NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL NOTE 2473

ON THE SPECTRUM OF ISOTROPIC TURBULENCE

By H. W. Liepmann, J. Laufer, and Kate Liepmann

California Institute of Technology

Washington
November 1951

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SUMMARY

Measurements of the spectrum and correlation functions at large Reynolds number $\left(RN \approx 10^5 \text{ based on the grid mesh}\right)$ have been made, as well as a series of accurate spectrum measurements at lower Reynolds number $\left(RN \approx 10^4\right)$.

The results are compared with the theoretical laws proposed in recent years. It is found that the measurements at large Reynolds numbers exhibit a range of frequencies where the spectrum is nearly of the form $n^{-5/3}$.

The largest part of the spectrum in the initial stage of decay at the lower Reynolds number was found to follow closely the simple spectrum $\frac{A}{B + n^2}$, where A and B are constants and n is the frequency of fluctuation. At $x/M = 1000$ (where x is the distance behind the the grid and M is the mesh size) the spectrum approaches a Gaussian distribution.

The second, fourth, and sixth moments of the spectrum have been computed from the measurements and are discussed in relation to theoretical results.

The significance of the number of zeros of the fluctuating velocity $u(t)$ is discussed and examples of measurements for the determination of the microscale of turbulence λ from zero counts are given.

INTRODUCTION

A field of turbulent fluctuations represents a dissipative system. Viscosity will tend to dissipate the energy of the fluctuations into heat and, if a stationary state of the system is to be kept, energy must be continuously supplied to the fluctuations. It is characteristic

of fluid flow phenomena that the rate of dissipation is strongly influenced by the inertia terms in the equation of motion. These convective, nonlinear terms do not describe any mechanism for the production of heat from the kinetic energy of the fluctuating field and their influcence upon the rate of dissipation is therefore indirect. The rate of dissipation is proportional to the mean square of the vorticity. The nonlinear inertia of the fluid motion usually tends to increase this mean square. A similar mechanism is well-known in fluid mechanics in the shock-wave formation where the tendency to steepen the shock front is balanced by the tendency to diffuse it because of the viscosity and heat conduction of the fluid.

The understanding and analytical formulation of the nonlinear effects upon the vorticity distribution is a central problem in turbulence research. In the search for an understanding of this mechanism Taylor's introduction of the concept of isotropic turbulence was very important (reference 1). Isotropic turbulence represents a much simpler type of turbulent motion than the general shear flow problem but does include many characteristic features of the general problem. Most of the recent progress in turbulence research came from investigations of isotropic turbulence. Kármán (reference 2) introduced the convenient formalism of the correlation tensors of different rank. Kármán and Howarth (reference 3) then gave the first equation in a suitable form relating the double correlation function to the triple correlation function. This equation shows the balance between inertia and viscous forces in a turbulent field. The triple correlation represents the nonlinear terms of the equation of motion. The Kármán-Howarth equation is clearly indeterminate as long as one does not have an additional analytical representation, in terms of the double correlation functions, of this fundamental process of vortex regrouping.

The theoretical research of the last few years has made some progress at least toward an understanding of isotropic turbulence. Kolmogoroff (reference 4) introduced the concept of local isotropy into turbulence, that is, the hypothesis that the motion of the smaller eddies in turbulent flow is always isotropic. This hypothesis, for which there is some experimental verification at present, makes an investigation of isotropic turbulence even more interesting. Kolmogoroff then proceeded, essentially on the basis of dimensional analysis and very simple physical reasoning, to arrive at some results concerning the correlation function and so forth. A thorough review of Kolmogoroff's work has been given by Batchelor (reference 5). Onsager (reference 6), Weizsäcker (reference 7), and especially Heisenberg (reference 8) attacked the problem from the point of view of the turbulent spectrum rather than the correlation function. Taylor had introduced the concept of the spectrum into turbulence as early as 1938 (reference 9). However, the one-dimensional form of the spectrum as given by Taylor is very convenient

for experimental measurements but not too convenient for theoretical study. Heisenberg therefore introduced a three-dimensional space spectrum function $E(k)$. The function $E(k)\,dk$ is defined as the fraction of the total turbulent energy with wave number between k and $k + dk$. Taylor's spectrum $\tilde{F}(k_1)\,dk_1$ is defined as the fraction of turbulent energy with components of the wave-number vector k in one fixed direction between k_1 and $k_1 + dk_1$. Heisenberg's and Taylor's spectrum functions are Fourier transforms of the correlation functions of Kármán Corresponding to the Kármán-Howarth equation, a similar equation for $E(k)$ may be written which obviously does not yield anything different from the equations for the correlation function. However, it is possible that the concept of spectrum is somewhat more intuitive than that of correlation function - probably because spectrum is a more familiar concept from other fields of physics - and most recent work operates essentially with the concept of spectrum. Knowledge of the spectrum function $E(k)$ is sufficient to describe the turbulent field completely. A rather complete survey of this group of recent theoretical investigations and the experimental evidence can be found in Batchelor's lecture to the Seventh International Congress for Applied Mechanics (reference 10). A brief discussion of some theoretical results for later reference is included in the section "General Considerations" of this report.

The general aim of the present experimental investigation is a study of the nonlinear exchange mechanism. There is not at present any straightforward and simple way to measure the exchange terms directly. In fact, the research has to start on a more basic level. A fairly broad investigation is needed to make sure that the flow conditions are close to those assumed by theory and that the experimental equipment is satisfactory. Finally, a rather thorough knowledge of various aspects of the turbulent field is needed before experimental results can be evaluated intelligently.

In the present work the following points were investigated:

(1) The correlation function and the spectrum at large Reynolds numbers

(2) The decay, microscale, and spectrum at intermediate and low Reynolds numbers

(3) The zeros of the fluctuating velocity components and their relation to the spectrum and moments of the spectrum

Part of these measurements can serve as an independent check of results obtained by Batchelor and Townsend and others, and part of the results are new. It should be emphasized here that the ultimate aim of

a more or less direct measurement of the exchange term has not been carried out yet and, in fact, was not even attempted. It was felt necessary first to improve and refine the methods, especially the measurement of the spectrum, and, as mentioned before, to gain a view of many aspects of the turbulent field.

The investigations were carried out at the Guggenheim Aeronautical Laboratory, California Institute of Technology, as part of the turbulence research conducted under the sponsorship and with the financial assistance of the National Advisory Committee for Aeronautics. The authors would like to acknowledge the cooperation of Mr. F. K. Chuang and the stimulating discussions with Professors Lagerstrom and DePrima. Mr. M. Jessey was responsible for the design and construction of most of the electronic equipment and this essential help is gratefully acknowledged.

SYMBOLS

u_i	turbulent velocity fluctuation $(i = 1, 2, 3)$
x_k	space coordinate $(k = 1, 2, 3)$
U	mean velocity
$\overline{u^2}$	mean square of velocity components
$R_i{}^j$	Kármán's double correlation tensor
$\Gamma_i{}^j$	Batchelor's spectral tensor
$\underline{k}$	wave-number vector
k_i	components of wave-number vector
n	frequency of fluctuation
$E(k)$	Heisenberg's spectrum density
$\tilde{F}(k_1)$	Taylor's spectrum density in space
$F(n)$	Taylor's spectrum density in time
$\sigma \equiv \frac{F(n)}{F(0)}$	dimensionless spectrum density parameter
$\zeta \equiv \frac{\pi}{2} F(0)n$	dimensionless frequency parameter

$f(r)$, $g(r)$	double correlation functions
$h(r)$	triple correlation functions
L	scale of turbulence
λ	microscale of turbulence
N_ξ	average number of "ξ values" per unit time of a random function
μ	viscosity
ν	kinematic viscosity
$G \equiv \lambda^4 f^{iv}(0)$	dimensionless parameter
$\xi_k = x_k' - x_k = \underline{r}$	distance between two points in field
ξ	value of random variable $u(t)$
η	value of random variable du/dt
ρ	density
p'	pressure fluctuation
θ, φ	spherical coordinates in k-space
$R_y = f(r)$	for case where r is measured in direction normal to u_1
ϵ	rate of turbulent energy dissipation
$P(N)$	probability distribution of N_ξ

GENERAL CONSIDERATIONS

Definition of Terms

Homogeneity, isotropy, and stationary state.- A field of turbulent fluctuations in space is given by the velocity $u = \left[u_i\left(x_j,t\right)\right]$. The field is called homogeneous if all mean values of u and their

derivatives do not depend on the value of x_j. It is called isotropic if these mean values are independent of reflection and rotation of the coordinate system. In most measurements of turbulence a velocity component u_i is observed as a function of time at a fixed position. For example, measurements are made in a wind tunnel at some distance behind the grid. In this case the process is called stationary; mean values of the square of $u(t)$ and its derivatives do not depend upon time and the correlation functions depend only upon the relative time interval. In the definition here one has to use the ensemble average to take a mean value. If the ordinary time average is used, the statement that the mean values do not depend on time becomes trivial. A stationary process in time is analogous in definition to homogeneity in space. Isotropy has no such counterpart since time is a single coordinate.

Time and space derivatives.- It is typical of present turbulence research that measurements are often made in a moving stream at a fixed position. On the other hand, theoretical considerations often deal with space distribution of a turbulent field at rest. To relate experiments and theory one uses the fact that the space coordinate in the direction of the mean motion is timelike. This fact is sometimes expressed by stating that the operator $\partial/\partial t$ can be replaced by $-U\,\partial/\partial x$ if U is the mean velocity in the direction x.

This statement is obviously never correct rigorously. In certain applications the difference, that is, the error, can be given or estimated. For example, for any stationary process $u(t)$ which possesses a derivative

$$\overline{u\,\frac{du}{dt}} = 0 \tag{1}$$

applying the exchange of operators simply gives

$$\overline{u\,\frac{\partial u}{\partial t}} = -U\overline{u\,\frac{\partial u}{\partial x}} = -U\,\frac{\partial \overline{u^2/2}}{\partial x}$$

But

$$-U\,\frac{\partial \overline{u^2}}{\partial x} \neq 0$$

since this term determines the decay of turbulence and for isotropic turbulence has the value

$$-U\,\frac{\partial \overline{u^2}}{\partial x} = 10\nu\frac{\overline{u^2}}{\lambda^2}$$

It is possible, however, to use the interchange between time and space variables in certain mean values. For example, it has been found experimentally by Townsend (reference 11), in agreement with the measurements presented here, that the mean square of the time derivative agrees closely with the mean square of the space derivative in the direction of mean motion; that is,

$$\overline{\left(\frac{\partial u}{\partial t}\right)^2} \approx \overline{U^2\left(\frac{\partial u}{\partial x}\right)^2} \tag{2}$$

To estimate here the degree of approximation within which this result is true is more complicated. One can write down the equations of motion and compute these terms and then try to estimate the difference; for example, for

$$\frac{\partial u_i}{\partial t} + U\frac{\partial u_i}{\partial x_1} + u_j\frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p'}{\partial x_i} + \nu\nabla^2 u_i$$

Hence,

$$\overline{\left(\frac{\partial u_i}{\partial t}\right)^2} = U^2\overline{\left(\frac{\partial u_i}{\partial x_1}\right)^2} + R$$

with

$$R = \overline{\left(U\frac{\partial u_i}{\partial x_1} + u_j\frac{\partial u_i}{\partial x_j} + \frac{1}{\rho}\frac{\partial p'}{\partial x_i} - \nu\nabla^2 u_i\right)^2} - U^2\overline{\left(\frac{\partial u_i}{\partial x_1}\right)^2}$$

The largest term in R appears to be

$$\overline{U\frac{\partial u_i}{\partial x_1}\cdot u_j\frac{\partial u_i}{\partial x_j}} \approx U\sqrt{\overline{u^2}}\,\overline{\left(\frac{\partial u_i}{\partial x_1}\right)^2}$$

Neglecting this term compared with $U^2\overline{\left(\frac{\partial u_i}{\partial x_1}\right)^2}$ hence would amount to an error of the order of the turbulence level $\sqrt{\overline{u^2}}/U$.

Mean values.- So far mean values have been used without any definition of what is meant by putting a bar over certain quantities. As a

matter of fact, the definition and use of mean values in turbulence is not an easy problem. Experimentally, the average is, in general, the time average provided by the response of a measuring instrument, for example, a thermo-cross-galvanometer combination. This time average is equivalent to $\frac{1}{\theta}\int_0^{\theta} ()\, dt$ where t is time and θ is the time interval for the averaging. For theoretical considerations an ensemble average is often more convenient. Experimentally, an ensemble average can be obtained by averaging over single measurements separated by time intervals sufficiently large to have zero time correlation. If the time θ in the averaging process is large compared with the correlation time scale T, then one should expect the time and ensemble averages to agree.

The finite extent of the measuring probe, that is, specifically, the length of the hot-wire, introduces an additional average over space in turbulence measurements. If the length of the hot-wire is not too great this can be accounted for by making corrections based on a rough estimate of the space characteristics of the turbulent field.

Correlation functions, spectrum.- The space distribution of a homogeneous turbulent field is characterized by giving the space correlations in the form of the well-known Kármán correlation tensors,

$$\left.\begin{aligned} R_i{}^j &= \frac{\overline{u_i(x_k,t)u_j(x_k',t)}}{\overline{u^2}} \\ \underline{r} &= \xi_k \end{aligned}\right\} \tag{3}$$

The space spectrum is then most conveniently defined as the Fourier transform of $R_i{}^j$ (references 12 and 13). Let $\underline{k}$ denote the wave-number vector, that is, the vector in the direction of wave propagation with absolute magnitude $k = |\underline{k}| = 2\pi/\Lambda$, Λ being the wave length. Then a spectral tensor $\Gamma_i{}^j$ can be defined by

$$\Gamma_i{}^j = \frac{1}{8\pi^3}\iiint_{-\infty}^{\infty} R_i{}^j(\underline{r})e^{i(\underline{k}\cdot\underline{r})}\, d\tau(\underline{r}) \tag{4}$$

Heisenberg's spectrum $E(k)$, that is, the energy density of turbulent fluctuations contained between k and $k + dk$, is then obtained as the mean value

$$E(k) = \frac{1}{2\pi} \int_0^{\pi} \int_0^{2\pi} \Gamma_i^{\,i}(k) k^2 \sin\theta \, d\theta \, d\varphi \tag{5}$$

For isotropic turbulence $R_i^{\,j}$ and $\Gamma_i^{\,j}$ are each expressible by a single function $f(r)$ and $E(k)$, respectively.

The spectrum function $F(n)$ introduced by Taylor was originally defined in time frequency n rather than in space. The relation between $F(n)$ and the Heisenberg spectrum thus involves an interchange between time and space variables. The equivalent definition of the Taylor spectrum in space $\tilde{F}(k_1)$ is the fraction of turbulent energy contained between k_1 and $k_1 + dk_1$ where k_1 denotes the component of k in a fixed direction. The direction is, in general, chosen to correspond to the direction of mean speed U in the experimental setup.

There exists a simple relation between $\tilde{F}$, the Taylor spectrum function, and E, Heisenberg's spectrum function, which is valid for homogeneous, isotropic turbulence,

$$\left.\begin{aligned} E(k) &= \frac{1}{3}\left[k^2\tilde{F}''(k) - k\tilde{F}'(k)\right] \\ \int_0^{\infty} E(k)\, dk &= 1 \end{aligned}\right\} \tag{6}$$

where from equation (5)

$$\left.\begin{aligned} \tilde{F}(k_1) &= \frac{2}{\pi} \int_0^{\infty} f(r) \cos k_1 r \, dr \\ f(r) &= \int_0^{\infty} \tilde{F}(k_1) \cos k_1 r \, dk_1 \end{aligned}\right\} \tag{7}$$

If one considers again the stationary time process, then one can define the corresponding time correlation and time spectrum as is standard practice. If $u(t)$ is a certain velocity component measured at one space position in the course of time, then one can define

$$\psi(\tau) = \frac{\overline{u(t + \tau)u(t)}}{\overline{u^2}} \tag{8}$$

where τ denotes a time interval. The corresponding spectrum $F(n)$ is then defined as the fraction of $\overline{u^2}$ contained in the frequency band n, $n + dn$, that is, the so-called "power spectrum." One has then:

$$F(n) = 4 \int_0^\infty \psi(\tau) \cos 2\pi n\tau \, d\tau \tag{9}$$

$$\psi(\tau) = \int_0^\infty F(n) \cos 2\pi n\tau \, dn \tag{10}$$

From equations (7) and (10), respectively, follow then the relations between spectrum, correlation function, and mean derivatives of the fluctuation which are so often used in turbulence. Specifically, the following quantities will be used:

(1) The space scale L and time scale T of turbulence defined by:

$$\begin{aligned} L &= \int_0^\infty f(r) \, dr \\ &= \frac{\pi}{2} \tilde{F}(0) \end{aligned} \tag{11}$$

$$\begin{aligned} T &= \int_0^\infty \psi(\tau) \, d\tau \\ &= \frac{F(0)}{4} \end{aligned} \tag{12}$$

(2) The so-called microscale λ or the corresponding mean-square frequency $\overline{n^2}$ defined by

$$\frac{1}{\lambda^2} \equiv \overline{k_1^2}$$

$$= \int_0^\infty k_1^2 \tilde{F}(k_1)\, dk_1 \tag{13}$$

$$\overline{n^2} = \int_0^\infty n^2 F(n)\, dn \tag{14}$$

From equations (7) and (10), clearly:

$$\overline{k_1^2} = -f''(0)$$

$$= \frac{1}{\overline{u^2}}\overline{\left(\frac{\partial u}{\partial x}\right)^2} \tag{15}$$

$$4\pi^2\overline{n^2} = -\psi''(0)$$

$$= \frac{1}{\overline{u^2}}\overline{\left(\frac{\partial u}{\partial t}\right)^2} \tag{16}$$

Triple correlations.- For a description of the dynamics of turbulence one needs also the triple correlation functions which are intimately connected with the nonlinear terms in the equations of motion. For isotropic, homogeneous turbulence the triple correlation tensor of Kármán can be expressed again by a single function, for example, $h(r)$ where

$$\left.\begin{aligned} h(r) &= \frac{\overline{u_2(x_i,t)u_2(x_i,t)u_1(x_1',t)}}{\left(\overline{u^2}\right)^{3/2}} \\ x_1' &= x_1 + \xi_1 \\ x_i' &= x_i \qquad\qquad i = 2,3 \end{aligned}\right\} \tag{17}$$

Functions equivalent to the spectral densities can be introduced here as was first done by Lin (reference 13). For example, one can define a function $H_1(k_1)$ equivalent to the Taylor spectrum for the double correlations and write

$$k_1 H_1(k_1) = \frac{2\left(\overline{u^2}\right)^{3/2}}{\pi} \int_0^\infty h(r) \sin k_1 r \, dr \tag{18}$$

Similar relations in time can also be written.

Equations of Motion

The equation for the correlation function in homogeneous, isotropic turbulence is the Kármán-Howarth equation which is, of course, a consequence of the Navier-Stokes equations:

$$\frac{\partial \overline{u^2} f}{\partial t} + 2\left(\overline{u^2}\right)^{3/2}\left(\frac{\partial h}{\partial r} + \frac{4h}{r}\right) = 2\nu\overline{u^2}\left(\frac{\partial^2 f}{\partial r^2} + \frac{4}{r}\frac{\partial f}{\partial r}\right) \tag{19}$$

where f and h are the double and triple correlation functions defined above.

The corresponding equation in the wave-number space has been given by Lin (reference 14):

$$\frac{\partial E(k)}{\partial t} + W(k,t) = -2\nu k^2 E(k) \tag{20}$$

with

$$W(k,t) = \frac{2}{3}k^2\left[k^2 H_1''(k) - kH_1'(k)\right]$$

A thorough discussion of these equations is given by Kármán and Lin (reference 15).

