle's mass. The ω_1 term is analogous to the kinetic or spatial hopping term. $\gamma_{1,2}$ are Yukawa-like couplings (not normally allowed for purely fermionic interactions). λ is the unique analogue in this system of a four-fermion coupling. The ω 's multiply linear terms in the field equations or Euler-Lagrange equations of motion for the Grassmann paths, while $\gamma_{1,2}$ and λ introduce nonlinearities.

In the presence of CTC's, there is no foliation of the spacetime (unique time ordering of spacelike surfaces) and hence no Hamiltonian evolution or Schrödinger equation. Nevertheless, both versions of quantum mechanics discussed below can be compactly characterized in terms of the action of the Hamiltonian H and, in particular, the unitary operator

$$U(t', t) \equiv e^{-iH(t'-t)}. \quad (4)$$

PATH INTEGRAL FOR PURE INITIAL STATES

The Feynman path integral offers a geometrically appealing generalization of quantum mechanics for a CTC-containing spacetime that does not admit a Schrödinger equation. One sums coherently over all field histories or paths defined over the spacetime, weighted by² $\exp(i \text{action})$. Correlations between observables are determined from the amplitudes obtained by projecting the paths onto the subsets that satisfy the possible observation outcomes. In principle, this defines all correlations, including those for measurements made during the CTC epoch. However, I will only consider observations made either before or after a compact CTC region. While the phenomena within the CTC epoch itself may offend our sensibilities, to dismiss their possibility on that basis may be premature.

²When considering particle dynamics, one must not confuse an essentially first quantized description in which particle world lines may wind any number of times around the CTC region from a second quantized description in which the fields (describing any number of particles) are single valued on the spacetime. The model considered here is of the latter type.

There are three logical steps to the full construction of the mechanics. (1) Consider an amplitude defined by initial and final states (before and after the CTC's) as boundary conditions on the path integral. We will find that systems with nonlinear equations of motion have a nonunitary initial-to-final map, as defined by the path integral. (2) To reestablish a probability interpretation of the (nonunitary) amplitudes squared, we renormalize the sum of the probabilities for all possible outcomes for each individual initial condition to one. Thus we determine the probability for any particular outcome relative to any particular initial condition. However, this renormalization depends on the initial state. Hence the linearity of quantum mechanics, i.e., the superposition principle, is lost. And (3) to maintain a consistent probability calculus in light of step 2, the path integral must include all paths extending to the future of the compact CTC region—even in computing correlations of only observations completed before the CTC's. It is this last step that has the most bizarre consequences: Pre-CTC quantum mechanics is nonlinear, nonunitary, and acausal, even in the absence of visits, real or virtual, from time travelers.

So, to begin I evaluate the path integral connecting states at $t = -\epsilon \equiv 0^-$ to states at $t = T + \epsilon$, with $\epsilon \rightarrow 0^+$. If z_2 is the spatial location of the time machine, then the field at z_2 satisfies $\psi(z_2, 0^-) = y(0^-) = \psi(z_2, T^+) = y(T^+)$, by construction. Hence, we need only evaluate how various values of $\psi(z_1, 0^-) = x(0^-)$ connect to $\psi(z_1, T^+) = x(T^+)$. In fact, in this path integral discussion, I will henceforth suppress the dependence on the field at z_2 outside the interval $0 \leq t \leq T$. Explicit restoration of that dependence would be totally straightforward: using a product basis, such as provided by Eq. (2), one simply multiplies the emerging $x(T^+)$ by the incident $y(T^-)$ for each basis state and then adds the components.

A simple strategy for explicit evaluation of the functional integral is to slice up the spacetime, do the integral over the slices with general boundary conditions, match the slices, and integrate over the boundary conditions of the interfaces to reconstruct the full integral. The integration over Grassmann positions x and y at a given t is really a sum over a complete set of states. Hence, for convenience we can use the spin basis defined by Eq. (2) to specify boundary conditions. And finally, the path integral over a finite time interval containing no CTC's is given by the action of U defined in Eq. (4). Putting all this together and defining the operator X as taking states of z_1 at $t = 0^-$ to z_1 at T^+ , one deduces the corresponding path integral to equal

$$\langle j|X|i\rangle = \sum_k \langle jk|U(T, 0)|ik\rangle, \quad (5)$$

where $i, j, k = \uparrow$ or \downarrow .

