SIMPLICITY WITH RESPECT TO CERTAIN QUADRATIC FORMS

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1. *Simplicity*.—Let n denote an arbitrary integer >0, and m an odd interger >0. Write $N[n = f]$ for the number of representations of n in the quadratic form f when this number is finite. If $N[n = f]$ is a polynomial $P(n)$ in the real divisors of n alone, $N[n = f]$ is, by definition, *simple;* similarly for $N[m = f]$. In previous papers,[1,2,3] I have considered simplicity when f is a sum of an even number of squares. For an odd number >1 of squares, no $N[n = f]$ is simple, although if n be restricted to be a square, say k^2, then $N[k^2 = f]$ is simple when f is a sum of 3, 5 or 7 squares, and for one case (k restricted) of 11 squares.

To discuss simplicity for forms f in an odd number of variables, the above definition was extended in a former paper[4] as follows. The extension leads to theorems that are actually useful in calculations of numbers of representations, and it was devised with this end in view. Let Σ refer to all positive, zero or negative integers t which are congruent modulo M to a fixed integer T, and which render the arguments of the summand >0. Let $P(n)$ denote a polynomial in the real divisors alone of n, and let c be a constant integer >0. Then if $N[n = f]$ is of the form $A\Sigma P(n - ct^2)$, where A does not depend upon n, it is called *simple*.

In the previous paper,[4] an exhaustive discussion of simplicity was given for sums of 3, 5, 7, 9, 11 and 13 squares; in the present note, we state general theorems for simplicity with respect to pure quadratic forms $S_{2r} + 2^{a+2}S_1$, where S_j denotes a sum of j squares, and r, a are arbitrary constant integers >0. These theorems are the first of their kind, and are the first instances of the like for forms $S_{2r} + 2^{a+2}S_{2h-1}$, where r, h, a are arbitary constant integers >0. As the proofs are short, they are sufficiently indicated in §3.

2. *Theorems*.—The summations refer to all positive, zero or negative integers t such that the arguments of the summands are >0; S_j is as in §1, and a is an arbitrary constant integer $\geqq 0$.

THEOREM 1. *If $A(r)$ is a function of r alone, and $m \equiv 1$ mod 4, and if*

$$N[m = S_{4r} + 2^{a+2}S_1] = A(r)\Sigma\zeta_s(m - 2^{a+2}t^2), \qquad (1)$$

where $\zeta_s(n)$ denotes the sum of the sth powers of all the real positive divisors of n, then, necessarily,

$$s = 2r - 1, A(r) = 8r.$$

The only values of r for which (1) *holds are $r = 1, 2$.*

THEOREM 2. *If $A(r)$ is a function of r alone, and $m \equiv 3 \bmod 4$, and if*

$$N[m = S_{4r} + 2^{a+2}S_1] = A(r)\Sigma\, \zeta_s(m - 2^{a+2}t^2), \qquad (2)$$

then, necessarily,

$$s = 2r - 1,\ A(r) = \frac{32r(2r-1)(4r-1)}{3(3^{2r-1}+1)}.$$

The only values of r for which (2) holds are $r = 1, 2$.

THEOREM 3. *If $A(r)$ is a function of r alone, and $m \equiv 1 \bmod 4$, and if*

$$N[m = S_{4r+2} + 2^{a+2}S_1] = A(r)\Sigma\xi_s(m - 2^{a+2}t^2), \qquad (3)$$

where $\xi_s(n)$ denotes the sum of the sth powers of the real positive divisors of the form $4h + 1$ of n minus the like sum for the divisors of the form $4h + 3$, then, necessarily,

$$s = 2r, \quad A(r) = 8r + 4.$$

The only values of r for which (3) holds are $r = 0, 1$.

THEOREM 4. *If $A(r)$ is a function of r alone, and $m \equiv 3 \bmod 4$, and if*

$$N\,[m = S_{4r+2} + 2^{a+2}\,S_1] = A(r)\Sigma\xi_s(m - 2^{a+2}\,t^2), \qquad (4)$$

then, necessarily,

$$s = 2r,\ A(r) = -\frac{32r(2r+1)(4r+1).}{3(3^{2r}-1)}$$

The only values for which (4) holds are $r = 0, 1, 2$.

THEOREM 5. *If $A(r)$ is a function of r alone, and m is odd, and if*

$$N[2m = S_{4r+4} + 2^{a+2}\,S_1] = A(r)\Sigma\zeta_s(m - 2^{a+2}\,t^2), \qquad (5)$$

then, necessarily,

$$s = 2r + 1,\ A(r) = 8\,(r+1)(4r+3).$$

The only values of r for which (4) holds are $r = 0, 1, 2$.

It is to be noticed that Theorems 1–5 hold for each particular value of a. The next differs from the preceding in that the integer n represented is arbitrary. It does not include 1–5, and is independent of them.

THEOREM 6. *If $A(r)$ is a function of r alone, and n is an arbitrary integer >0, and if*

$$N[n = S_{2r} + bS_1] = A(r)\,\Sigma\,\phi\,(an-bt^2),$$

where a, b are constant integers >0, and ϕ is simple (as first defined), then, necessarily $r = 1, 2, 3$ or 4.

3. *Proofs.*—All follow immediately from the results for S_{4r}, S_{4r+2}, S_{4r+4}, S_{2r} in the papers[1,2,3] cited, on an obvious application of the next.

LEMMA 1. *If $n \equiv j$ mod 4, so that $n - 2^{a+2}t^2 \equiv j$ mod 4, and if*

$$N[n = S_j + 2^{a+2}t^2] = A\Sigma\psi(n - 2^{a+2}t^2), \tag{6}$$

where A is independent of n and ψ is simple as first defined, then (6) *implies*

$$N[n = S_j] = A\psi(n), \tag{7}$$

and (7) *implies* (6).

That (7) implies (6) is obvious. To see that (6) implies (7), equate the generating functions of the functions on the left and right of (6), to get an identity in the parameter q of the generating function. Divide both sides of this identity by

$$\Sigma q^{2^{a+2}t^2}(t = 0, \pm 1, \pm 2, \ldots),$$

and thus get the identity which implies (7). Clearly the lemma holds also for ψ not simple.

LEMMA 2. *If n is an arbitrary integer* >0, *and A is independent of n, each of the following implies the other,*

$$N[n = S_j + t^2] = A\Sigma\psi(n - t^2), N[n = S_j] = A\psi(n).$$

[1] E. T. Bell, *Bull. Am. Math. Soc.*, **35**, 695 (1929).

[2] E. T. Bell, *J. London Math. Soc.*, **4**, 279 (1929).

[3] E. T. Bell, *J. für die r.u.a. Math.*, to appear shortly.

[4] E. T. Bell, *Amer. J. Math.*, **42**, 168 (1920).

THE DIFFERENTIAL GEOMETRY OF A CONTINUOUS INFINITUDE OF CONTRAVARIANT FUNCTIONAL VECTORS[1]

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1. *Introduction.*—A general theory of function space affinely connected manifolds has been developed by the author in several publications.[2] In this paper I propose to give a number of new results pertaining to the differential geometry and invariant theory of a continuous infinitude of contravariant functional vectors. An application is made of these results to the differential geometry of functional group vectors of infinite groups of functional transformations. It is my intention to publish the complete results and proofs elsewhere.

2. *Infinitude of Contravariant Functional Vectors.*—Let $\eta_i[y]$ be a covariant functional vector, then the pair of functionals $\xi^{\alpha}[y]$, $\xi^{\alpha}_i[y]$ will