Form of Spectrum Functions

The aim of a theory of isotropic turbulence is the determination of the function $f(r)$ or $E(k)$. Earlier attempts to obtain $f(r)$ are

found, for example, in the Kármán and Howarth paper (reference 3). In recent years it has become more common to operate with the spectrum function rather than with the correlation function.

In the present state of theoretical analysis of $E(k)$ or $f(r)$ it is best to distinguish two steps. First, one can arrive at some conclusions regarding the general form of E or f in certain regions without making any explicit statement about the nonlinear term $W(k,t)$ or $h(r)$. Second, one can arrive at conclusions by assuming a specific form of W or h.

The first set of theoretical results includes Kolmogoroff's theory of locally isotropic turbulence and the results obtained concerning E and f, especially the $k^{-5/3}$ law for $E(k)$. This has also been obtained independently by Weizsäcker (reference 7) and Onsager (reference 6). The set includes also the k^4 law for small wave numbers obtained by Lin (reference 14), Batchelor (reference 12), and others and, further, Loitsianskii's (reference 16) result concerning the invariance of $\overline{u^2}\int_0^\infty r^4 f(r)\,dr$ during decay. Finally, the behavior of the turbulent field at very low Reynolds numbers as already discussed by Kármán and Howarth, Loitsianskii, and recently again by Batchelor and Townsend (reference 17) belongs to this group.

On the other hand, the k^{-7} law of Heisenberg (reference 8) for large wave numbers and all relations obtained for medium wave numbers depend on assumed forms of the exchange term $W(k,t)$, $h(r,t)$, or corresponding terms. Kármán (reference 18) has recently systematized these assumptions. An analysis of all these various laws and assumptions can be found in Batchelor's lecture to the Seventh International Congress for Applied Mechanics (reference 10) and in the recent paper by Karman and Lin (reference 15).

From the point of view of experiment these two goups of results differ essentially. Verification of results of the first group mainly means a verification of rather broad and essentially kinematic assumptions. For example, the theory has until now always been based on the consideration of an unlimited turbulent field with zero mean velocity. Experiments are carried out with turbulence which is produced by a specific mechanism and which is certainly somewhat inhomogeneous turbulence. Furthermore, the maximum length or minimum wave number in an experimental setup is limited. The range or validity in Reynolds number, for example, for the $k^{-5/3}$ law, is not easily predictable and, similarly, the range of wave numbers for the k^4 law.

Consequently, the aim of experiments here is to establish the applicability of the theory. Formally, there is little doubt that these laws apply in the assumed kinematical model since the derivation from the equations does not involve any additional assumptions.

The aim of experiments for the verification of the second group of results goes deeper because, in addition to the applicability of the theories to the experimental setup, the physical mechanism of exchange is involved here. It is consequently impossible to work on the second group of problems fruitfully without having gone to a certain extent through the first.

A further question to the experimenter is the consistency of measurements. Quite often the theoretically defined quantity cannot be measured directly. Finally, the sensitivity of the form of the measured quantity, as, for example, $E(k)$, toward the assumptions made is important. This often makes a decision in favor of a certain specific assumption very dubious.

These simple considerations are included here to justify the approach to the problem. Even if the ultimate aim is quite clearly recognized it is not always possible to direct the experiments immediately toward this aim since any result will be ambiguous. Clearly, a direct experimental approach to the turbulent exchange problem is most desired but this approach does not seem very fruitful before the background is somewhat cleared up.

Remarks on Turbulent Fluctuations as Stationary Random Processes

If a turbulent velocity component u is measured at a fixed position as function of time, $u(t)$ denotes a stationary random process. Processes of this general type have been considered in physics and mathematics before and it is interesting to see how turbulence fits into these known processes. Brownian motion and the shot effect are examples of processes of this type. Often the term "random noise" is applied to processes of this kind. If one measures the turbulence spectrum in the standard fashion by analyzing the output of a single stationary hot-wire, or if one measures the time derivative in the same way by differentiating the output current of a single hot-wire, and so forth, in all these cases one analyzes a stationary one-dimensional random process and application of these results to the three-dimensional space field of turbulence requires additional steps, such as the interchange of space and time variables discussed above.

Before the results of measurements are given in this report it appears useful to discuss briefly two aspects of the problem of

stationary random process which are used in the measurements, namely, the power spectrum and the number of zeros of $u(t)$ per unit time.

Power spectrum.- If $u(t)$ denotes one turbulent velocity component, not necessarily the one in the direction of mean motion, one can define a power spectrum $F(n)$ as discussed in the section "Definition of Terms." The necessary conditions underlying the concept can be found in the literature, for example, in references 19 and 20.

Dryden (reference 21), who carried out one of the earliest measurements of the turbulent spectrum, found that the measurements of $F(n)$ could be represented rather well by the function

$$F(n) = \frac{F(0)}{1 + c^2n^2} \tag{21}$$

where c and $F(0)$ are characteristic constants related by the normalization of $F(n)$. The corresponding time correlation $\psi(\tau)$ is then

$$\psi(\tau) = e^{-|\tau|/T} \tag{22}$$

Clearly, a function of this form cannot represent the spectrum for arbitrary high values of n since then

$$\int_0^\infty n^2F(n)\ dn = \infty$$

Thus the dissipation of turbulence - if for the moment one takes the relation $\overline{\left(\frac{\partial u}{\partial t}\right)^2} \approx \overline{\left(\frac{\partial u}{\partial x}\right)^2}$ for granted - becomes infinite. Indeed, Dryden's measurements showed that for large values of n the measured spectrum falls below the values given by equation (21).

The measurements of $F(n)$ made during the course of the present investigations are accurate enough to obtain

$$\overline{n^2} = \int_0^\infty n^2F(n)\ dn \qquad \text{the second moment of } F$$

as well as

$$\overline{n^4} = \int_0^\infty n^4 F(n)\, dn \qquad \text{the fourth moment of } F$$

both of which are found finite - as expected - and in general agreement with measurements of Batchelor and Townsend of the mean square of the first and second derivatives of $u(t)$. However, in agreement with Dryden it was found that by far the largest part of the turbulent energy is contained in a portion of the spectrum $F(n)$ for which the simple form of equation (21) gives a good approximation. Hence, one may take at present this form of $F(n)$ as, at least, an empirical interpolation formula from experiments and one may inquire into the general nature of such a spectrum. Without implying too much about the physical significance of the simple spectrum function the following properties are interesting and possibly significant. Dryden noted that this spectrum plays a specific role in random processes. Today one can formulate the significance of this spectrum as follows:

(1) If the probability distribution of $u(t)$ is Gaussian - as is known with good accuracy from experiments of Simmons and Salter (reference 19) and Townsend (reference 11) - and the spectrum has the form $\frac{A}{B + n^2}$, then the process is Markoffian; that is, it is the simplest random process for a continuous variable (see reference 20).

(2) Processes leading to a spectrum of this type are well-known and fairly common. For example, the so-called random telegraph signal, that is, a variable which attains the values a and $-a$ jumping from the one value to the other at random times, has this spectrum. The number of changes of sign is then given by a Poisson distribution.

Hence, there may be a possibility of relating some part of the turbulence to the random shedding of vortices by the grid. This appears worth a further investigation. Furthermore, it is interesting to note that - assuming again that the exchange of time and space variables is closely valid - the simple spectrum satisfies the kinematic requirement for low wave numbers. The corresponding Heisenberg spectrum behaves like k^4 for $k \rightarrow 0$. Indeed, it follows simply from equations (6) that the Heisenberg spectrum $E(k)$ corresponding to $F(n) = \frac{F(0)}{1 + c^2 n^2}$ has the form

$$E(k) \approx \frac{L^5 k^4}{\left(1 + L^2 k^2\right)^3}$$

where L is the scale of turbulence and hence $E(k) = L^5k^4$ as $k \rightarrow 0$. Furthermore, for large values of k, that is, $kL \gg 1$

$$E(k) \approx \frac{L}{(Lk)^2}$$

$$\approx \frac{k^{-6/3}}{L}$$

which makes it experimentally very difficult to distinguish this spectrum behavior from the $k^{-5/3}$ law.

It is not intended here to overemphasize the importance or physical significance of the simple spectrum correlation functions. It is only pointed out that the simple spectrum is an excellent interpolation formula with possible physical significance which has to be studied further and that one should be quite careful in the experimental verification of power laws such as the $k^{-5/3}$ law. Using the simple spectrum Loitsianskii's invariant becomes simply proportional to

$$\overline{u^2}\left[UF(0)\right]^5 \approx \overline{u^2}(UL)^5 \tag{23}$$

a form which can be experimentally determined. Results are given later in this report.

Number of zeros of $u(t)$.- Another rather interesting problem arising in stationary random processes is the expected number of passages through zero (or any other constant value) per unit time. Questions of this nature have been discussed in recent years, for example, by Rice (reference 22).[1]

Let $p(\xi,\eta)\, d\xi\, d\eta$ denote the probability at a given time of finding $u(t)$ between the values ξ and $\xi + d\xi$ and du/dt between η and $\eta + d\eta$. Then $p(\xi,\eta)\, d\xi\, d\eta$ denotes the time the trace spends in the interval ξ, $\xi + d\xi$ having a derivative between η and $\eta + d\eta$. To obtain the number of passages through the interval $d\xi$, regardless of the time spent at a crossing, $p(\xi,\eta)\, d\xi\, d\eta$ has to be divided by the

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[1]Further references to other investigations into the subject, especially the work of Kac, are given in Rice's paper.

time spent in the interval at a crossing to reduce all passages to the same statistical weight. Let τ be this time; then clearly

$$\tau = \frac{d\xi}{|\eta|}$$

since η is the "velocity" in crossing the interval $d\xi$ and $|\eta|$ has to be used since τ does not depend on the sign of η. Hence, the number of passages through ξ, $\xi + d\xi$ with a specific value of the derivative η is given by

$$\frac{p(\xi,\eta)\ d\xi\ d\eta}{\tau} = |\eta| p(\xi,\eta)\ d\eta$$

and, consequently, the total number of passages through the interval regardless of η, by

$$N_\xi = \int_{-\infty}^{\infty} |\eta| p(\xi,\eta)\ d\eta \tag{24}$$

This is the formula given by Rice in reference 22 where a more rigorous proof will be found.

So far the assumptions underlying N_ξ are very general. To obtain a simple relation between the spectrum and N_ξ additional assumptions are necessary. These assumptions are justified for linear processes, such as electrical noise problems and so forth, but certainly not rigorously true for turbulence. Still it is of considerable interest to study the effect of possible deviation from these assumptions on N_ξ. Since $u(t)$ is a stationary random process

$$\overline{u \frac{du}{dt}} = 0$$

that is, there is no correlation between u and du/dt. If both u and du/dt have Gaussian distributions, then it is directly follows that

$$N_\xi = \frac{e^{-\frac{\xi^2}{2\overline{u^2}}}}{\pi\left(\overline{u^2}\right)^{1/2}} \sqrt{\overline{\left(\frac{du}{dt}\right)^2}} \tag{25}$$

Specifically, the number of zeros N_0 is given by

$$N_0{}^2 = \frac{1}{\pi^2}\frac{1}{\overline{u^2}}\overline{\left(\frac{du}{dt}\right)^2}$$

$$= 4\int_0^\infty n^2F(n)\ dn$$

$$= 4\overline{n^2} \tag{26}$$

In this form formula N_0 is quite similar to the harmonic value for which the number of zeros N_0 is given by

$$N_0 = 2\bar{n}$$

The distribution of du/dt is, however, in general, not Gaussian. This fact is essentially a consequence of the nonlinearity of the equations of motion and was verified experimentally by Townsend (reference 11). Still, it is interesting to investigate the deviation of N_0 from the mean-square frequency as obtained by other means, for example, from $F(n)$ or $\overline{\left(\frac{du}{dt}\right)^2}$, and to extend this comparison to Reynolds numbers so low that the influence of the nonlinear terms should become unimportant. Unfortunately, an accurate comparison cannot be given as yet. It turned out that it is experimentally easy to count the number of zeros with fair absolute accuracy, that is, of the order of ±10 percent. To increase the accuracy beyond this point proved to be more difficult because of the very large range of passage time which the counting apparatus had to reproduce accurately. Consequently, only preliminary measurements are briefly reported here.

APPARATUS AND METHODS

Wind Tunnel

The wind tunnel used for most measurements was the small 20-inch-square GALCIT "Correlation" tunnel built many years ago for similar investigations. Repeated improvements in the tunnel have reduced the free-stream turbulence level to the order of 0.03 percent. Traversing equipment along the tunnel axis for about 200 centimeters' length is

provided. The GALCIT 10-foot tunnel was used for some investigations at high Reynolds numbers reported here. Sketches of the two tunnels are included as figures 1 and 2.

Grids

The grids used in the present investigation differed somewhat from previous ones. They consisted of precision, woven-brass screen mounted in wooden frames. The screens, obtained from Tyler and Company in Cleveland, are very uniform. The geometry of the grids is given below.

Grid	Rod diameter, d (cm)	Mesh size, M (cm)
1	0.413	1.68
2	.184	.818
3	.108	.422
4	.0445	.141

Only in the 10-foot tunnel was a grid of the two-plane type used. The dimensions for this grid were $M = 4$ inches and $d = 0.75$ inch.

Hot-Wire Apparatus

Except for the investigations in the 10-foot tunnel, wires of 0.00005-inch diameter were used. In the 10-foot tunnel the wire diameter was 0.00025 inch. The wires varied in length, depending on the problem, between about 0.3 centimeter and 0.1 centimeter. The silver cover of the platinum wire was in all cases etched off before soldering the wires to the tips of sewing needles.

The amplifier used was rebuilt for the investigation. The frequency response is flat within a few percent between 1/2 cycle and 25,000 cycles. A cut-off filter at approximately 10,000 cycles is provided to reduce noise and pickup whenever there is no need for an investigation of higher frequencies. This was the case most of the time in the present set of experiments.

A resistance-capacitance compensator and a stage of differentiation using a second compensated amplifier is incorporated. The output is read with a thermocross potentiometer arrangement. The compensation is adjusted by means of a square wave arrangement.

Frequency Analyzer

A wave analyzer from Hewlett and Packard with an adjustable band width from nominal 30 cycles to 145 cycles was used for spectrum measurements. The analyzer was slightly modified to allow reading the output on the thermocross meter.

The measurements in the 10-foot tunnel were made with a General Radio Sound Analyzer which is not so suitable for turbulence measurements as the wave analyzer. Part of the very marked improvement in accuracy between earlier spectrum measurements in the 10-foot tunnel and the recent measurements in the 20-inch tunnel is due to improvement in the equipment and part is due to the fact that the flow in the smaller tunnel is steadier than that in the 10-foot tunnel.

Zero Counting

The number of zeros or ξ values was first counted with the help of a photomultiplier cell. The cell was mounted behind a narrow slit in front of an oscilloscope screen and the passage of the trace on the screen thus produced pulses which, after suitable amplification, were counted by means of a 2^9 scaler counter. Every 512th pulse was thus registered on a mechanical counter. An average count comprised about 10^5 passages. Recently, an electronic gate circuit was used for the same purpose.

Length Correction[2]

The measured values of λ^2 were corrected for the length l or the hot-wire by means of the relation $\lambda^2 = \left(\lambda^2\right)_{measured} - \frac{G - 3}{18} l^2$ where $G = \lambda^4 f^{iv}(0)$. The values for G given in this report are uncorrected since their limited accuracy does not warrant an elaborate correction involving still less known higher derivatives of f.

A length correction should be applied to the spectrum measurements. However, it appears difficult to obtain an analytical expression for such a correction. Check measurements were therefore made with wires of

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[2]Length correction formulas were derived by the present authors in a previous unpublished report submitted to the NACA. The procedure is straightforward and similar derivations have been published in the meantime by Frenkiel. Thus there is no need to give the derivation here.

different length and the correction was evaluated semiempirically. It turned out that the correction is unimportant for $F(n)$ itself for the wire length used. For the higher moments the correction becomes more important but at the same time the accuracy of the measurements decreases. Thus, it was felt that correction should be applied at present only to λ^2. The rest of the data are presented uncorrected for the length of the wire.

RESULTS OF MEASUREMENTS

Decay

A few measurements of the decay of turbulence behind a grid were made and figure 3 presents a sample. The decay measurements agree with Batchelor and Townsend's (reference 23) measurements in the initial stage of decay; they show the nearly linear relation between $1/\overline{u^2}$ and t or x/M for $\frac{x}{M} \lesssim 100$ (approx.) and the subsequent increase in the rate of decay.

λ^2 Measurements

The value of λ^2 during decay was determined by Townsend's method from the mean square of the differentiated wire output making use of the interchange between time and space derivatives. Then

$$\frac{1}{\lambda^2} = \frac{1}{\overline{u^2}}\overline{\left(\frac{\partial u}{\partial x}\right)^2}$$

$$\approx \frac{1}{U^2\overline{u^2}}\overline{\left(\frac{\partial u}{\partial t}\right)^2}$$

Also λ^2 was determined from the measured spectrums. The relation here is

$$\frac{1}{\lambda^2} \approx \frac{1}{U^2}\frac{1}{\overline{u^2}}\overline{\left(\frac{\partial u}{\partial t}\right)^2}$$

$$= \frac{4\pi^2}{U^2}\int_0^\infty n^2F(n)\ dn$$

A set of results is shown in figures 4 and 5. The estimated errors are indicated by the extent of the lines from the points. The straight line with slope $10\nu/U$ is plotted in the figures to allow a comparison with the measurements. It is seen that the measured $d\lambda^2/dx$ is near $10\nu/U$. The measured slope is a little smaller than $10\nu/U$ and the agreement is not so close as that given by Batchelor and Townsend. However, in the initial stage of decay there is no doubt that $d\lambda^2/dx$ is much closer to $10\nu/U$ than to $7\nu/U$. The slope $10\nu/U$ is that which follows from the assumption of isotropic turbulence if the relation of $1/\overline{u^2}$ is linear with x/M or t. The slope $7\nu/U$ follows from certain other similarity assumptions (e.g., reference 24). The question of the slope of $\lambda^2(x)$ has been discussed sufficiently by Batchelor and Townsend (references 23 and 25), Frenkiel (reference 24), and others, and there is no need to repeat the arguments here.

Spectrum and Correlation Measurements in GALCIT 10-Foot Tunnel

More extensive measurements were made of the spectrum of turbulence. These measurements include earlier investigations of the spectrum at large Reynolds numbers in the 10-foot tunnel. These latter measurements were done in order to check on the existence of a range of frequencies for which the $k^{-5/3}$ spectral law applies at Reynolds numbers which can be reached in a wind tunnel. As mentioned before, these measurements do not have the precision of the later spectrum measurements at lower Reynolds numbers. The results are shown in figure 6 plotted in a reduced scale

$$\frac{F(n)}{F(0)} \equiv \sigma \quad \text{against} \quad \frac{\pi}{2}F(0)n \equiv \zeta$$

The simple spectrum

$$\sigma = \frac{1}{1 + \zeta^2}$$

is shown for comparison. The scatter of the points is evident and any conclusions therefore are not too convincing. Here the simple spectrum does not agree so well as in the measurements at smaller Reynolds numbers. Figure 7 shows the same measurements in a logarithmic plot. The line with a slope corresponding to $n^{-5/3}$ is included. These same measurements were used by Kármán (reference 22) for a similar comparison with the $n^{-5/3}$ law and his interpolation formula which here fits the measurements very well indeed. A range of fair agreement with the

$n^{-5/3}$ law can be found and, similarly, a range of agreement between the corresponding $r^{-2/3}$ law and the measured correlation curve (cf. fig. 8 and reference 22). The range and degree of agreement is quite similar to the one found in the meantime by Townsend (reference 26) at large Reynolds numbers. However, in view of the inaccuracy of these measurements and the arbitrariness in fitting these power laws to measured points, agreements such as that presented in figures 7 and 8 should be accepted with reserve. Figures 9 to 11 show the change in the correlation curve with Reynolds number. During the past year the accuracy of spectrum measurements has been very much improved and it is hoped that future measurements at large Reynolds numbers can be made with more definite results.