It is a simple exercise to see that the linear field theory defined by $\omega_{0,1} \neq 0$ and $\gamma_{1,2} = \lambda = 0$ in Eq. (3) yields a unitary X .

Note that "unitarity" should include the possibility that $X^\dagger X$ is proportional to and not only equal to the identity because in that case a single, state-independent factor, which can be absorbed into the functional mea-

sure, can restore literal unitarity without altering any observable correlations. A proof that all such linear theories produce unitary X 's goes as follows. The most general Hamiltonian with linear equations of motion is

$$H = r_1 + c_1 a + c_1^* a^\dagger + c_2 b + c_2^* b^\dagger + r_2 a^\dagger a + r_3 b^\dagger b + c_3 a b + c_3^* b^\dagger a^\dagger + c_4 a^\dagger b + c_4^* b^\dagger a,$$

where r_i and c_i are real and complex coefficients. Without loss of generality, we can set $c_1 = c_2 = 0$; this corresponds to shifting the operators by Grassmann c -numbers or mixing a with a^\dagger and mixing b with b^\dagger to define a new basis for each individual spin. Written in four-by-four matrix form, the resulting H is block diagonal, i.e., $H = h \otimes h'$, where h acts in the $(\uparrow\uparrow, \downarrow\downarrow)$ space and h' acts in the $(\uparrow\downarrow, \downarrow\uparrow)$ space. Also, $\text{tr}h = \text{tr}h'$, which is a further crucial consequence of the linearity. Hence, the unitary evolution in the four-dimensional space is of the form $U = u \otimes u'$, where u and u' are unitary two-by-two matrices on the above-mentioned subspaces, and $\det u = \det u'$. It is this equality of determinants that ensures the unitarity of X upon performing the partial trace prescription of Eq. (5). (To do the traces explicitly one can write u and u' as the appropriate sums of the Pauli and unit matrices and implement the equality of determinants.)

The nonlinear couplings, e.g., $\gamma_{1,2}$ and λ , produce a nonunitary X . A very simple example of a nonunitary X follows from $\lambda \neq 0$ but $\omega_{0,1} = \gamma_{1,2} = 0$. I will use this for illustrative purposes later with a maximally nonunitary value $\lambda T = \pi + 2\epsilon \approx \pi$; in this case,

$$X = \begin{pmatrix} 1/2(1+e^{-i\lambda T}) & 0 \\ 0 & 1 \end{pmatrix} \approx \begin{pmatrix} i\epsilon & 0 \\ 0 & 1 \end{pmatrix}. \quad (6)$$

In general, if a nonunitary evolution operator X maps an initial state Ψ onto a final state $X\Psi$, then we must renormalize the final state to become $X\Psi / (\Psi^\dagger X^\dagger X \Psi)^{1/2}$. For nonunitary X , the evolution is, consequently, nonlinear in Ψ . Given that Ψ is a pure quantum state, this final state is also pure (as one may confirm by computing its density matrix).

The implementation of initial-state-dependent renormalizations to generate a probability interpretation as outlined in step (2) above is straightforward. What may not be so clear is the necessity of Hartle's [8] assertion that even correlations before the CTC's are influenced by the nonunitarity. For example, in the system at hand, why could not the correlation of the spins at t_2 with their values at t_1 for $t_{1,2} < 0$ (i.e., before the CTC) be computed using the path integral from t_1 to t_2 with the appropriate states as boundary conditions? (This calculation would give the results of conventional quantum mechanics.) Instead, we are instructed to evaluate a path integral that traverses the CTC region. The most compelling argument for this procedure is not, I believe, given explicitly by Hartle but is implicit in the desire to produce a consistent probability calculus. In particular, if it is possible to make observations after the CTC epoch, then it is reasonable to require that observations made before the CTC's are consistent with a larger set of observations

that includes measurement after as well as before. I offer the following example to illustrate the construction and its consequences.

It is sufficient to consider a single spin, up or down, whose path integral from $t=0$ to $t=T$ is given by a nonunitary X , e.g., that of Eq. (6). In this example I take the dynamics of the single spin before $t=0$ and after $t=T$ to be trivial, i.e., given by the identity operator; so up and down are degenerate outside $0 \leq t \leq T$. Consider times $t_1 \leq t_2 \leq 0$ and $t_3 > T$. Given the initial condition that the state at t_1 is

$$|+\rangle \equiv \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle),$$

what are the probabilities for \uparrow and \downarrow at t_2 ? Without X , they would both be $1/2$. However, if we consider continuing paths to t_3 and measuring \uparrow versus \downarrow at that time, we would find the probabilities $\mathcal{P}(i,j,k)$, where i,j,k are the spins at t_1, t_2, t_3 , respectively:

$$\mathcal{P}(+, \uparrow, \uparrow) = \epsilon^2 / (1 + \epsilon^2),$$

$$\mathcal{P}(+, \uparrow, \downarrow) = 0,$$

$$\mathcal{P}(+, \downarrow, \uparrow) = 0,$$

$$\mathcal{P}(+, \downarrow, \downarrow) = 1 / (1 + \epsilon^2).$$

These follow from evaluating the path integral with boundary conditions i and k and projecting onto j , squaring, and then renormalizing the sum to one. (X would be unitary were $\epsilon = 1$.) We now require that the probability of spin j at t_2 given spin i at t_1 is given by

$$\mathcal{P}(i,j) = \sum_k \mathcal{P}(i,j,k).$$

Therefore,

$$\mathcal{P}(+, \uparrow) = \epsilon^2 / (1 + \epsilon^2),$$

$$\mathcal{P}(+, \downarrow) = 1 / (1 + \epsilon^2).$$

This is, perhaps, disconcerting, particularly as we could have chosen t_2 close to, or even equal to, t_1 . However, it is not only logically consistent but also in keeping with the original requirement of only allowing consistent world histories [11]. The situation is simplest to describe in words for the extreme case of $\epsilon = 0$. For $\epsilon = 0$, trajectories on which the state \uparrow enters the time machine interfere destructively with each other. Mathematically, we can always decompose the state $+$ into a superposition of \uparrow and \downarrow . However, if we actually perform an \uparrow or \downarrow measurement (i.e., a physical process) on the state $+$, the full world histories for which the outcome of the measurement is \uparrow self-destruct and together carry vanishing probability.

We can now resolve the apparent discontinuity at $\epsilon = 0$. For $\epsilon \neq 0$, if a state approaches $t=0$ as pure \uparrow , it will leave $t=T$ as pure \uparrow with probability one. Yet for $\epsilon = 0$, there is no amplitude for an \uparrow component at $t=T$, irrespective of initial state. However, from the previous discussion, we learned that as $\epsilon \rightarrow 0$, it becomes harder and harder to prepare a pure \uparrow state at $t=0$. Hence, the

physics of $\epsilon=0$ connects smoothly to the behavior as $\epsilon \rightarrow 0$.

Clearly, with this sort of mechanics, suitable experiments before the CTC epoch could determine that CTC's will be in their future. Presumably, with a time machine of finite extent within a continuous space, the strength of this effect would be proportional to the fraction of the experiment's future light cone that intersects the time machine. Hence, the observed linearity of quantum mechanics, although verified in experiments of enormous precision, does not in practice tell us much about future CTC's, except that they are not in our immediate future.