Spectrum Measurements in 20-Inch Tunnel

Measurements of the spectrum of turbulence in the 20-inch (correlation) tunnel were made with a very much improved technique. Figure 12 first shows a typical spectrum plotted in the conventional logarithmic scale to allow a comparison of the amount of scatter with earlier measurements. Then a series of spectrums plotted in the reduced form of $\sigma = F(n)/F(0)$ against $\zeta = \frac{\pi}{2}F(0)n$ is shown in figures 13 to 16 for various speeds U and grid sizes. All these measurements were made in the initial phase of decay. In each figure the spectrum $\sigma = \left(1 + \zeta^2\right)^{-1}$ is shown for comparison. The measured values lie evidently quite close to the simple spectrum for values of ζ less than about 6. For higher values the measured values fall below the simple spectrum, as has been discussed in the section "General Considerations." In any case the largest part of the turbulent energy is doubtless contained in a region where $\sigma = \left(1 + \zeta^2\right)^{-1}$ is a very good approximation. The value of $F(0)$ had to be found from the experimental data. This was done by fitting a curve of the form $\frac{A}{B + n^2}$ to the measurements of the not-normalized spectrum. In this way one finds a value $e(0)$ from which $F(0)$ is obtained by normalization, that is, by the division of $e(0)$ by the area $\int e(n)\,dn$. The correctness of $e(0)$ could be checked somewhat by comparing the area $\int e(n)\,dn$ with the result of a measurement of the the mean square of the total output (i.e., without frequency analysis) which should be equal to the area. This was checked and found to be true within ± 3 percent in all cases.

Moments of Spectrum

The spectrum measurements in the 20-inch tunnel were accurate enough to allow the computation of the second and fourth moments of the spectrum density $F(n)$, that is, the quantities

$$\overline{n^2} = \int_0^\infty n^2 F(n)\ dn$$

and

$$\overline{n^4} = \int_0^\infty n^4 F(n)\ dn$$

The second moment is used to determine λ^2, as has been mentioned above; the fourth moment furnishes

$$G = \lambda^4 f^{iv}(0)$$

A set of curves showing $n^2F(n)$ and $n^4F(n)$ is given in figures 17 to 22. It is seen that the accuracy of the measurements and the frequency range is sufficient to evaluate the moments by graphical integration. The results are tabulated below:

U (cm/sec)	M (cm)	x (cm)	λ^2 (cm^2)	G	C
1130	1.68	48.5	0.054	13.2	2.6
1130	1.68	107.5	.123	13.8	3.0
1130	1.68	160.0	.172	11.9	2.7
630	1.68	49.5	.088	10.7	3.0
630	1.68	109.5	.229	10.2	3.00
630	1.68	156.5	.321	9.39	3.1
300	1.68	108.5	.463	8.4	
1500	1.68	108.5	.057	15.6	
1130	.818	108.5	.150	10.22	
1130	.422	108.5	.133	8.20	

During decay G is nearly constant in the initial range and then decreases. This agrees with the results of Batchelor and Townsend.

The measured absolute values for G here appear slightly higher. In the last column in the table above, the quantity

$$C = \left(\frac{\overline{u^2}}{\overline{u_o^2}}\right)^{1/5} F(0)U$$

is included and C is proportional to Loitzianskii's invariant in the case where the simple spectrum applies. Of course, F(0) and $\overline{u^2}/\overline{u_o^2}$ vary considerably during decay. As seen from the table, C is nearly constant, that is, does not show a definite trend. Clearly, the accuracy in the determination of C is not very good.

Behavior of Spectrum at High Frequencies

It has been mentioned before that Heisenberg's theory of the spectrum of isotropic turbulence leads to an n^{-7} law for high frequencies. It is therefore of interest to compute the sixth moments of the spectrum

$$\overline{n^6} = \int_0^\infty n^6 F(n)\, dn$$

If $F(n) \sim n^{-7}$, then the sixth moment should diverge. Plots of $n^6F(n)$ for two speeds are presented in figures 23 and 24. For the lower speed, U = 630 centimeters per second, the area appears to be finite; for the higher speed, U = 1130 centimeters per second, the frequency range of the measurements is evidently not large enough to decide at all whether or not the curves approach zero.

However, it should be kept in mind that the turbulent energy in the frequency range that matters in the $n^6F(n)$ curve is extremely small. Hence, the electrical noise and the pickup in this range are of the order from 20 percent to 80 percent of the total output and the measured values of F(n) are therefore the difference between two fairly close numbers. This difference is then multiplied by 10^{18} to 10^{24} in the important range and the resulting curve is therefore not very trustworthy. The only definite result which can be asserted so far with confidence is that F(n) does not vary more slowly than n^{-7} in the high frequency range. Batchelor has recently reported during a visit at GALCIT that Townsend

has obtained measurements of $\overline{\left(\frac{\partial^3 u}{\partial t^3}\right)^2}$. Since

$$\overline{\left(\frac{\partial^3 u}{\partial t^3}\right)^2} \approx \overline{n^6}$$

a measurement of $\overline{\left(\frac{\partial^3 u}{\partial t^3}\right)^2}$ is equivalent to n^6, the sixth moment from the spectrums. Townsend finds the sixth moment finite. This agrees with the present measurements of the spectrum within the limitations discussed above. Batchelor interprets this result as confirming the n^{-7} power law plus a rapid cut-off at high frequencies. This conclusion appears premature because of the difficulties stressed above.

Spectrum at Very Large Distances from Grid

Very far downstream from the grid, in the so-called final stage of decay, the spectrum and the correlation curve should approach a Gaussian distribution. Measurements of the time correlation curve were made by Batchelor and Townsend (reference 17) and the conclusion was reached that the final stage of decay is rapidly approached for $x/M > 300$.

The spectrum of turbulence was therefore measured at $x/M = 1000$ behind a fine mesh grid. The measurements are plotted in figure 25 compared with the Gaussian spectrum compatible with the measured value of λ; that is, the spectrum was evaluated as usual and λ was determined. Then the Gaussian spectrum was plotted which would give the same λ.

The measured points at very low frequency are, as mentioned before, not too trustworthy. The rest of the curve approaches the Gaussian. Actually this latter fact becomes more obvious if one tries to plot these measurements in the same form as the spectrums at lower values of x/M. Figure 26 shows that type of plot and evidently little similarity with the spectrum $\sigma = \left(1 + \zeta^2\right)^{-1}$ can be found here. Similarly, one can convince oneself that with increasing x/M the measured spectrums tend toward the Gaussian. Still, from the present set of measurements the Gaussian form is not reached within the accuracy of measurement. Here future work is needed to see whether $x/M = 1000$ is still too small or whether other factors enter.

Spectrum of Derivative

The function $F(n)$ is defined as the power spectrum of $u(t)$. Similarly, one can define a power spectrum $W(n)$ of the derivative $\partial u/\partial t$. If $W(n)$ is not normalized one has

$$\overline{\left(\frac{\partial u}{\partial t}\right)^2} = \int_0^\infty W(n)\ dn$$

On the other hand,

$$\frac{1}{\overline{u^2}}\overline{\left(\frac{\partial u}{\partial t}\right)^2} = 4\pi^2 \int_0^\infty n^2 F(n)\ dn$$

and, consequently,

$$\frac{W(n)}{\overline{u^2}} = 4\pi^2 n^2 F(n)$$

Figure 27 shows measurements of $W(n)$ plotted in comparison with $n^2F(n)$. The agreement is seen to be excellent. This is a welcome check on the consistency of the measurements.

Zero Counts

A sample set of λ^2 values obtained from the counting of zeros using an electronic gate circuit is included in figures 4 and 5. The value of λ^2 was obtained from the number of zeros assuming a Gaussian distribution for both the function $u(t)$ and the derivative $\partial u/\partial t$. Then λ^2 is given simply by

$$\lambda^2 = \left(\frac{U}{\pi N_0}\right)^2$$

Earlier measurements using the photocell method gave essentially the same result. As computed from the zeros, λ^2 behaves as functions of x quite similar to the λ^2 values obtained from the other methods, but the absolute values of λ^2 as obtained from the counts are consistently larger, especially at the high speed. How much of this difference in absolute value can be traced to the deviation from the Gaussian

distribution and independent probabilities of $u(t)$ and $\partial u/\partial t$ remains to be investigated. That N_ξ is nearly Gaussian in ξ is shown in figure 28. This is, however, obviously only necessary but not sufficient for independence between $u(t)$ and $\partial u/\partial t$. It was first believed that the better agreement for larger values of λ reflected the lack of resolution of the system. However, the resolution of the new circuit should be very much better and the difference between the λ measurements and the zero counts probably has to be traced to other reasons.

CONCLUDING REMARKS

The results of an investigation of the spectrum of isotropic turbulence led to the following conclusions:

1. The spectrum and correlation functions in the turbulent flow behind a grid at large Reynolds numbers $\left(RN \approx 10^5 \text{ based on the grid mesh}\right)$ show a range of frequencies for which the $n^{-5/3}$ or $r^{-2/3}$ laws, respectively, apply. Kármán has shown that these measurements can be represented well by an interpolation formula for the spectrum of the form

$$\frac{F(n)}{F(0)} = \left(1 + \zeta^2\right)^{-5/6}$$

where

$$\zeta = \frac{\pi}{2}F(0)n$$

2. The measurements at smaller Reynolds numbers of the order of 10^4 show that in the initial stage of decay the bulk of the turbulent energy lies in a frequency range in which the spectrum is closely approximated by the simple relation

$$\frac{F(n)}{F(0)} = \left(1 + \zeta^2\right)^{-1}$$

corresponding to an exponential correlation function.

3. The mean square of the frequency $\overline{n^2}$ or the corresponding microscale λ was measured by differentiation and from the spectrum. The measurements agree closely and the slope of $\lambda^2(x)$ is nearly linear in the initial stage of decay. The value of the slope was found to be a

little less than $10\nu/U$. The turbulent energy was found to vary downstream according to the law $\overline{(u')^2} \approx 1/t$ for distances up to about 100 mesh. The quantity $G = \lambda^4 f^{iv}(0)$ was obtained from the spectrums and G was found nearly constant in the initial stage of decay. It decreases in the later stage; G increases with speed. These results are in general agreement with Batchelor and Townsend's measurements.

4. The sixth moments of the spectrums were computed. Here the limit of accuracy of the measurements is reached and conclusions have to be tentative. The sixth moments appear to be finite and hence $F(n)$ at very high frequencies should vary faster than n^{-7}. Certainly $F(n)$ does not vary slower than n^{-7}.

5. The spectrum at $x/M = 1000$ was found to approach a Gaussian curve.

6. Using some simplifying assumptions, λ was also obtained from a count of the average number of zeros of the fluctuating velocity component. The general trend of $\lambda^2(x)$ agrees with the measurements by other methods; the absolute values of λ^2 as obtained from the zero counts are somewhat larger. It was not possible yet to decide whether this difference is significant.

California Institute of Technology
Pasadena, Calif., August 16, 1949

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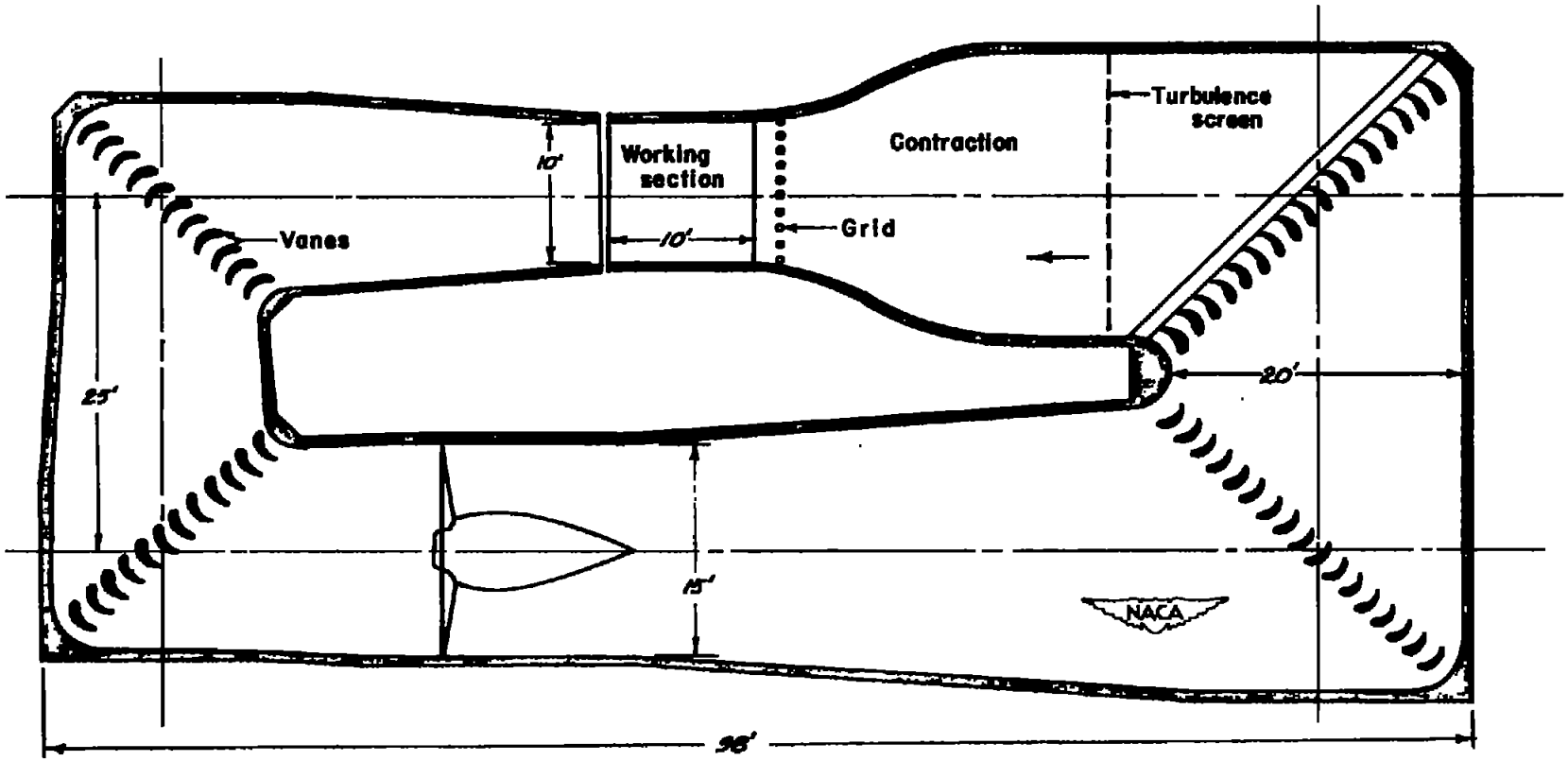

Figure 1.- Vertical section through GALCIT 10-foot wind tunnel.

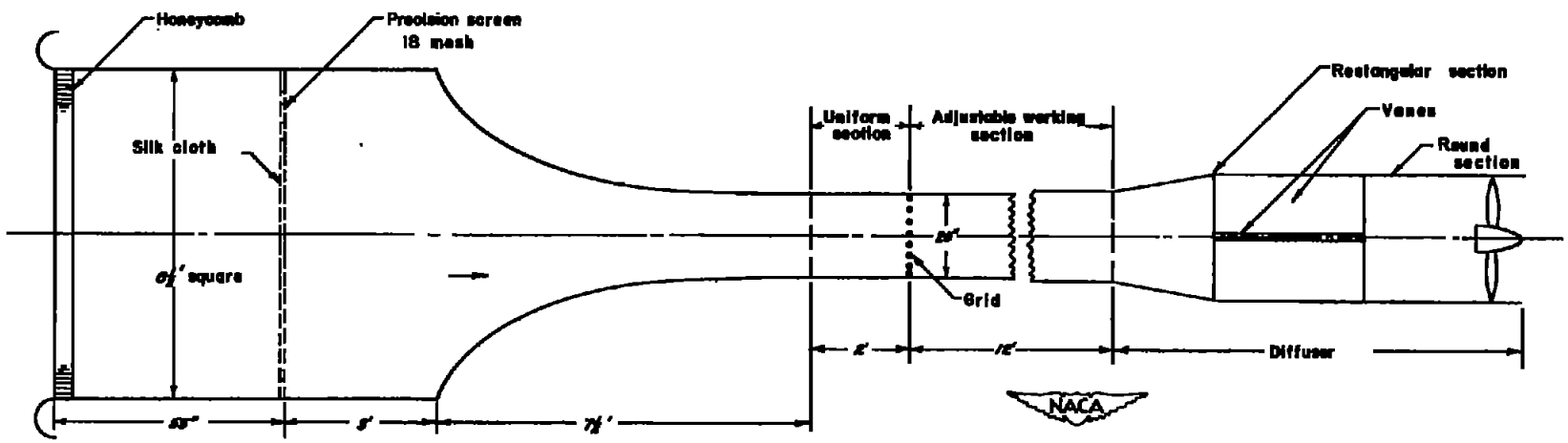

Figure 2.- Sketch of GALCIT 20-inch tunnel.

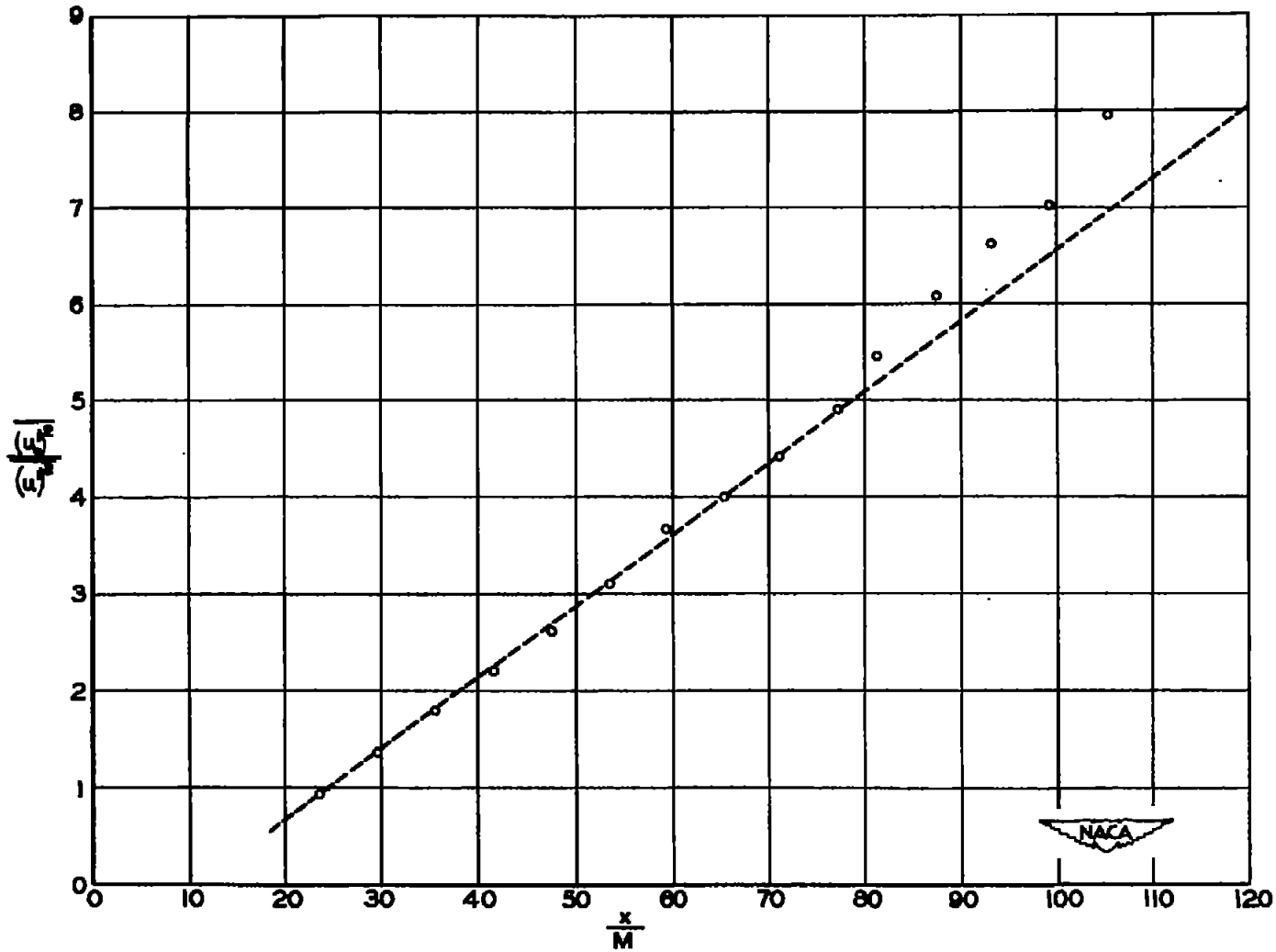

Figure 3.- Energy decay behind grid. M = 1.68 centimeters; U = 1130 centimeters per second.

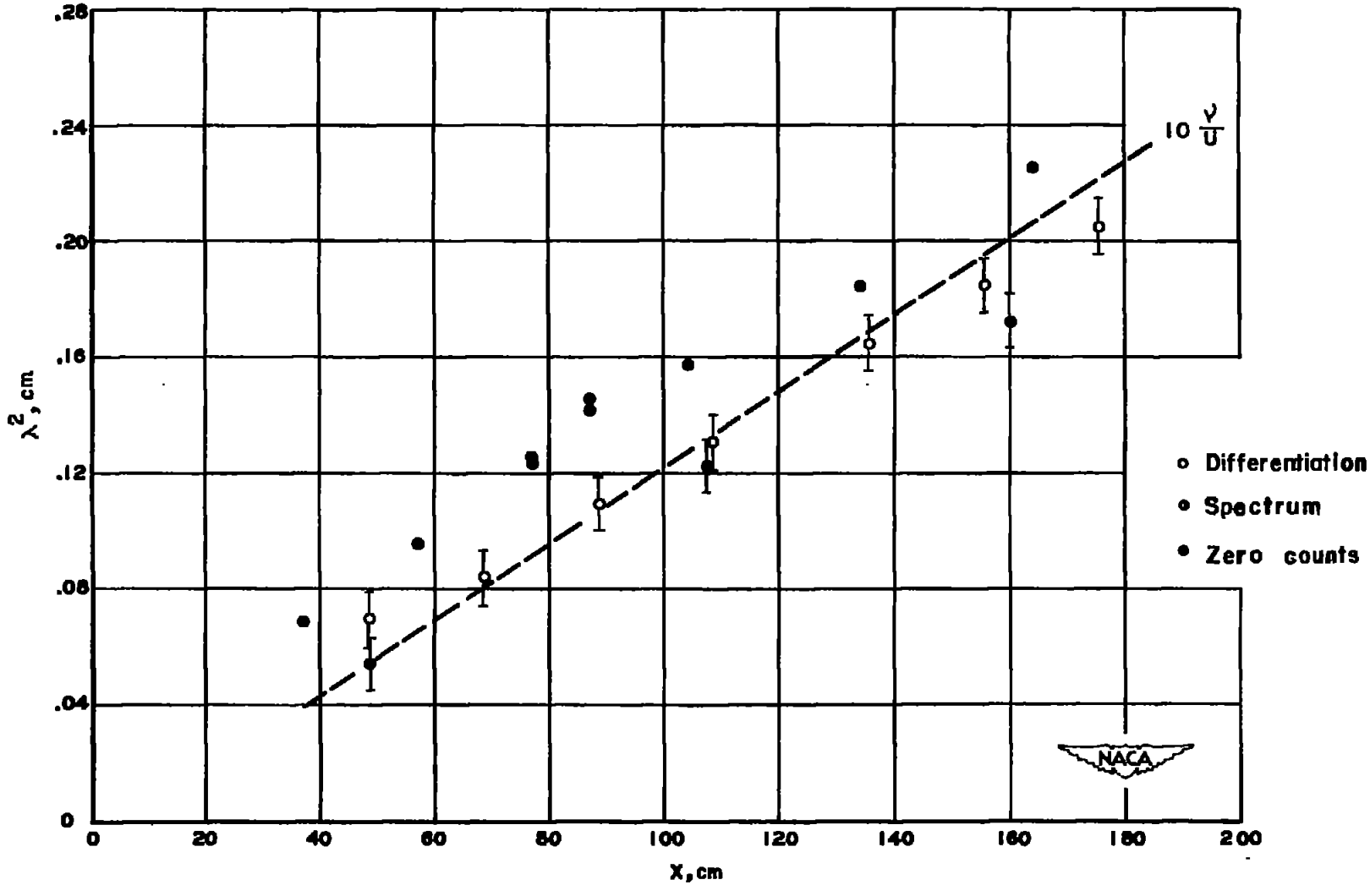

Figure 4.- Plot of λ^2 against x. U = 1130 centimeters per second; M = 1.68 centimeters.

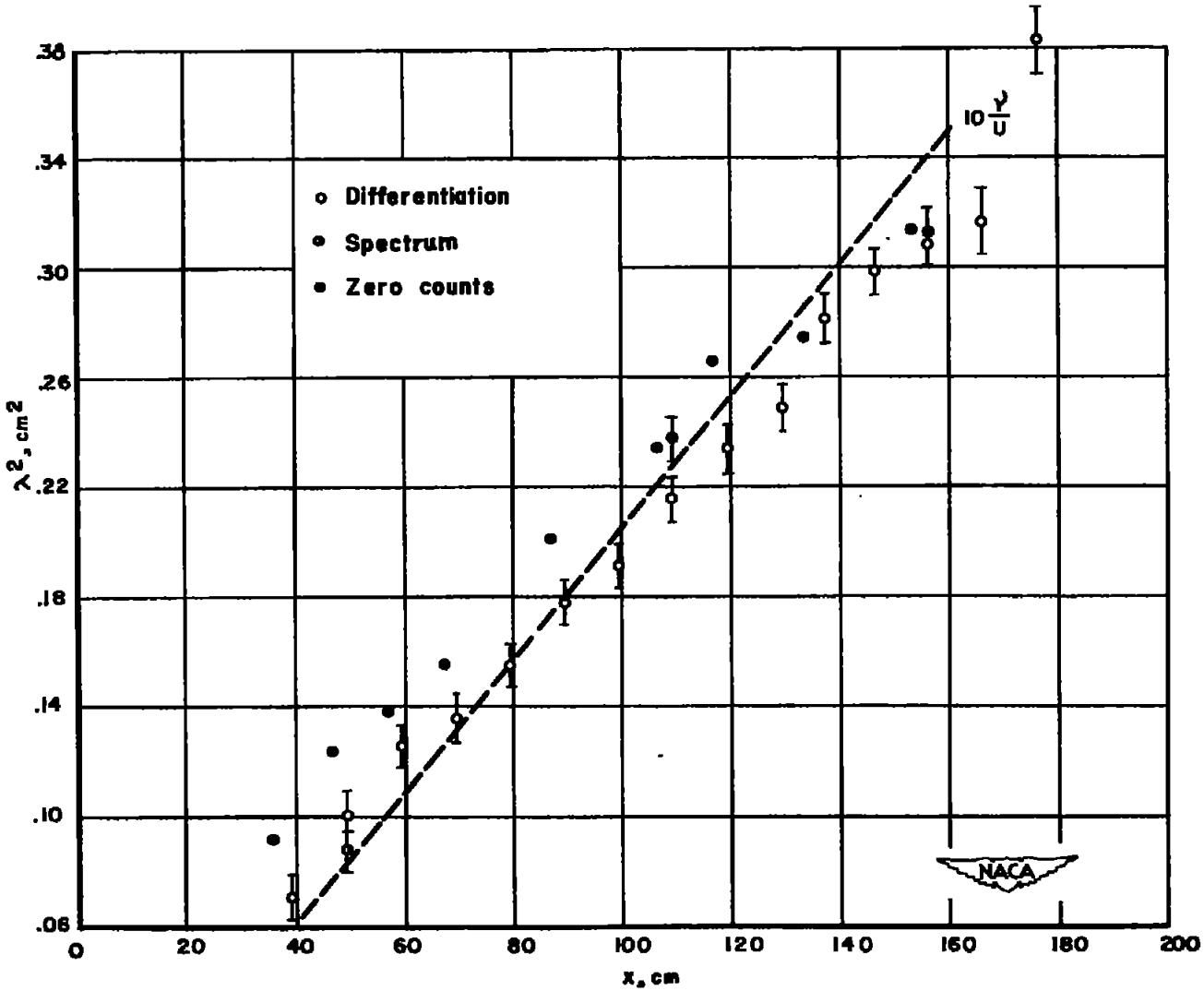

Figure 5.- Plot of λ^2 against x. U = 630 centimeters per second; M = 1.68 centimeters.

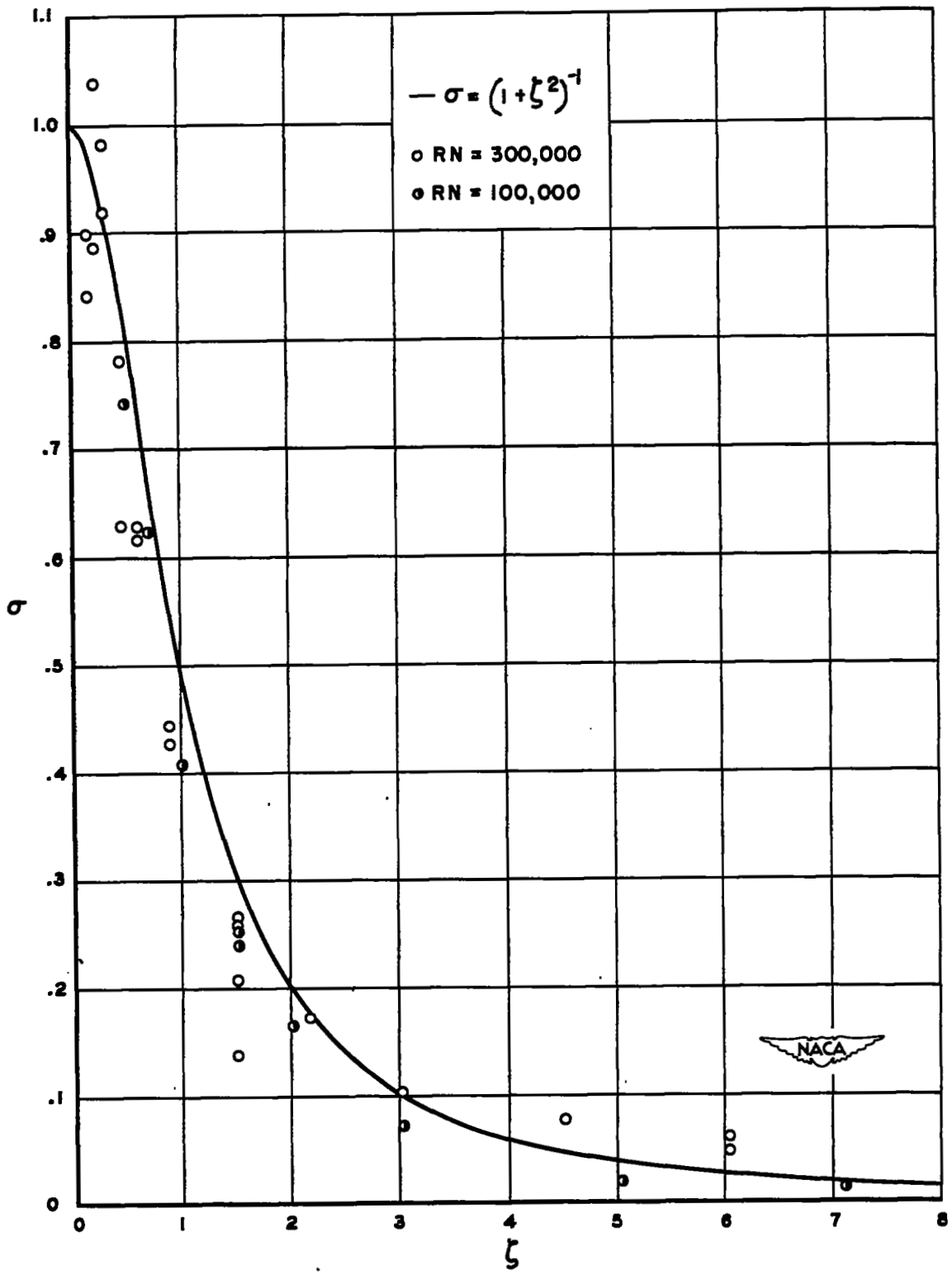

Figure 6.- Reduced spectrum.

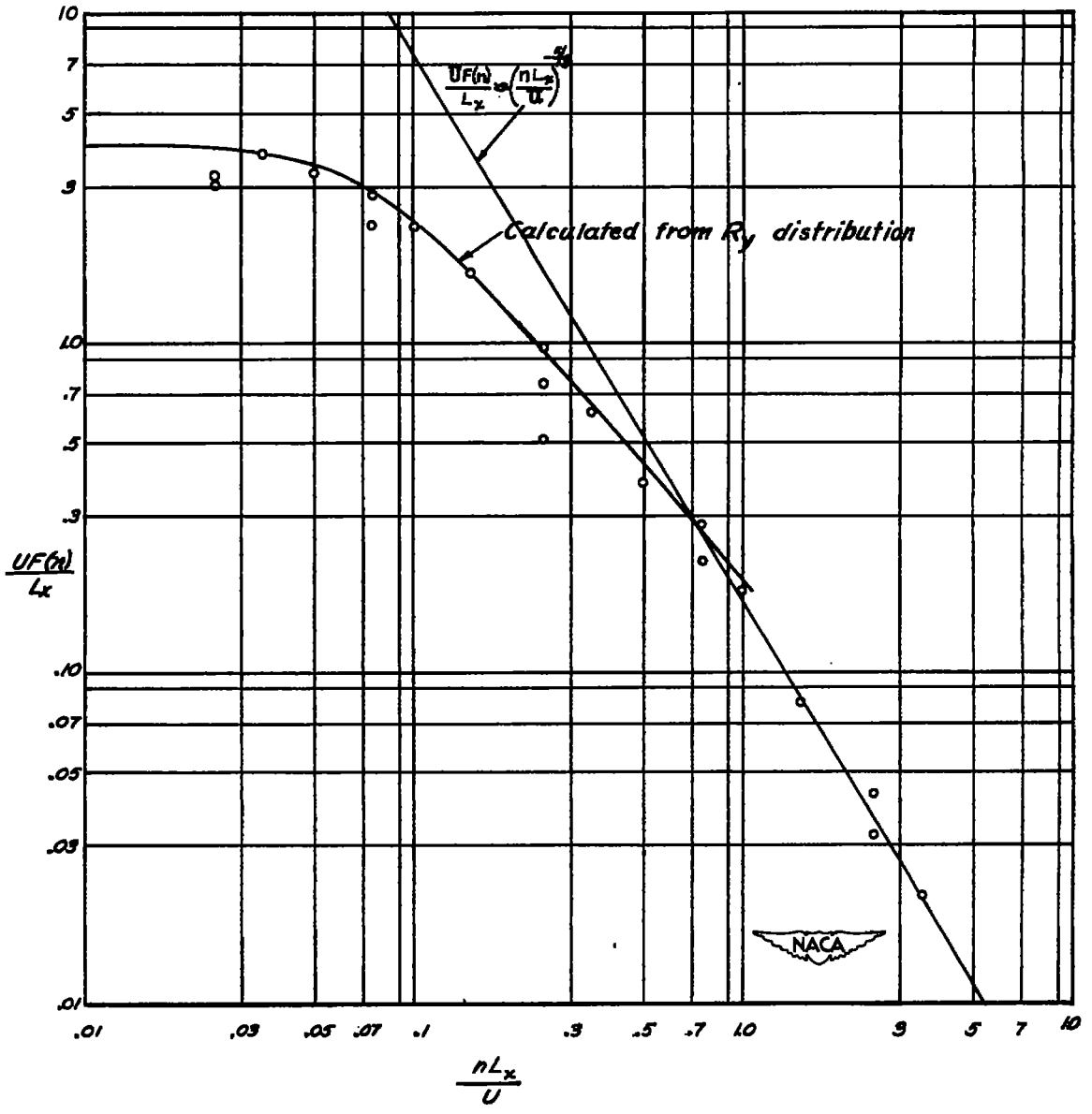

Figure 7.- Energy spectrum. RN = 300,000; 4-inch grid.

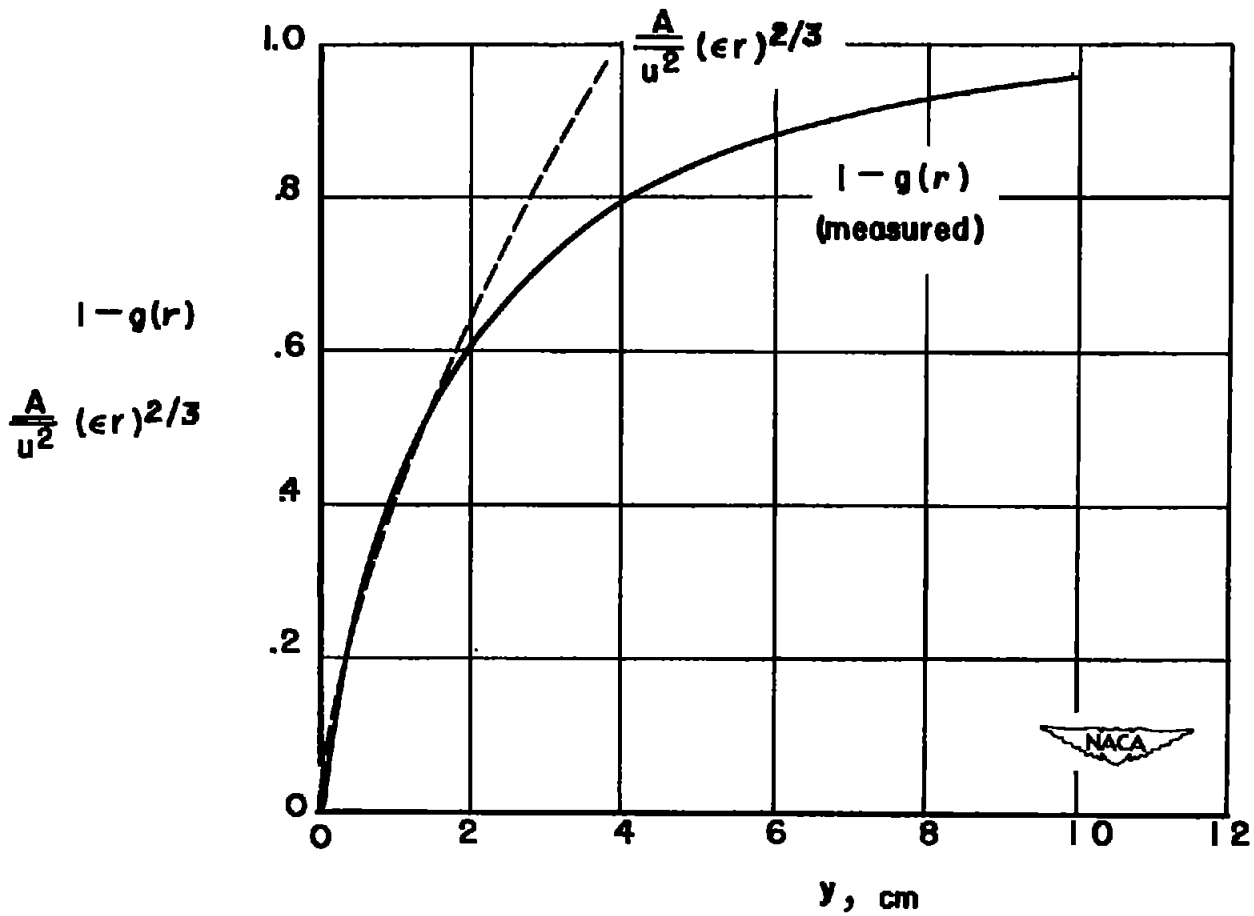

Figure 8.- Comparison of two-thirds law with experimental correlation curve. $\epsilon = 10^5$ square centimeters per second^3; A = 1.0; RN = 300,000.