DENSITY MATRIX MECHANICS

Deutsch proposed [2] a profoundly different mechanics that likewise reduces to ordinary quantum mechanics in the absence of CTC's. One considers a density matrix ρ for the entire system that evolves in the standard fashion:

$$i\partial_t \rho = [H, \rho]$$

or

$$\rho(t') = U\rho(t)U^\dagger,$$

where U is given in Eq. (4). So the dynamics is linear in ρ , and Hermiticity of H ensures that ρ remains normalized to $\text{tr}\rho=1$. The CTC's enter in boundary and consistency conditions on ρ . If the degrees of freedom are in two disjoint sets labeled 1 and 2 (in our simple model they are just the two spins) and set 2 refers to the CTC's, then we require

$$\text{tr}_1\rho(0^+) = \text{tr}_1\rho(T^-), \quad (7)$$

where tr_i , $i=1$ or 2 , is the partial trace over the labeled subset. I will return for successive clarifications on the handling of initial conditions, but, roughly speaking, we will do something like, for a given incoming $\rho(0^-)$,

$$\text{tr}_2\rho(0^-) = \text{tr}_2\rho(0^+), \quad (8)$$

$$\text{tr}_1\rho(0^-) = \text{tr}_1\rho(T^+). \quad (9)$$

$\text{tr}_1\rho(0^+)$ is not set by initial conditions but rather by the consistency of information around the CTC, implemented by Eq. (7), which gives as many linear constraints as there are free parameters in $\text{tr}_1\rho(0^+)$. As determined by the evolution equations, the dynamical output is, roughly,

$$\text{tr}_2\rho(T^+) = \text{tr}_2\rho(T^-). \quad (10)$$

The evolution of ρ outside a CTC epoch is identical to ordinary quantum mechanics, and CTC's do not generically impose any *a posteriori* constraint on physics in their past [2]. However, if the density submatrix $\text{tr}_1\rho(T^-)$ is not pure,³ the matching provided by Eq. (7) does not require

the matching of states, amplitudes, or paths around the CTC. In general, pure states will evolve into mixed states, as illustrated in examples that follow.

First consider an initial density matrix $\rho_1(0) \equiv \text{tr}_2\rho(0^-)$ that is pure, and ignore the 2 variable at point z_2 for $t=0^-$. What is $\rho_1(T) \equiv \text{tr}_2\rho(T^-)$ using the time evolution and Eq. (7) for various H 's or U 's?

To proceed, one needs a simple theorem: given a system composed of two subsystems, 1 and 2, the most general total density matrix ρ compatible with $\text{tr}_2\rho$ being pure is $(\text{tr}_2\rho) \otimes \rho_2$ where ρ_2 is the most general density matrix for subsystem 2.

As a first example, consider H and, therefore, U which are diagonal in the basis defined by Eq. (2). The necessary calculation is a simple exercise in four-by-four matrix multiplication, taking partial traces, and solving linear equations. For this particular class of U 's, one finds that the submatrix $\text{tr}_1\rho$ is not fully determined by the matching condition, Eq. (7). (The implied linear conditions are not all linearly independent.) Nevertheless, if the spin at z_1 is initially in the pure \uparrow state, it necessarily emerges at T as pure \uparrow , and pure initial \downarrow emerges as pure \downarrow . For other pure initial states, there exists a solution of Eq. (7) for which they emerge at T unchanged. Hence, quantum coherence is not necessarily destroyed. However, there exist solutions of Eq. (7) for which initial non-trivial linear combinations of \uparrow and \downarrow emerge as mixed states; i.e., $\text{tr}_2\rho(T^-)$ is not pure. Deutsch proposes to resolve such ambiguities by maximizing the entropy. One could just as soon choose a unique solution by minimizing the entropy. Rather than either of these, I would stress that this indeterminacy arises for a set of measure zero in the parameter space of generic H 's. The neighborhood of that set will either provide a unique limiting behavior, or the set will be a separatrix of different qualitative behaviors. In either case, the dynamics would provide a resolution or interpretation of the ambiguity.