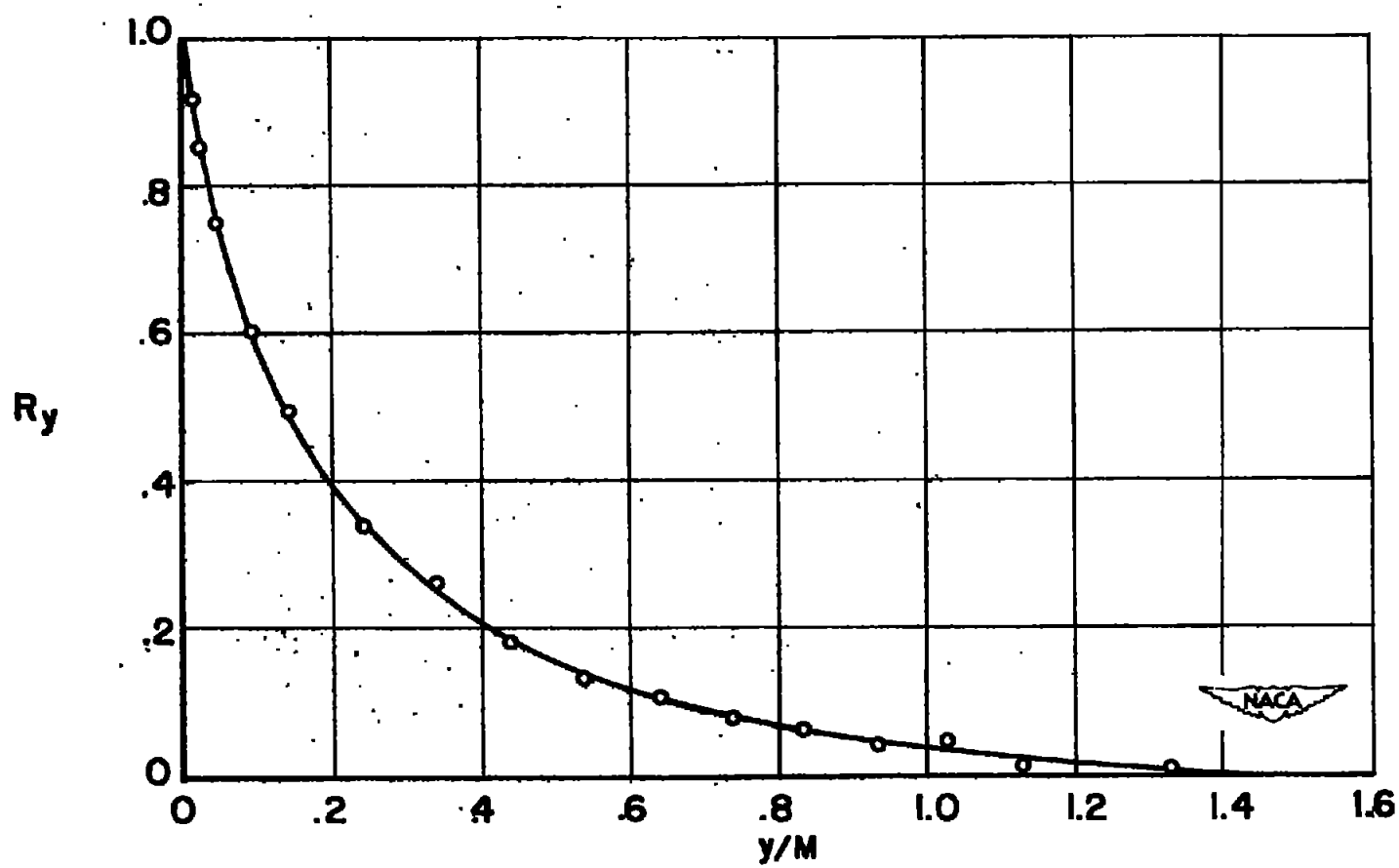

Figure 9.- Distribution of R_y. M = 4 inches; x/M = 40.4; RN = 300,000.

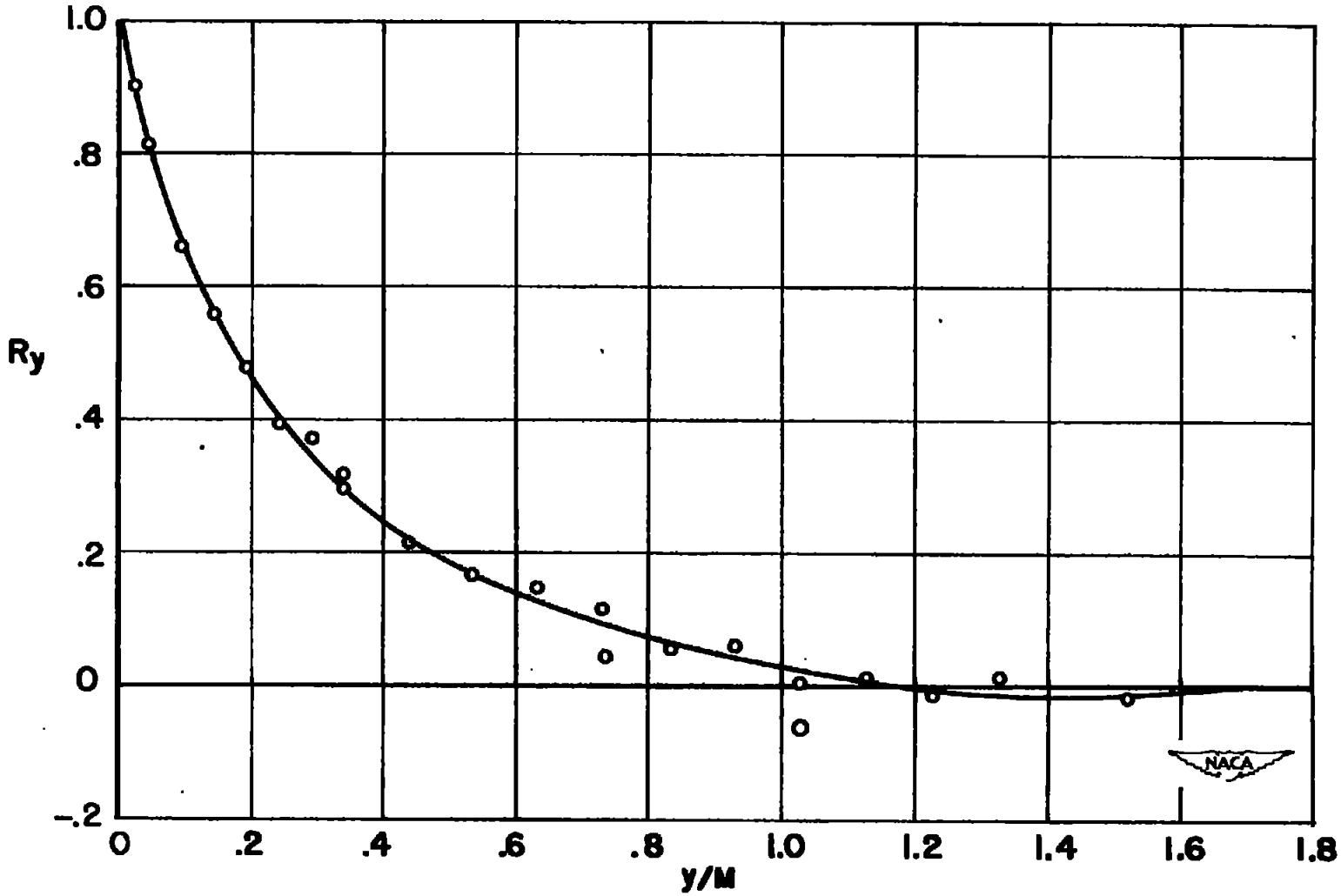

Figure 10.- Distribution of R_y. M = 4 inches; x/M = 40.4; RN = 100,000.

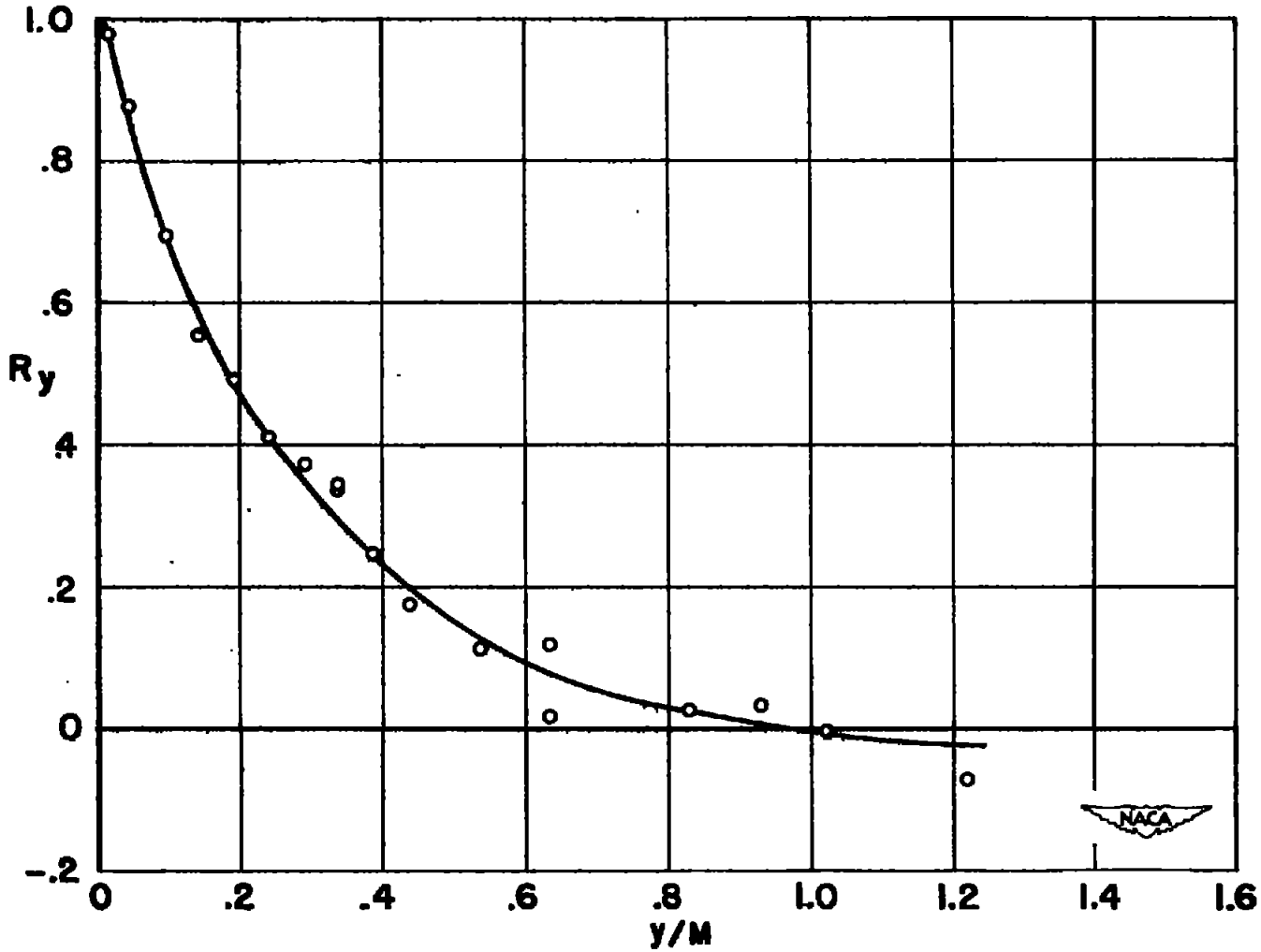

Figure 11.- Distribution of R_y. M = 4 inches; x/M = 40.4; RN = 40,000.

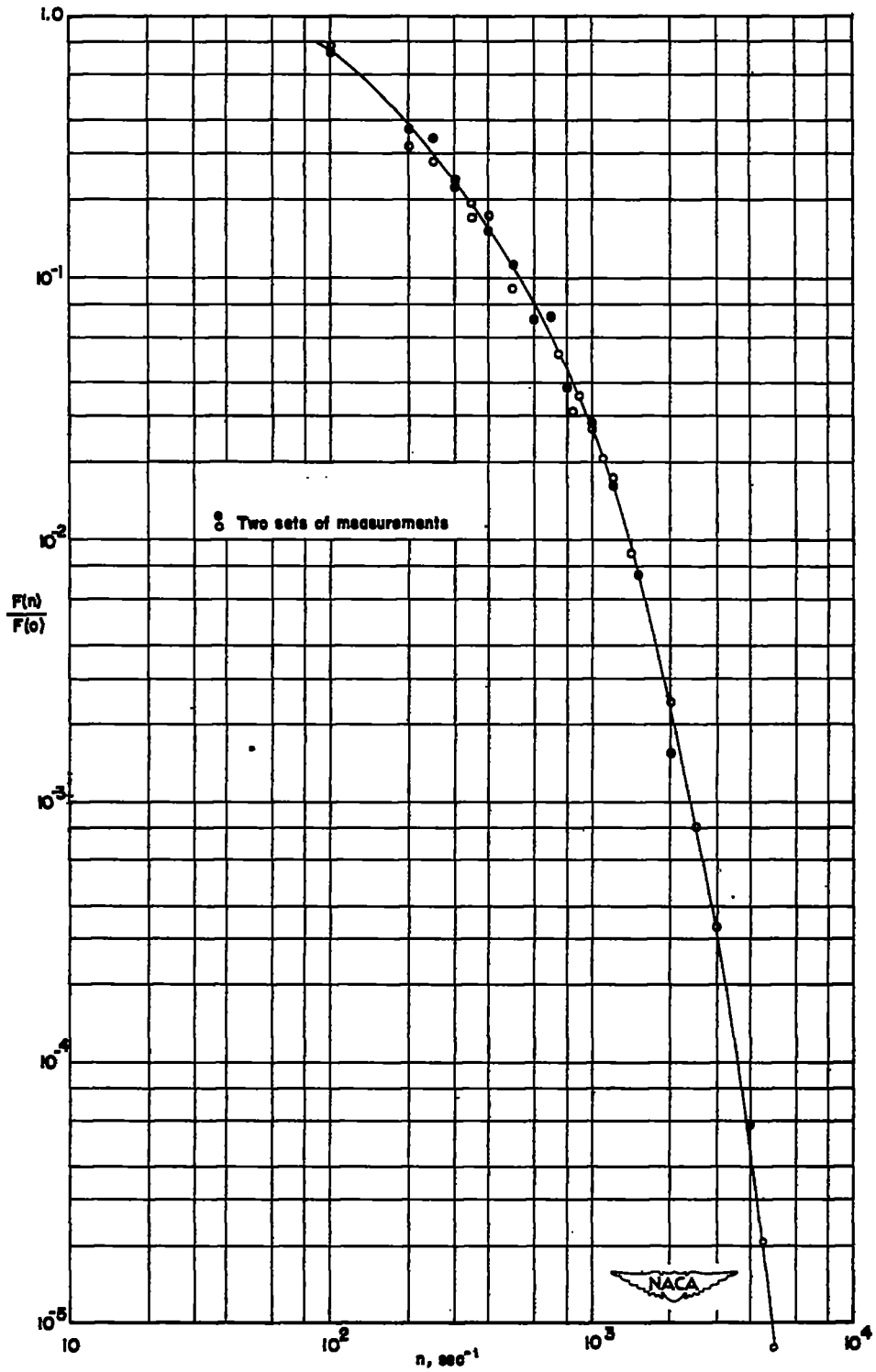

Figure 12.- Energy spectrum of turbulence produced behind grid. U = 1130 centimeters per second; x/M = 81; M = 1.27 centimeters; F(0) = 4.0×10^{-3} second.

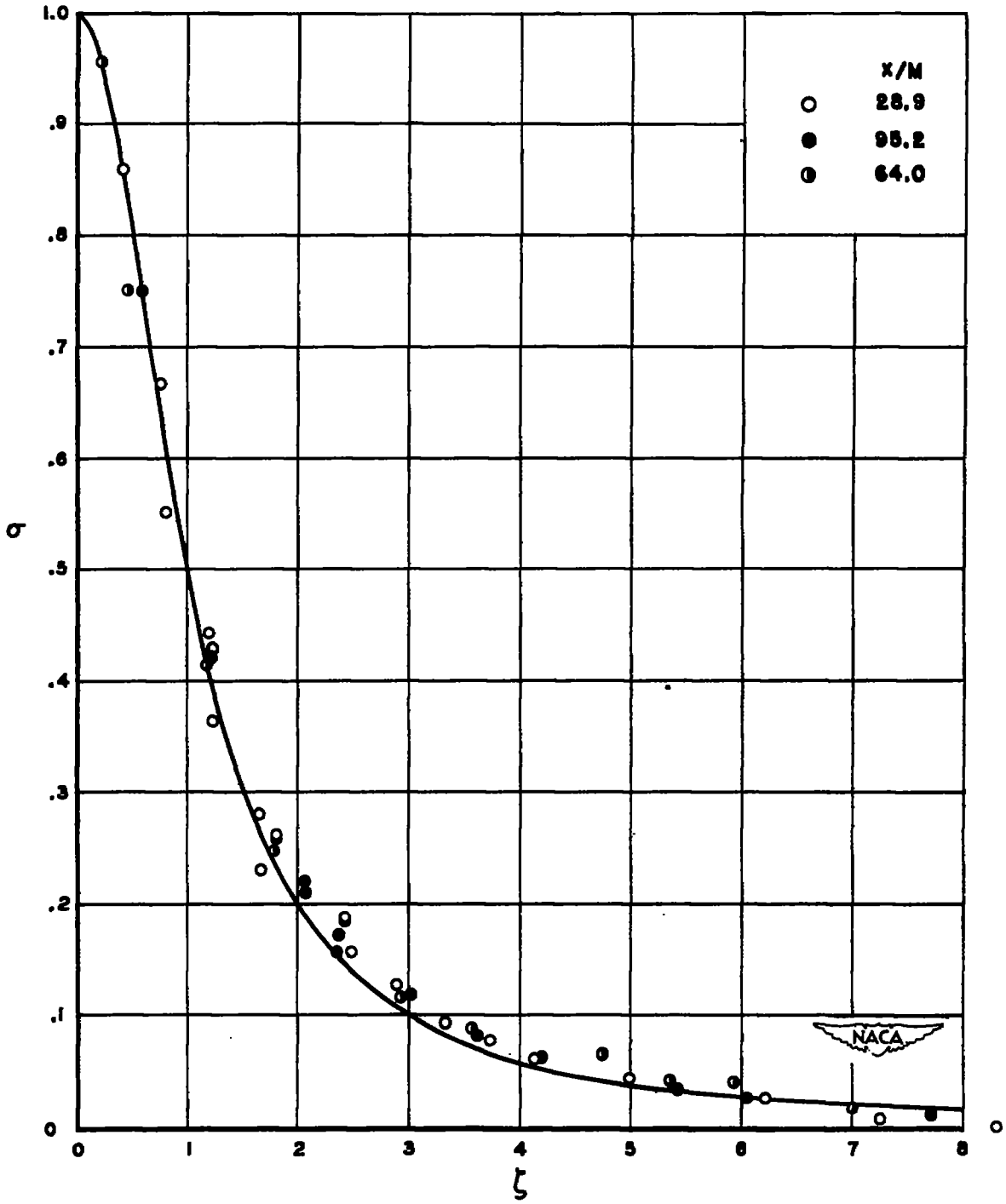

Figure 13.- Reduced spectrum. U = 1130 centimeters per second; M = 1.68 centimeters; RN = 12,400.