With a richer class of interactions, quantum decoherence is a necessary consequence of the proposed density matrix mechanics. A simple explicit example is provided by

$$U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (11)$$

which could arise, say, from H of Eq. (3) with $\omega_0 = \gamma_{1,2} = \lambda = 0$ and $\omega_1 T = \pi/4$, an example with linear equations of motion and unitary evolution according to the path integral approach. Again, the pure states \uparrow and \downarrow evolve into themselves. However, if the spin at z_1 is initially in the pure $+$ state $(|\uparrow\rangle + |\downarrow\rangle)/\sqrt{2}$, its final density matrix is

$$\rho_1(T^+) = \text{tr}_2\rho(T^-) = \begin{pmatrix} 1/2 & 1/2\sqrt{2} \\ 1/2\sqrt{2} & 1/2 \end{pmatrix},$$

which is decidedly mixed.

Turn now to the case of an input density matrix that is mixed, which could arise either because of the initial

³A pure density matrix ρ satisfies $\text{tr}\rho^2=1$ and can be written $\rho = |\Psi\rangle\langle\Psi|$ where $|\Psi\rangle$ is some pure state.

correlations of spins 1 and 2 in a pure state before the CTC or because the total state is mixed. It is here that I will argue for a departure from Deutsch's calculus. To make the comparison clearest, I begin with a system identical to that analyzed in Ref. [2]. Deutsch describes the system as a $1/2$ integer quantum spin carried along a prescribed classical trajectory that intersects itself once as it loops once around a time machine. Hence, the Hilbert space outside the CTC epoch consists of a single spin. The evolution of the two spins upon contact within the CTC epoch is given by some general unitary transformation, which can be thought of as the product of the U matrix discussed above with the unitary matrix

$$V = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

which interchanges 1 and 2. This label switching arises because of Deutsch's parametrizing the classical trajectory by its proper time and then describing the spins as "younger" and "older." The CTC allows the outgoing younger spin to enter as the older one. In contrast, my labels 1 and 2 preserve their meaning around the CTC.

The initial conditions are completely characterized by the $t=0$ density matrix for spin number 1, $\rho_1(0)$. If $\rho_1(0)$ is mixed, then the most general density matrix $\rho(0)$ for the 1,2 system such that $\text{tr}_2\rho(0)=\rho_1(0)$ is no longer of the product form, $\rho_1(0)\otimes\rho_2(0)$, and, in fact, it has many more free parameters. Deutsch argues that we should nevertheless assume a product form and proceed as before. [Following this prescription of assuming a product form, for generic values of the parameters, $\text{tr}_1\rho(0)$ will be determined by the consistency condition, Eq. (7), and the output $\rho_1(T)=\text{tr}_2\rho(T)$ will be unique and, in fact, mixed.] The justification of the assumed restriction on $\rho(0)$ to the product form is the rhetorical, "How could spin 2 be correlated with spin 1 at $t=0$ when they had not yet had a chance to interact?"

Deutsch's construction yields a logically self-consistent dynamics. However, I will now argue in favor of an alternate approach that has the virtue of restoring a traditional ensemble and probability interpretation to the density matrix that was otherwise lost. The rhetorical retort is that at $t=0$ spin 2 had interacted with spin 1 in its past, even if those interactions were not in spin 1's past.

In normal quantum theory, the density matrix can be viewed as describing a statistical ensemble of pure states. One can combine ensembles by adding together their ρ 's, weighted by relative probabilities. Also, one can decompose an ensemble into subsets. This corresponds to dividing ρ into pieces, a process limited only by the restriction that the component ρ 's are themselves legitimate density matrices.⁴ Such composition or decomposition of ensem-

bles commutes with the time evolution because the dynamical equations are linear in ρ . Since the proposed time evolution of ρ is the conventional, linear one, can this ensemble and probability interpretation be preserved?