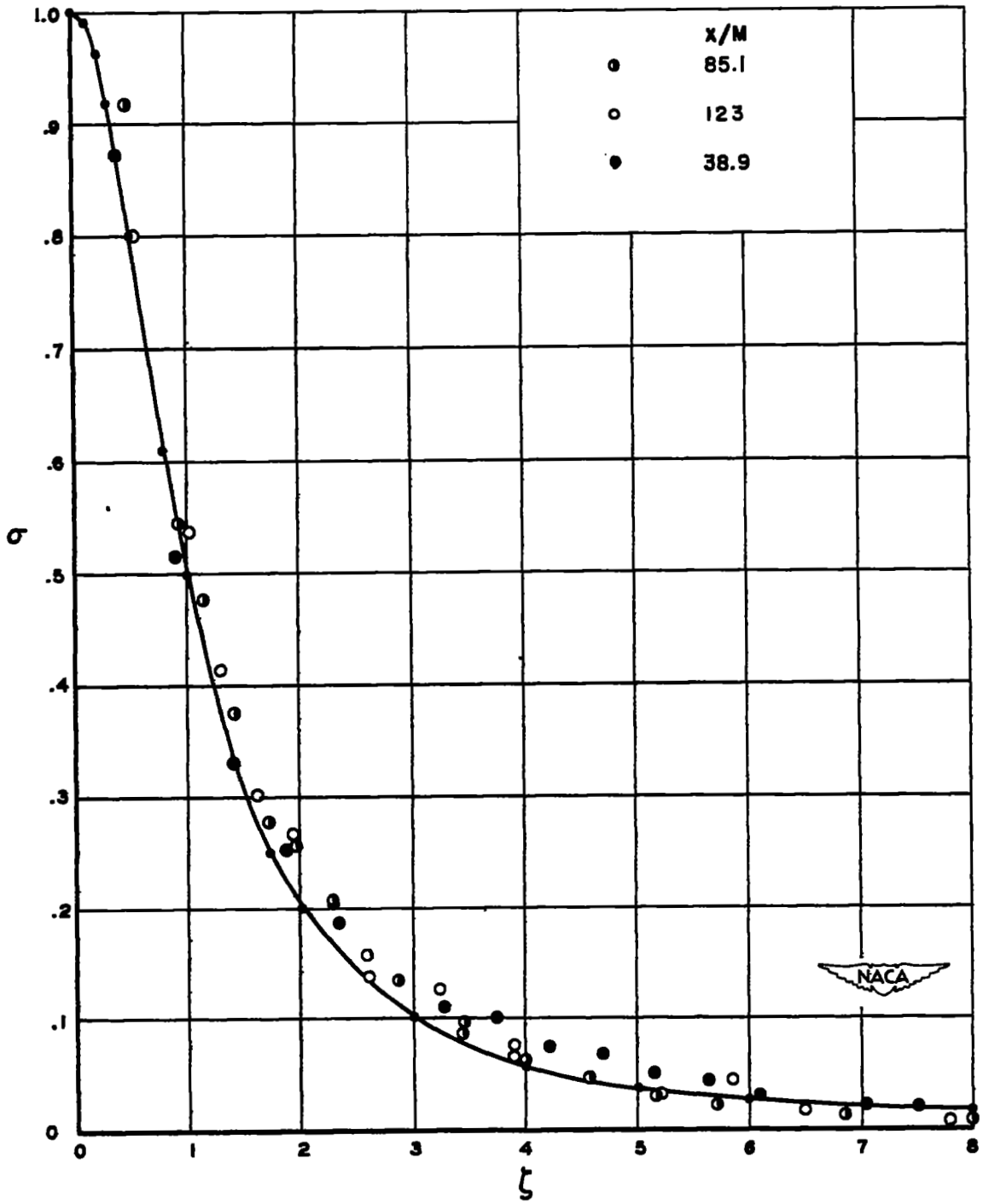

Figure 14.- Reduced spectrum. U = 630 centimeters per second; M = 1.68 centimeters; RN = 6900.

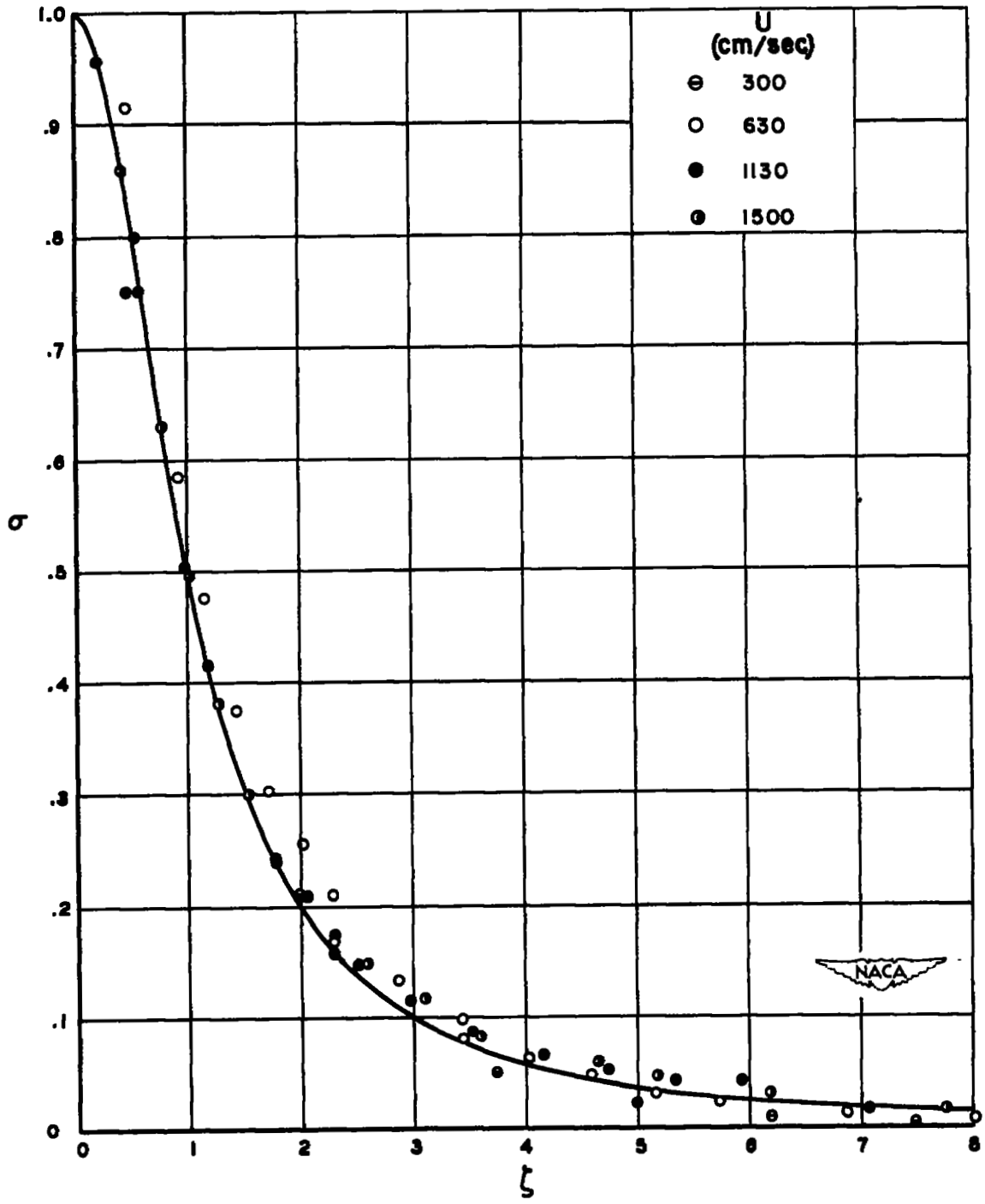

Figure 15.- Reduced spectrum. x/M = 65; M = 1.68 centimeters.

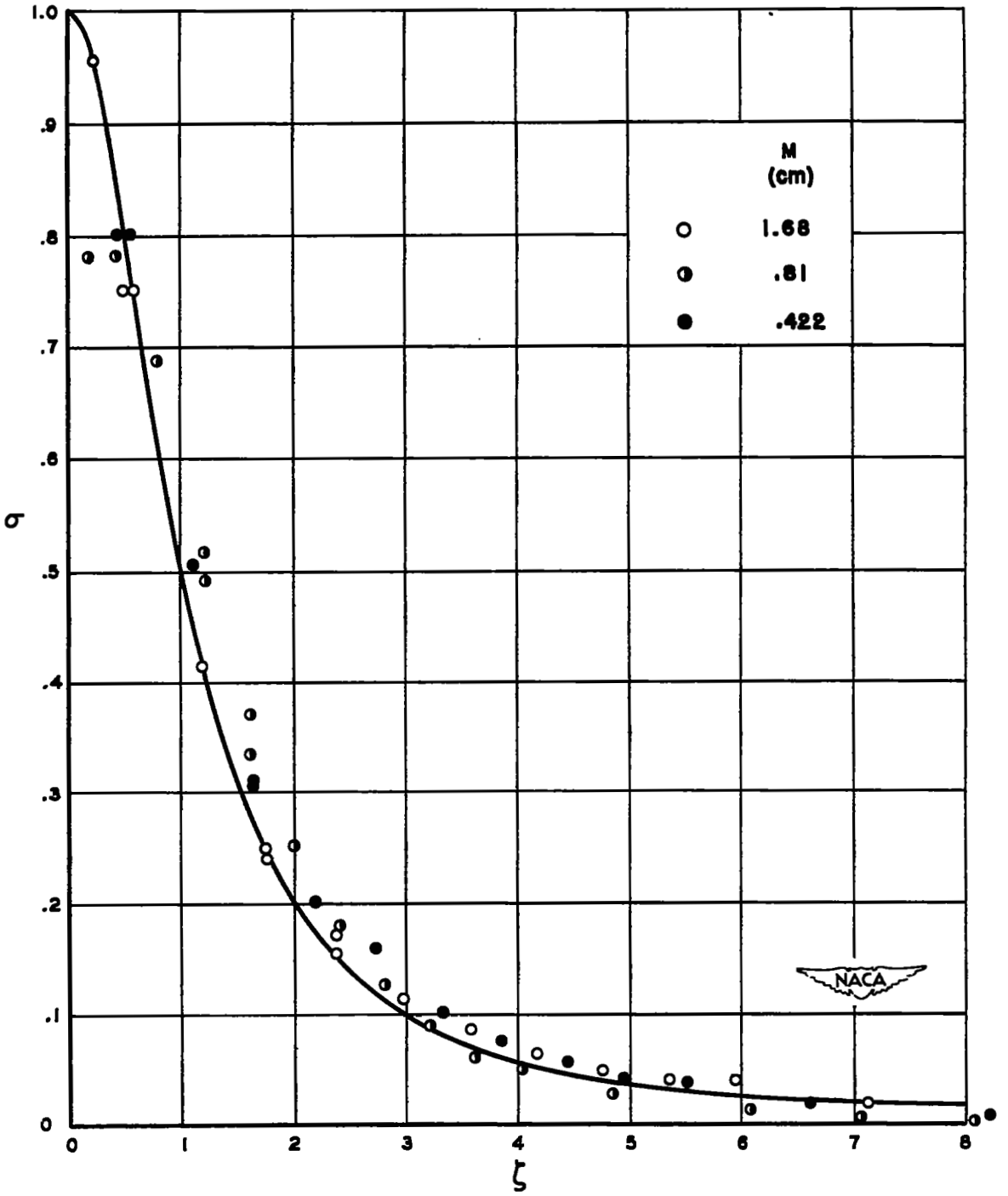

Figure 16.- Reduced spectrum. U = 1130 centimeters per second; x = 108 centimeters.

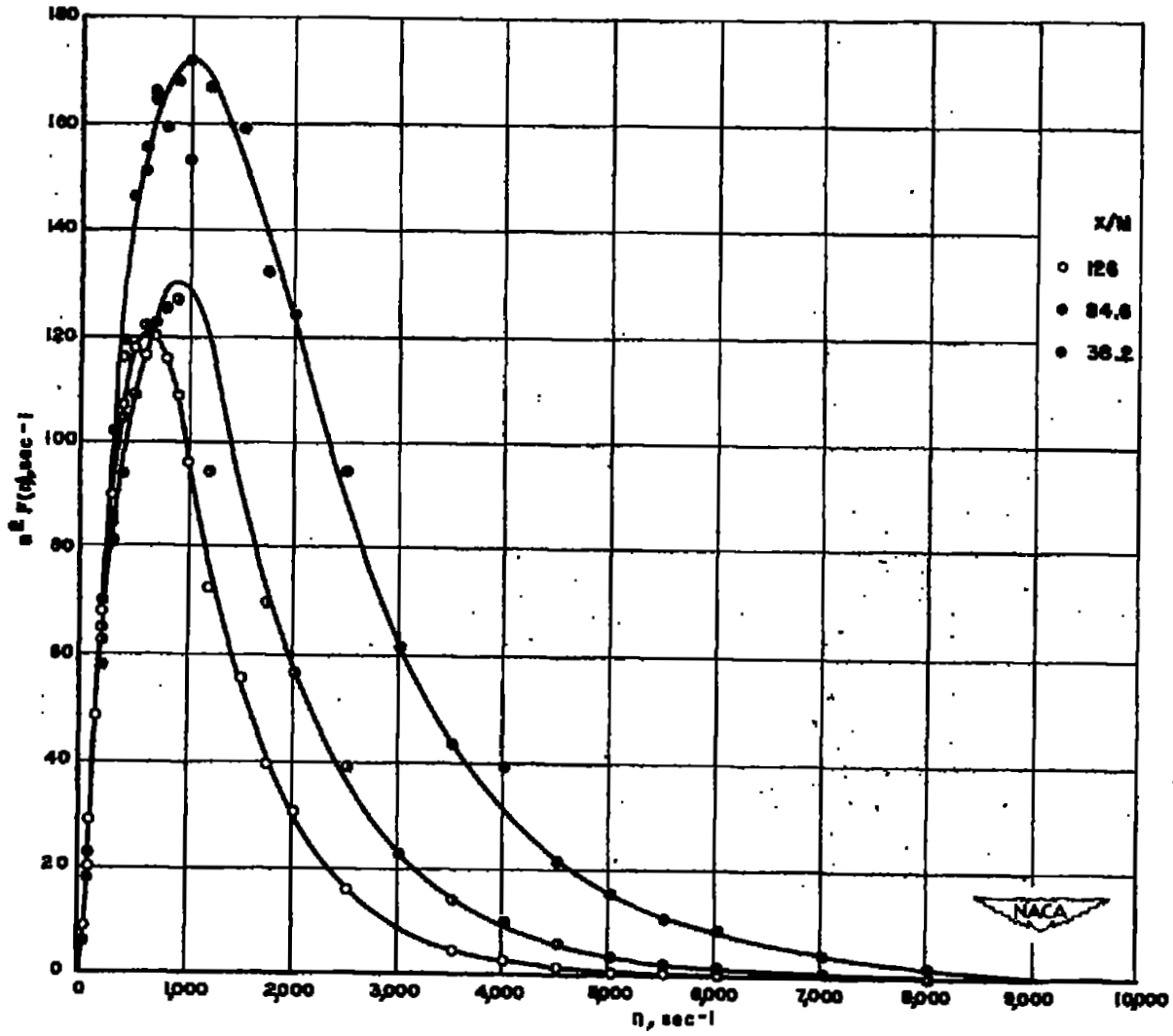

Figure 17.- Second moments. U = 1130 centimeters per second; M = 1.68 centimeters.

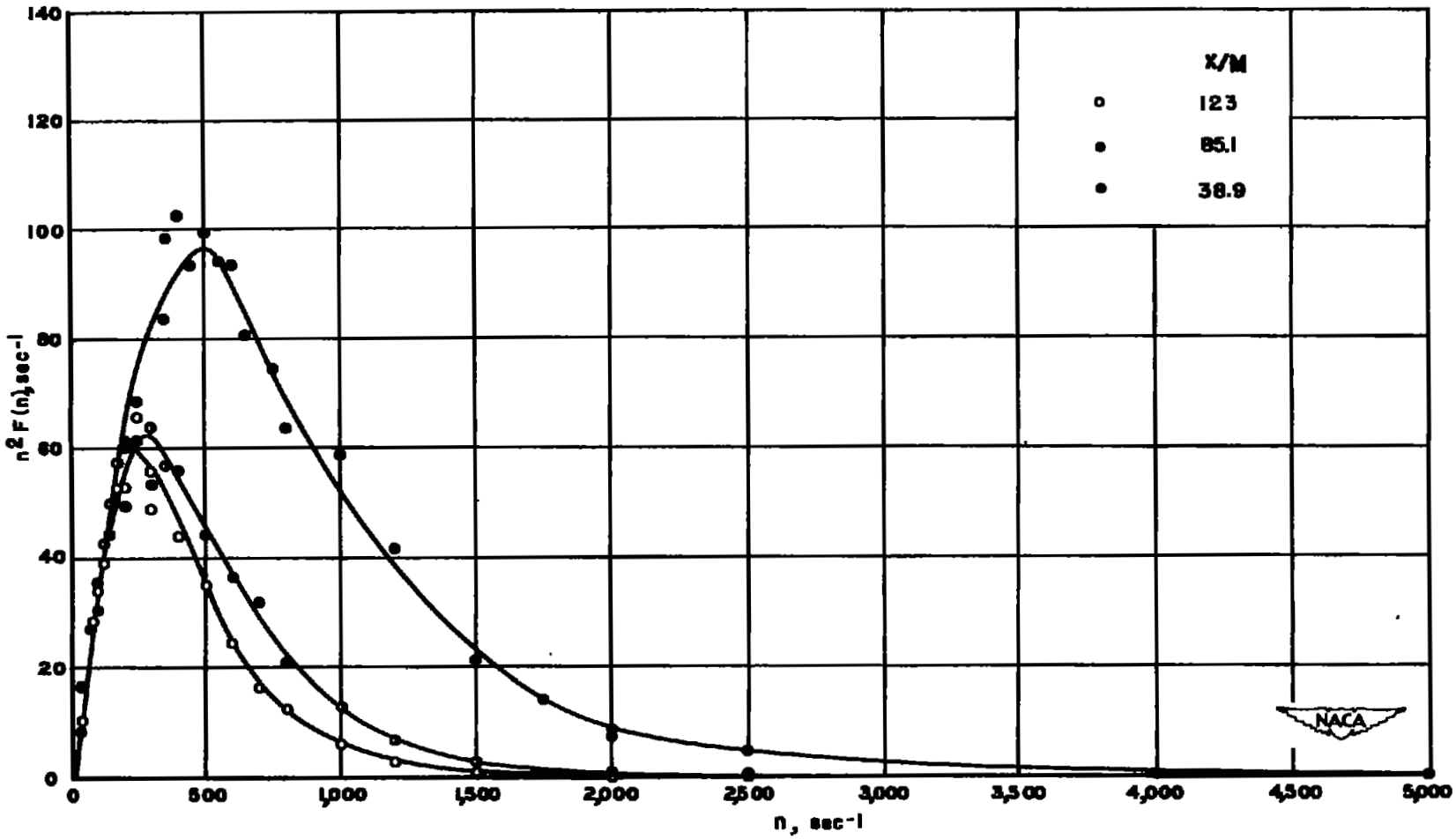

Figure 18.- Second moments. U = 630 centimeters per second; M = 1.68 centimeters.