A natural decomposition of a mixed ρ is to consider it as a statistical, incoherent ensemble of its eigenstates, each with a probability given by its respective eigenvalue. Each individual eigenstate, of course, defines its own pure ρ , and we discussed earlier the density matrix mechanics propagation of a pure initial ρ across a CTC epoch. In that case, the absence of correlation between 1 and 2 was a mathematical necessity, not an additional assumption. If we consider a mixed $\rho_1(0)$ as an ensemble of its pure eigenstates, propagate each of them separately to $t=T$ to obtain a (generically) unique $\rho_1(T)$, and then add those together weighted by the initial probabilities, we get a final answer which is itself a solution of the total evolution and matching conditions. It differs from Deutsch's answer because the total $\rho(0)$ is not of the pure product form $\rho_1(0)\otimes\rho_2(0)$. Instead, it is a particular sum of products. Certainly not the most general $\rho(0)$ satisfying $\text{tr}_2\rho(0)=\rho_1(0)$, its value is, nevertheless, determined by $\rho_1(0)$. It is this $\rho(0)$, and not the pure product form, that allows ρ to be interpreted in terms of an ensemble with additive probabilities. Deutsch's pure product algorithm determines a ρ that encodes the predictions for all possible observations, but the simple linear mathematics representing the physical acts of combining and dividing ensembles is lost.

Return, now, to the system originally defined in this paper, in which the spin at z_2 has an existence outside the CTC epoch, and consider pure initial states of the two-spin system such that $\rho_1(0^-)$ is mixed. Were we to treat $\rho_1(0^-)$ as an initial condition as described in the preceding paragraphs, we would by construction be destroying the coherence of the initial total state. It would be equivalent to replacing the initially pure total $\rho(0)$ by $\text{tr}_2\rho\otimes\text{tr}_1\rho(0)$, which is, typically, mixed. While this may be a tenable proposal for CTC physics, I will pursue here the construction of a dynamics that does not discard information willy-nilly but only loses it when absolutely necessary.

A natural way of keeping track of at least some of the pure initial state 1-2 correlations upon entering the CTC epoch is given by the Schmidt decomposition, which provides the following. A pure state $|\Psi\rangle$ of a system composed of subsystems 1 and 2 can be written

$$|\Psi\rangle = \sum_a c_a |a\rangle_1 \otimes |a\rangle_2,$$

where the set of states $|a\rangle_2$ form an orthonormal basis for the subspace 2 (chosen to be the smaller of spaces 1 and 2, if they have different dimensionality); the set of $|a\rangle_1$ are an equal number of orthonormal states in space 1; and c_a are complex coefficients. (The proof by construction begins by choosing the $|a\rangle_2$ as the eigenvectors of $\text{tr}_1|\Psi\rangle\langle\Psi|$. One then writes $|\Psi\rangle$ in the most general possible form using the $|a\rangle_2$'s and some basis on subspace 1. Finally, one requires that $\text{tr}_1|\Psi\rangle\langle\Psi|$ has the ini-

⁴A density matrix is Hermitian, has unit trace, and has eigenvalues (and hence, diagonal elements) between 0 and 1.

tially assumed form.)

Apply this decomposition to the problem at hand. If $\rho(0^-) = |\Psi\rangle\langle\Psi|$, then

$$\text{tr}_2\rho(0^-) = \sum_a |c_a|^2 |a\rangle_1\langle a|.$$

This density matrix for the spin at z_1 is not in general pure. However, it is here written in diagonal form which can be interpreted, as argued previously, as representing an ensemble of pure states, labeled by a and with probabilities $|c_a|^2$. Recalling the Schmidt decomposition of $|\Psi\rangle$, we know that each $|a\rangle_1$ comes with its particular $|a\rangle_2$. Each pure subensemble $|a\rangle_1\langle a|$ at $t=0$ is mapped by the dynamical evolution to an (in general, mixed) state specified by some $\rho_1^a(T^+)$. So we can take the final total density matrix to be

$$\rho(T^+) = \sum_a |c_a|^2 \rho_1^a(T^+) \otimes |a\rangle_2\langle a|. \quad (12)$$

There is an important lesson implicit in Eq. (12), our prescription for propagating general, pure initial states. While it was always clear that the matching of $\text{tr}_1\rho$ at $t=0^+$ and T^- was a potential source of incoherence (because we deal only with information encoded locally in density matrices), the matching of $\text{tr}_1\rho$ at $t=0^-$ and T^+ likewise destroys coherence. Those initial correlations between the two spins that are encoded as relative phase information in the language of wave functions or in certain off-diagonal elements of the total density matrix are simply lost when crossing the CTC epoch if we insist on using only density matrices, partial traces, and differential time evolution. A simple example is provided by the pure initial state

$$|\Psi\rangle = (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)/\sqrt{2}$$

or

$$\rho(0^-) = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Even under the trivial dynamics of $U=1$ (for which nothing happens to any state), the output $\rho(T^+)$ is mixed. Were we to simply propagate the two partial traces, since each individually is maximally random, the total $\rho(T^+)$ would be maximally random. In contrast, the construction that led to Eq. (12) allows us to preserve the fact that \uparrow comes with \downarrow and vice versa,

$$\rho(T^+) = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

but we have lost the phase information that distinguishes $|\Psi\rangle$ from $(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)/\sqrt{2}$.

Mixed initial states can be decomposed as a sum of pure ensembles, given by the $\rho(0^-)$ eigenvectors and weighted by their eigenvalues. Each pure state can be

propagated as described above and the ensemble reassembled, thus defining a density matrix mechanics which minimizes information loss and preserves the ensemble interpretation.

MIXED STATES IN THE PATH INTEGRAL APPROACH

The evolution of pure states within the path integral approach was specified in our earlier discussion. In particular, if nonunitarity is confined to path integrals over a particular epoch, $0 \leq t \leq T$, then an initial state specified at some $t_0 < 0$ can serve as the initial boundary condition on the path integral. Unitarity within the $t < 0$ epoch ensures that starting the path integral from a yet earlier time and keeping only those paths consistent with the specification at t_0 would yield identical predictions. In contrast, the specification of a mixed initial state and, more generally, many considerations regarding information flow must be altered if the dynamics includes a nonunitary epoch.

If X is the nonunitary operator that maps states at $t=0$ to states at $t=T$, then one might imagine that a density matrix $\rho(0)$ characterizing a mixed state at $t=0$ is mapped to $t=T$ according to

$$\rho(T) = X\rho(0)X^\dagger / \text{tr}X\rho(0)X^\dagger.$$

However, this is untenable. It suffices to consider a two-state system as an example, with X given by Eq. (6). If the system at $t=0$ has equal probabilities to be \uparrow and to be \downarrow , then the system evolves into the statistically identical mixed state at $t=T$ because X maps \uparrow onto \uparrow and \downarrow onto \downarrow . However, if the system is equally likely to be $+$ or $-$ (with $|\pm\rangle = (|\uparrow\rangle \pm |\downarrow\rangle)/\sqrt{2}$) at $t=0$, then the system at $t=T$ is nearly pure \downarrow . These two ensembles have the same density matrix at $t=0$, but differ at $t=T$. In contrast with normal quantum mechanics, knowing the density matrix at one time (and, therefore, the mean value of all observables at that time) does not determine the density matrix at a future time, even if the evolution of all pure states is known. Hence, a statistical ensemble is not completely characterized by its density matrix but rather must be specified in terms of probabilities for particular (not necessarily orthogonal) states.

In light of these considerations, with nonunitary evolution the traditional entropy

$$\sigma \equiv -\text{tr}\rho \log\rho$$

cannot be an adequate measure of disorder or lack of information because ρ does not characterize the statistical state of the system. Rather than attempt to propose an adequate alternative, I comment here on two antithetical perspectives, again posed in the explicit context of the sample provided by Eq. (6).

The nonunitary X of Eq. (6) maps nearly all pure states onto states that are nearly pure \downarrow . Does this imply that the information in a typical situation increases in going from $t=0$ to $t=T$? The answer depends on what is a suitable measure (characterizing uniform distribution of probability) in the space of states. Were the states actually half-integer intrinsic angular momenta, then the Haar measure on the rotation group would provide a natural

measure in the spin space. The situation is not so clear if we are dealing with occupied and unoccupied modes of a Fermi field. "Totally random" does not necessarily mean equal probabilities for occupied and unoccupied. "Randomness" in a physical system must be defined relative to some physical process or situation, e.g., high temperature. Furthermore, given any measure at $t=0$, X itself defines a new measure at $t=T$ (dynamical transporting the $t=0$ measure) such that X is volume preserving or, equivalently, information preserving.