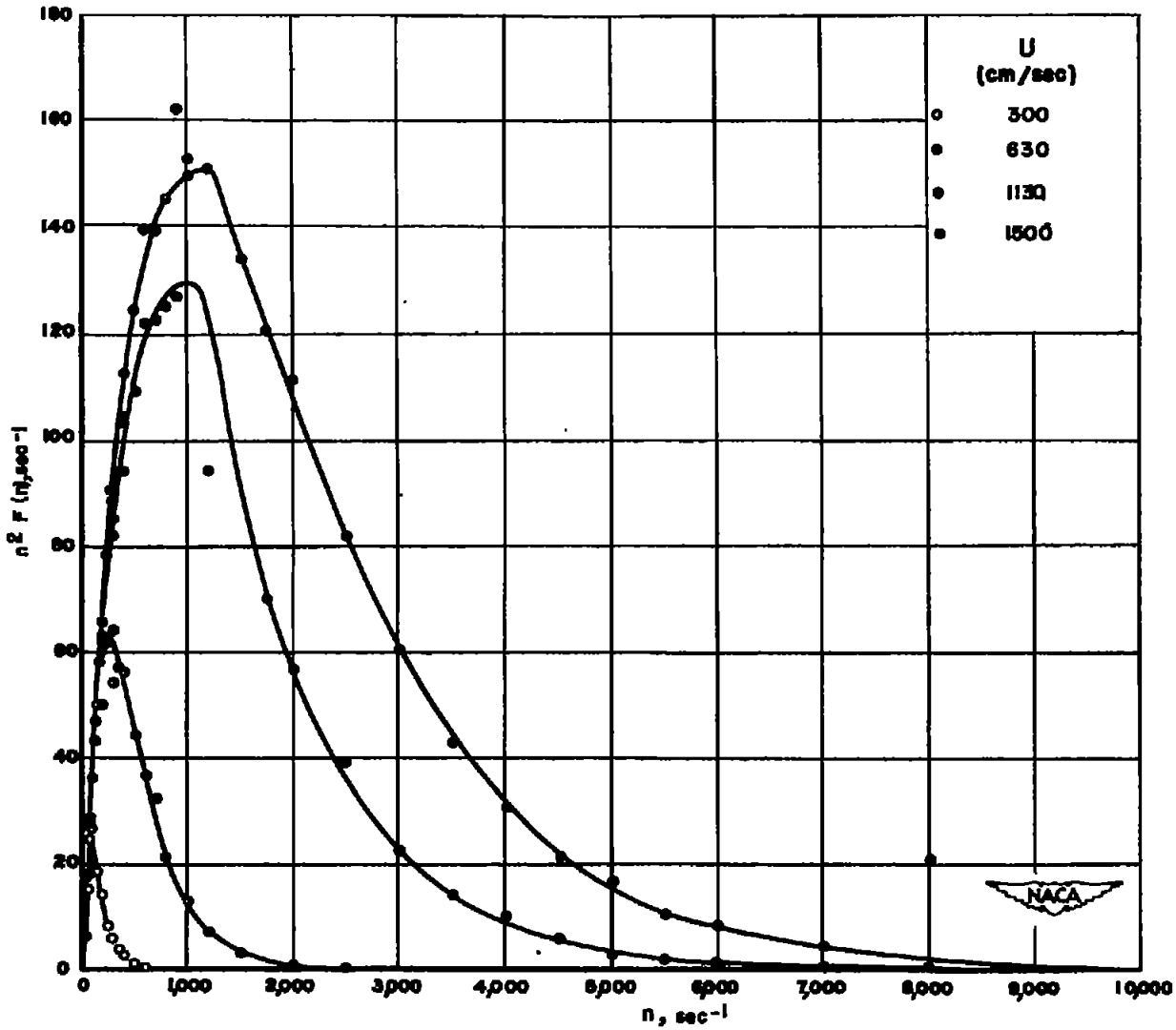

Figure 19.- Second moments. x/M, approximately 85.

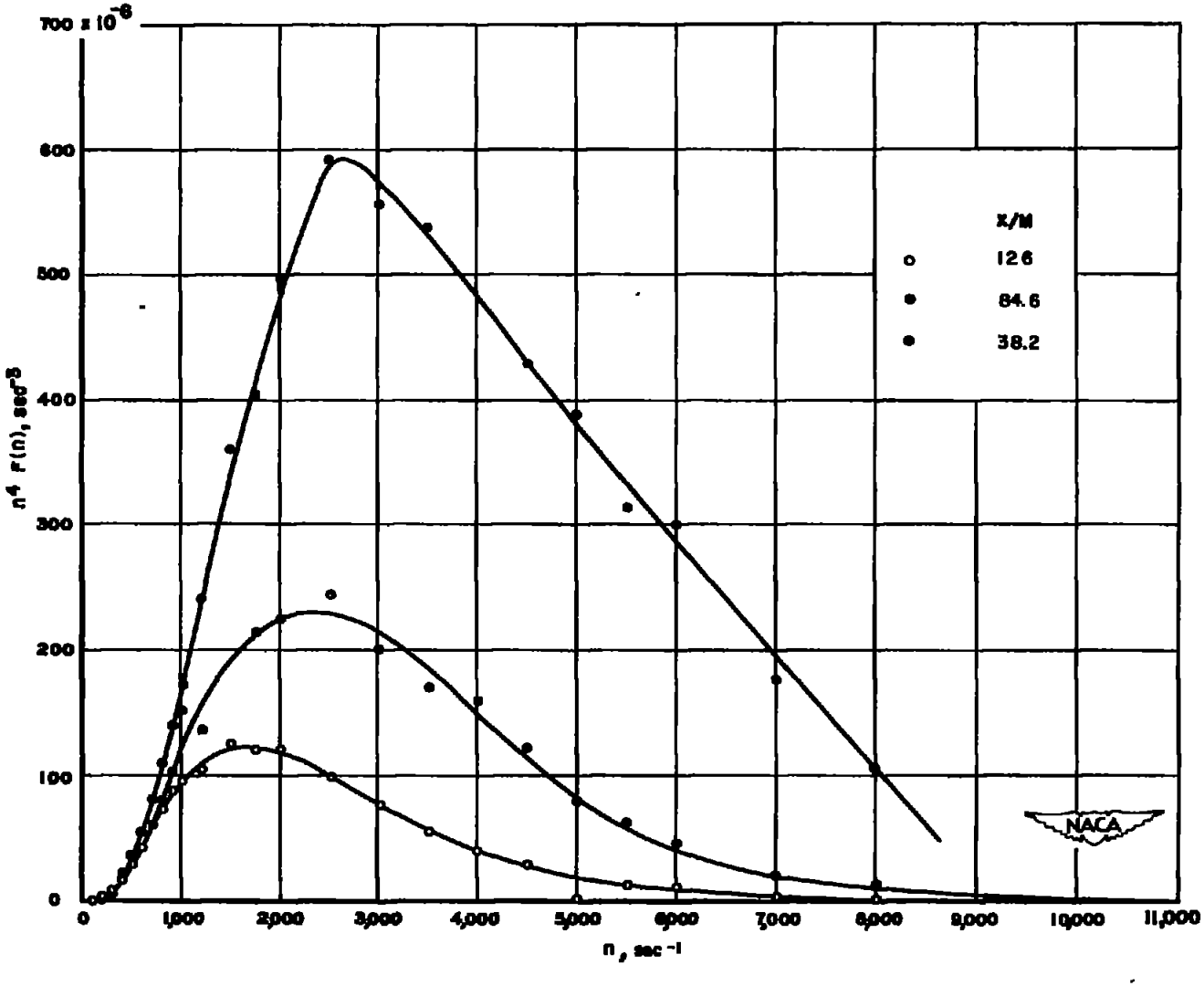

Figure 20.- Fourth moments. U = 1130 centimeters per second; M = 1.68 centimeters.

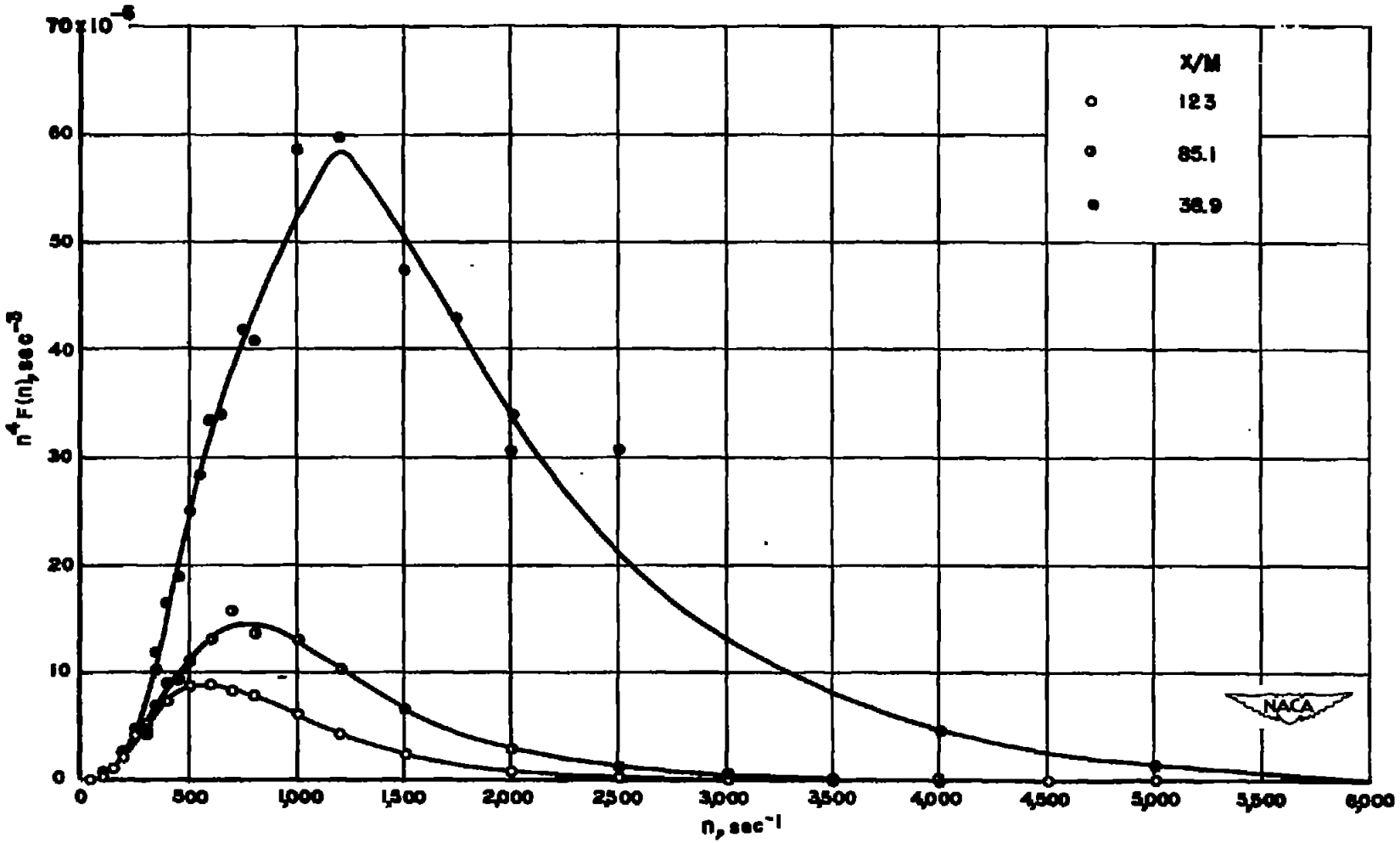

Figure 21.- Fourth moments. U = 630 centimeters per second; M = 1.68 centimeters.

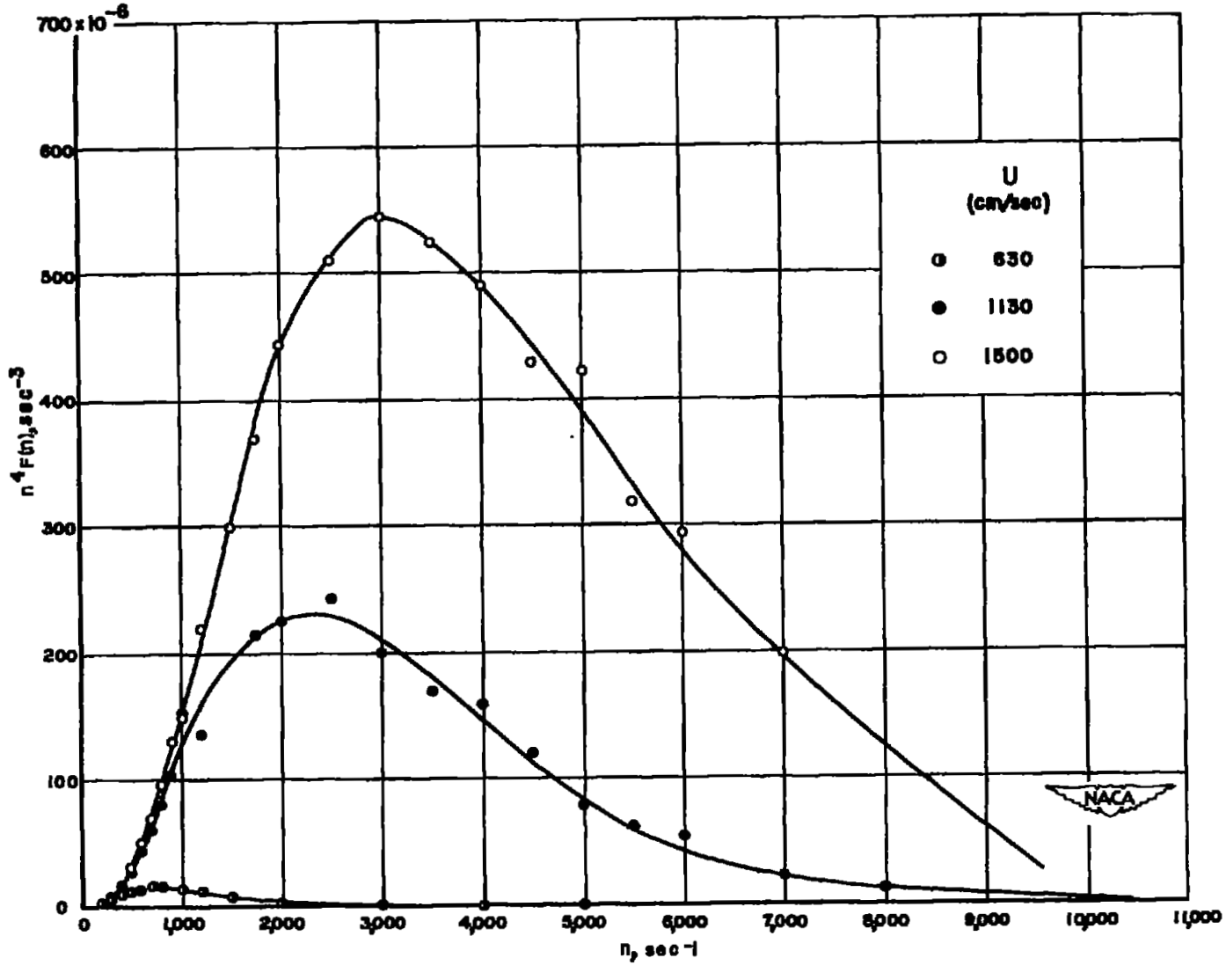

Figure 22.- Fourth moments. x/M = 85.

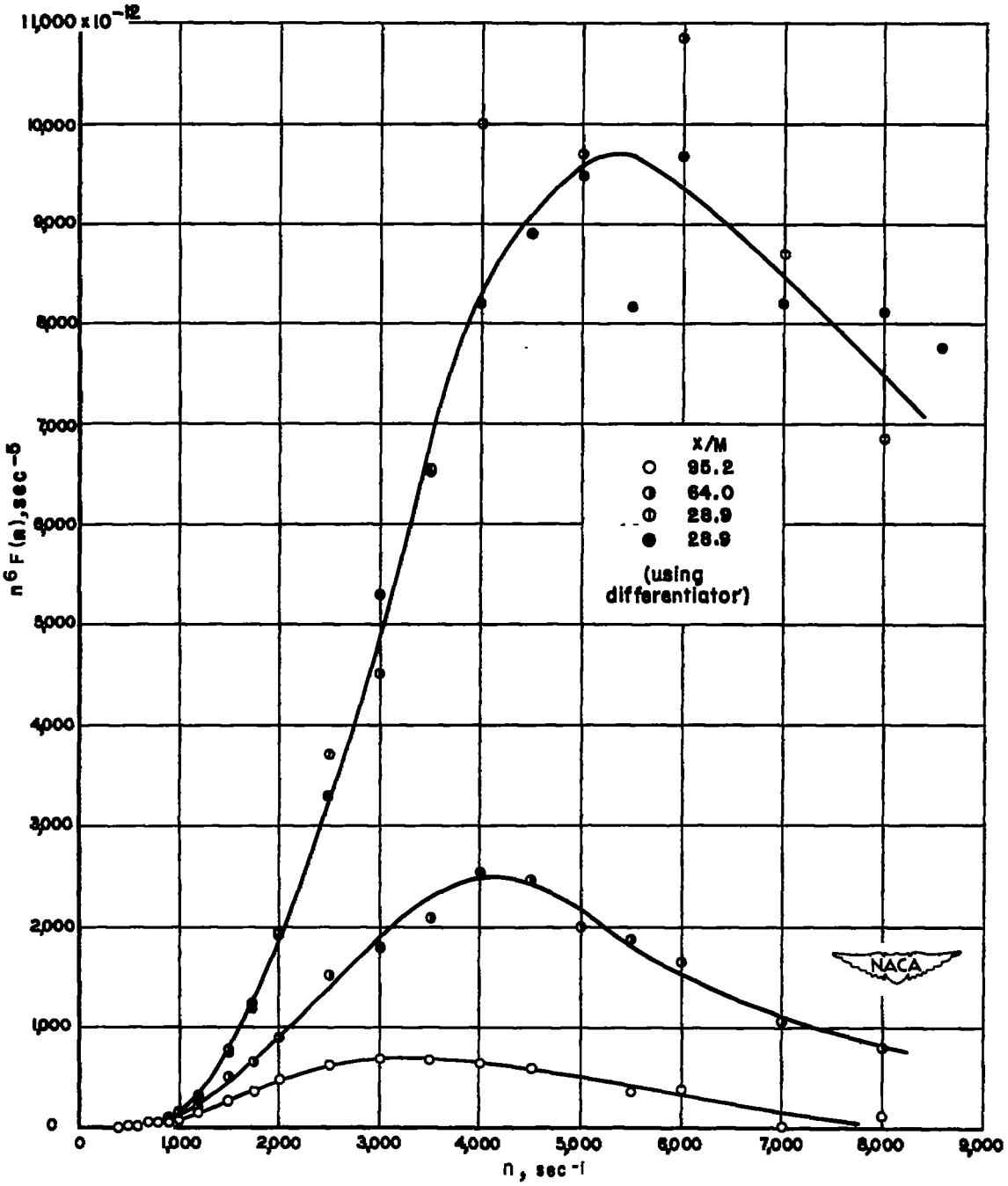

Figure 23.- Sixth moments. U = 1130 centimeters per second; M = 1.68 centimeters.