On the other hand, one may be concerned that X typically destroys information in the following sense: if a message is encoded in a particular state (or sequence of states) before $t=0$, what arrives at $t=T$ is nearly independent of the information encoded. What started out as orthogonal alternatives are now nearly identical, and unambiguous reading of the message seems hopeless. However, the key word here is "nearly," because if CTC's are, indeed, possible, the "arbitrarily technologically advanced civilization" [11] could construct a 100% efficient message decoder (essentially containing a version of X^{-1}). Hence, X can again be viewed as information preserving.

CONCLUSIONS

The initial motivation for studying possible dynamics in the presence of CTC's was to search for irresolvable paradoxes which might preclude the existence of any consistent dynamics and, hence, of CTC's themselves. Yet, rather than identifying some fatal, irresolvable contradiction, we encountered a proliferation of alternative dynamical schemes, each reducing to ordinary quantum mechanics in the absence of CTC's.

The method used here was to consider a very simple system that could be solved explicitly but was rich enough to provide examples of interesting phenomena and had a structure that would make the generalization to more sophisticated systems fairly self-evident. The two main directions explored here had been suggested previously. One uses a path integral and has coherent states without a Schrödinger equation. The other uses density matrices and has Hamiltonian evolution without coherent states. In each case, the CTC's produce bizarre (and different) phenomena, even outside the CTC epoch.

There are certainly potentially interesting questions

that one cannot address within models as simple as those analyzed here. In a spacetime with a nonuniform metric there can be particle production. What now happens to pure initial states, e.g., the ground state, in the various schemes? What sort of conservation laws survive in the presence of CTC's?

The dynamical systems studied here may provide helpful examples regarding issues of complexity. Many discussions of complexity in quantum mechanics focus on features that can emerge only in large dimensional vector spaces, such as the statistics of level spacings. However, in the above discussion we have examples of four-by-four matrix Hamiltonians that fall into qualitatively different dynamical classes. What, in a small matrix, can distinguish simple from complex behavior? In the path integral investigation, the aspect that distinguished unitary from nonunitary (with its decidedly eerie consequences) was linearity of the equations of motion. In the density matrix formulation, incoherence arose because of a clash between the eigenstates of the Hamiltonian and the basis for the Hilbert space that was natural relative to the spacetime. To handle the CTC, which treated spatial points z_1 and z_2 differently, it was convenient to use a Hilbert space basis that factored into products of the spin at z_1 with the spin at z_2 . If we are allowed to consider (as in this example) the spacetime and the concept of locality as specified independently of the dynamics, then the extent to which the dynamics respects that locality is a measure of its simplicity. If spacetime is itself part of the dynamics (as it must ultimately be), then the only useful analogous notion of simplicity may be relative to common-versus-uncommon situations and phenomena—likely a subjective criterion.

Of course, the real desideratum is the design of a time machine, presumably an amalgam of relativistic, astronomical bodies to produce large curvature and virtually zero temperature to ensure quantum coherence and allow regions of negative energy density.

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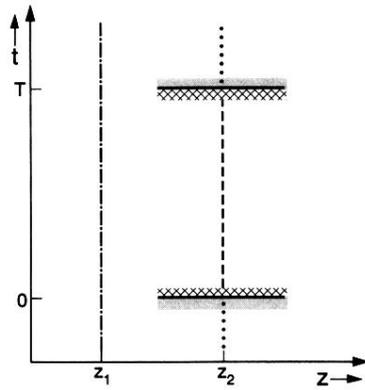


FIG. 1. A class of CTC spacetimes is defined by identifying the like-shaded time boundaries of two spatial regions at times $t=0$ and T . Further simplification comes from restricting the space to two points, z_1 and z_2 , whose respective world histories (broken lines) are connected and disconnected.