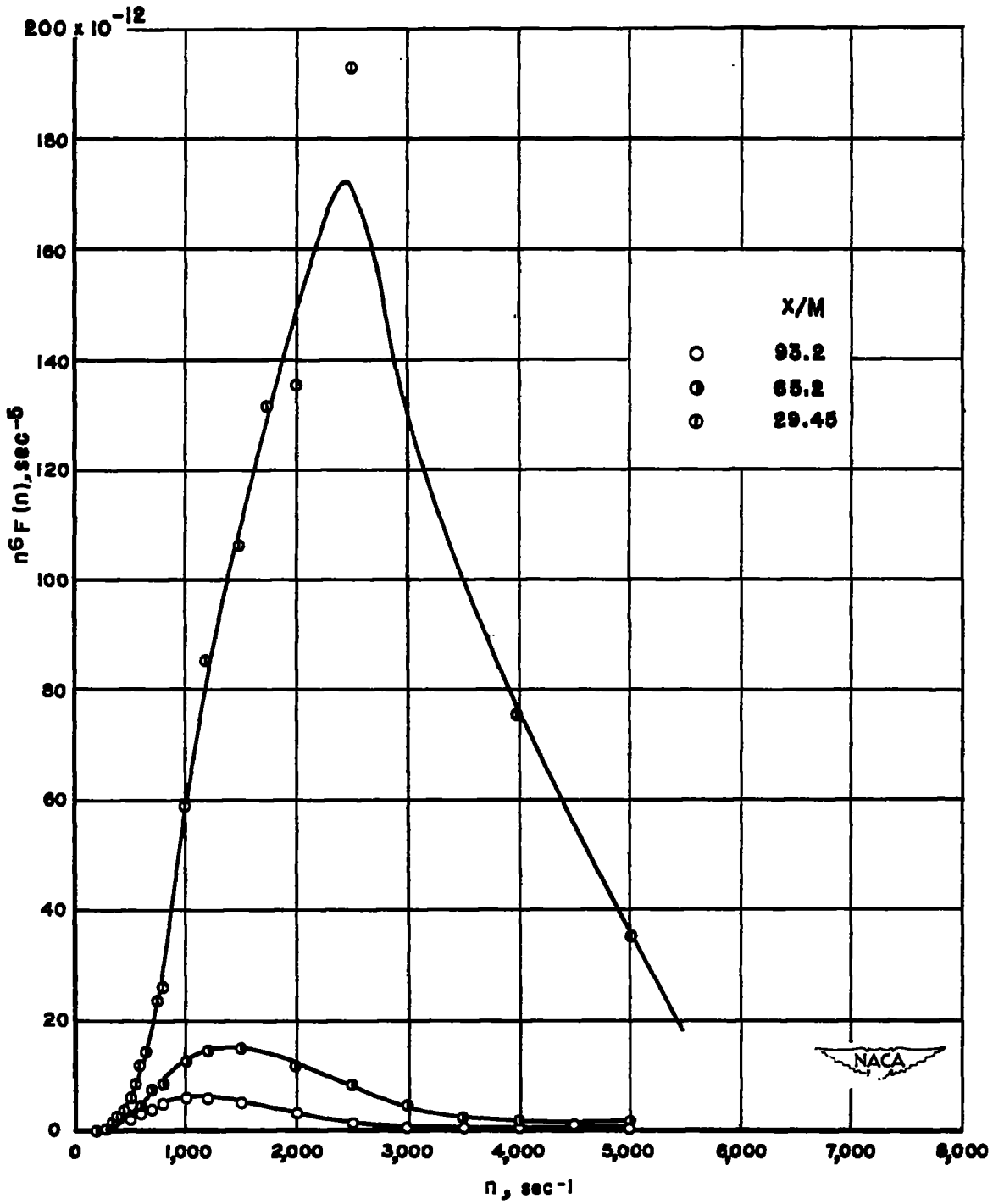

Figure 24.- Sixth moments. U = 630 centimeters per second; M = 1.68 centimeters.

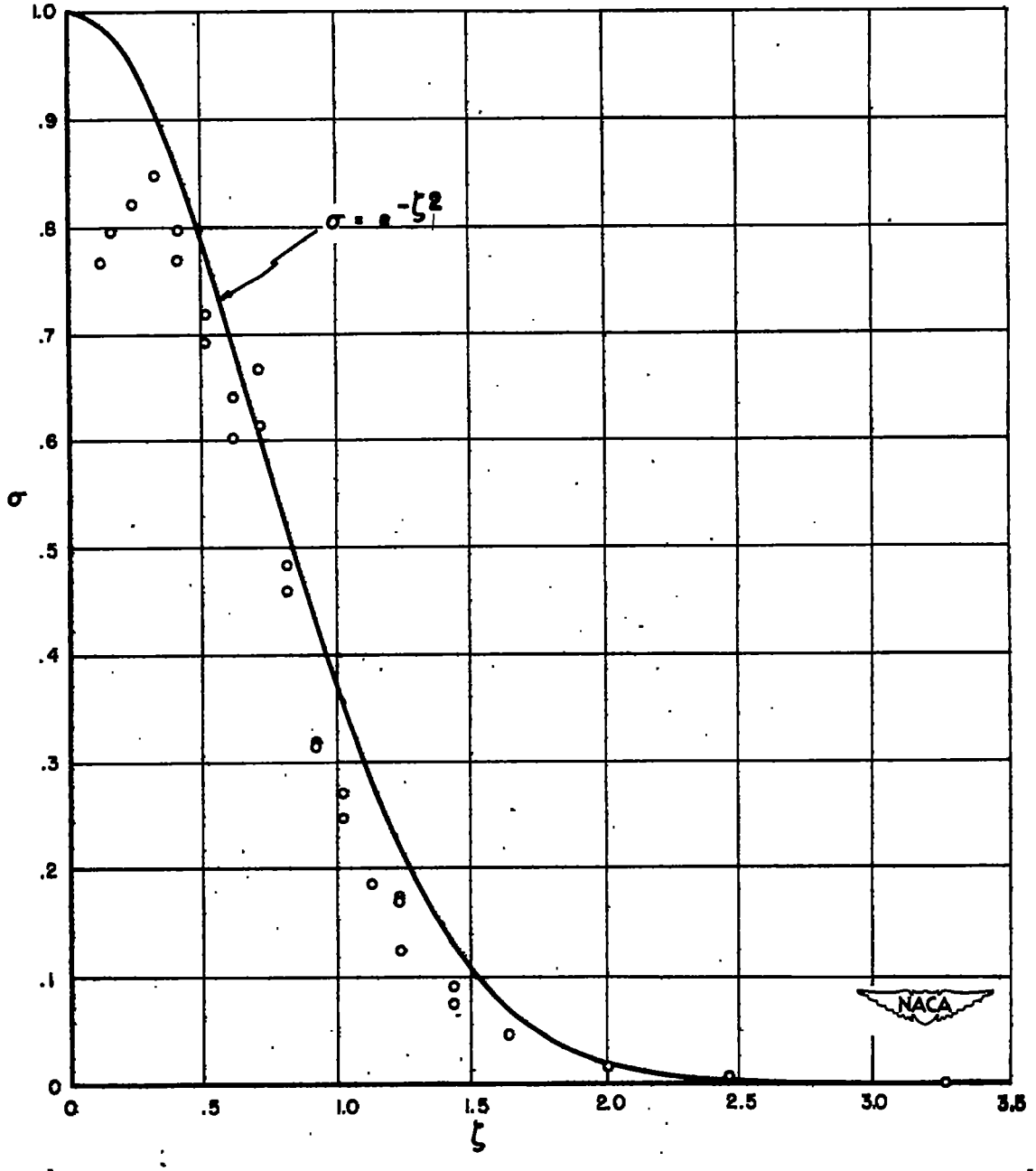

Figure 25.- Plot of measurements of spectrum of turbulence at $x/M = 1000$ behind a fine mesh grid and comparison with $e^{-\zeta^2}$. $U = 630$ centimeters per second; $M = 0.141$ centimeters.

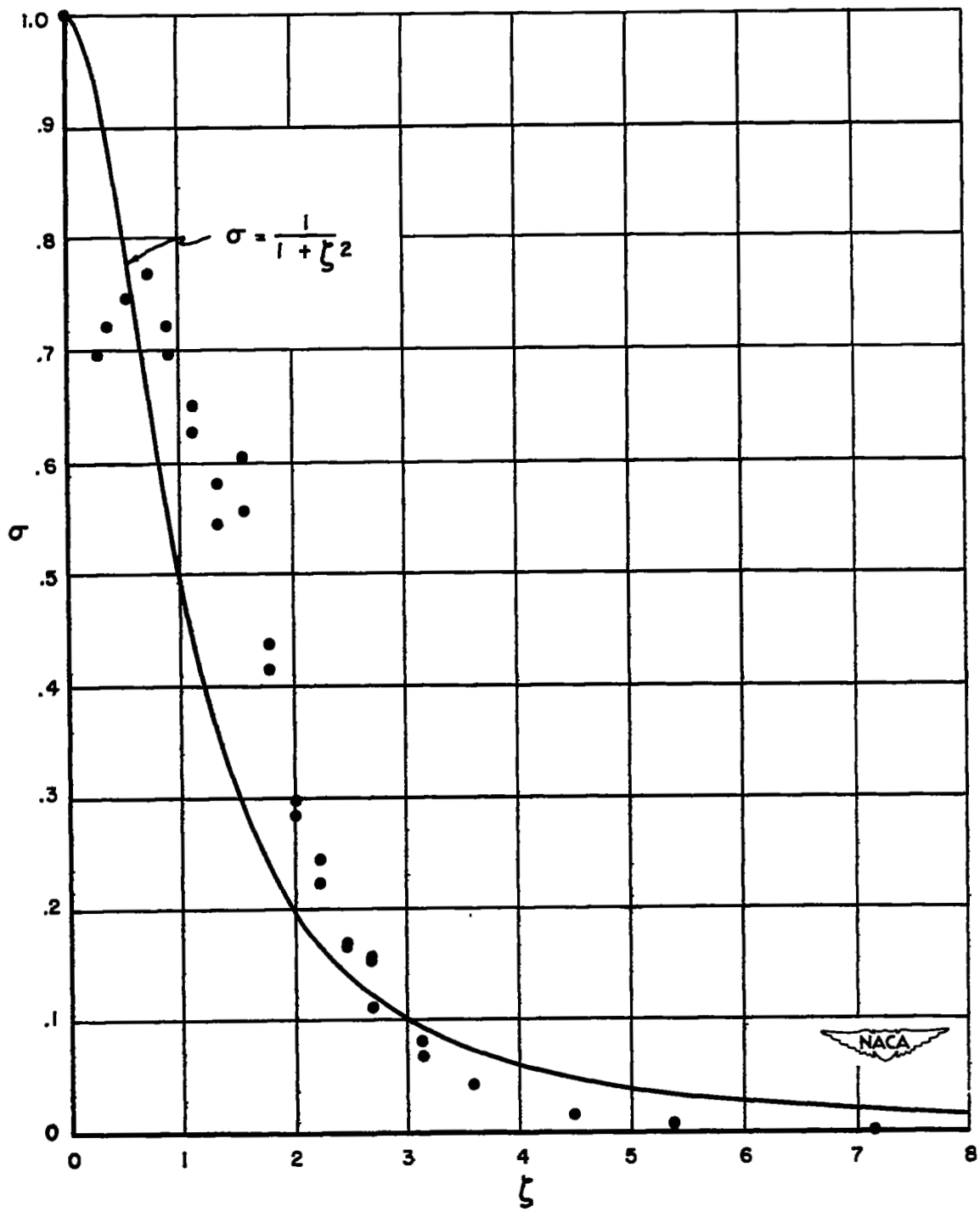

Figure 26.- Plot of measurements of spectrum of turbulence at $x/M = 1000$ behind a fine mesh grid and comprarison with $\frac{1}{1 + \zeta^2}$. $U = 630$ centimeters per second; $M = 1.41$ centimeters.

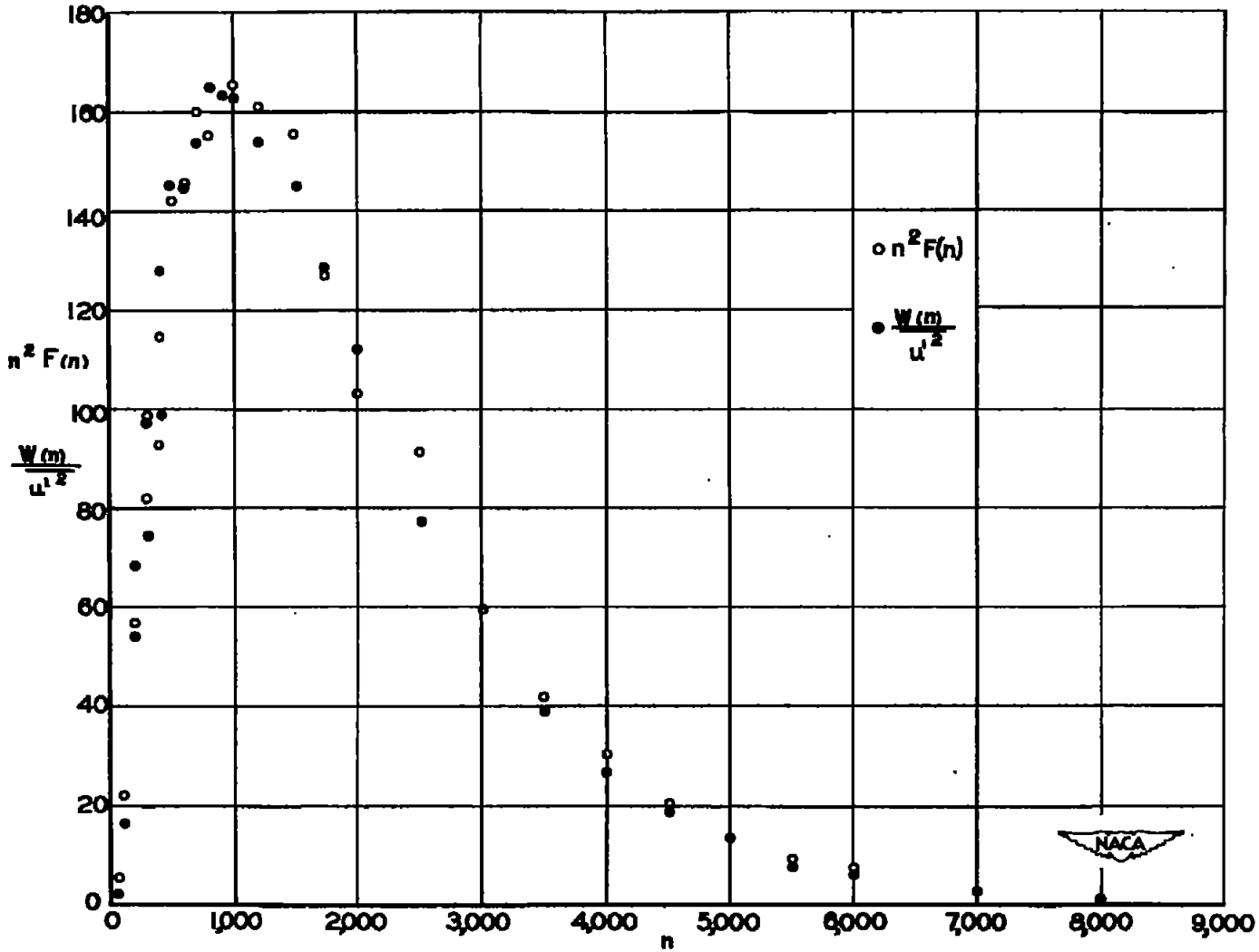

Figure 27.- Comparison of measurements of $W(n)$ and $n^2F(n)$.
$U = 1130$ centimeters per second; $x/M = 138.5$.

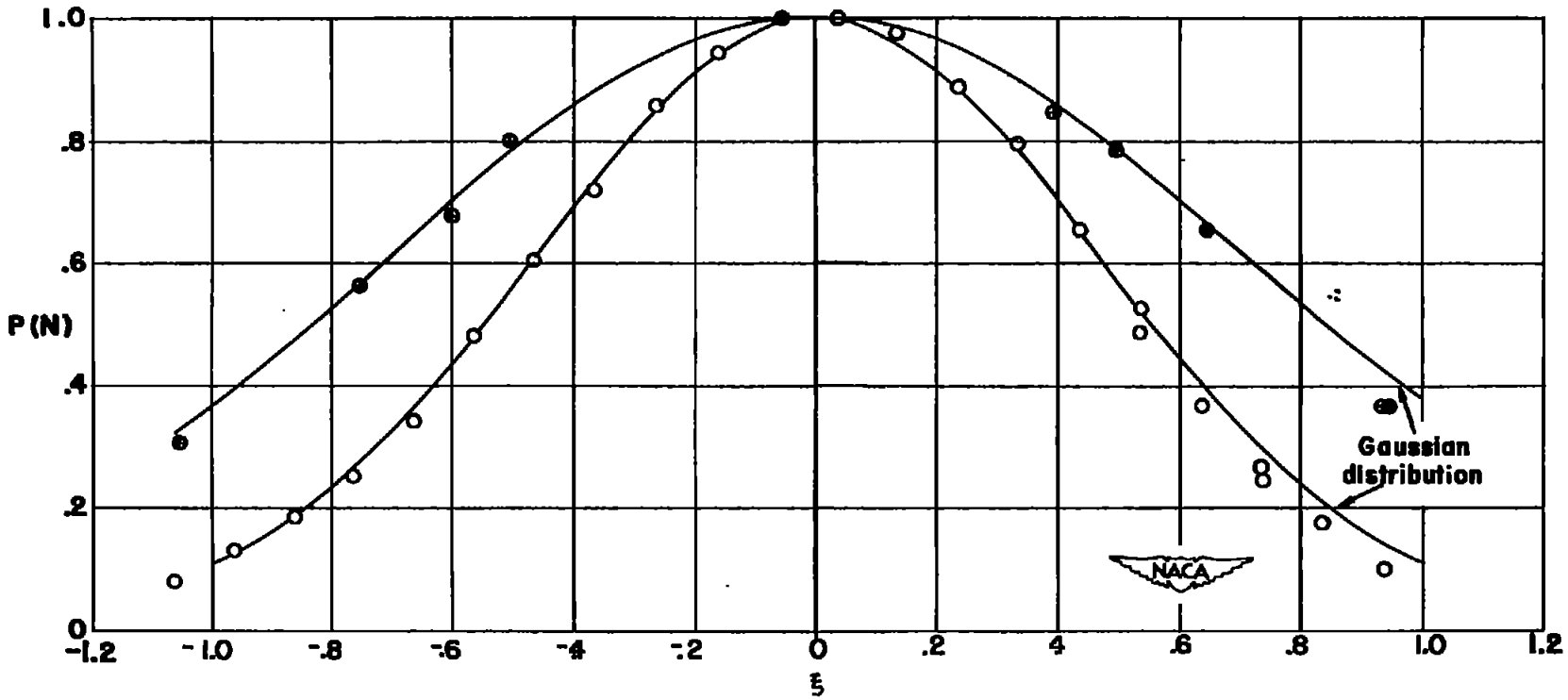

Figure 28.- Probability distribution of N_ξ for two turbulence intensities. U = 1133 centimeters per second; x/M = 138.5.