extended to other cases and can be developed into a powerful tool for examining permanent currents as well as changes produced by changing winds. Efforts in this direction are being continued.

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THE PROBLEMS OF CONGRUENT NUMBERS AND CONCORDANT FORMS

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1. Four Related Problems.—All letters in formulas denote rational integers, and solution means the *complete* solution in such integers. The problem of solving the simultaneous diophantine equations

$$rX^2 + mY^2 = rZ^2, \quad sX^2 + nY^2 = sW^2$$

includes as special cases two classical problems.

Problem 1.—If $r = s = Y^2 = 1$, $n = -m$, where m is a given constant, the problem is that of congruent numbers. It goes back to Diophantus in the third century, the Arabs of the tenth and eleventh centuries, and Leonardo of Pisa (Fibonacci) in the early thirteenth century. For m arbitrarily assigned it is still unsolved.

Problem 2.—For $r = s = 1$ the problem is Euler's (1780) of concordant forms, also unsolved.

Many special cases of these two have been investigated. Thus Fermat proved by his method of descent that if $m = n = -1$ in Problem 2, there are no integers X, Y, Z, W all different from zero satisfying the equations. From this his theorem for fourth powers follows. Modern work originating in these problems has been concerned with cubics and quartics having at most a finite number of sets of values of the indeterminates satisfying the equations. Some of this has used the theory of the units in special algebraic number rings. From the results it is possible, by the method applied to Problem 3, to derive much information on new diophantine systems of degrees higher than the second. This will be considered elsewhere. For the present, the inherent complexity of the solution of Problem 3 may suggest why these two old and apparently simple problems are still not completely solved.

Problem 3.—To state necessary and sufficient forms of r, m, s, n in order that there shall exist X, Y, Z, W all different from zero satisfying the equations.

A special case that has been frequently discussed may be noted. In Problem 1, the required form of m is given by

$$4m = xyz^2w^2(x^2 - y^2), \ w(x + y) \text{ even}.$$

The corresponding X, Z, W are given by

$$4X = zw(x^2 + y^2), \ 4Z = zw(x^2 + 2xy - y^2), \ 4W = zw(x^2 - y^2).$$

For m squarefree, $zw = \pm 1$, giving a known criterion. The proof is immediate by the method used for solving Problem 3. Although it is not included in Problem 3, another, somewhat similar problem, dating from the Arabs and usually included with questions on congruent numbers is

Problem 4.—To state a necessary and sufficient form of n in order that X, Y, Z all different from zero shall exist satisfying

$$n + X^2 = Y^2, \ n - X^2 = Z^2.$$

The solution is given by

$$4n = x^2(a^2y^4 + b^2z^4), \ ab = 2;$$

the corresponding X, Y, Z are given by

$$X = xyz, \ 2Y = x(ay^2 + bz^2), \ 2Z = x(ay^2 - bz^2).$$

This is equivalent to

$$ab = 2, \ fgh^2 = 4, \ a^2y^4 + b^2z^4 = fu, \ n = guv^2;$$

$$X = ghyzv, \ 2Y = ghv(ay^2 + bz^2), \ 2Z = ghv(ay^2 - bz^2).$$

2. *Solution of Problem 3.*—If r, m, s, n, X, Y, Z, W are indeterminates, the equations are homogeneous cubics, each of which is separable and hence (completely) solvable. The result of equating the parametric expressions for X and those for Y in the solutions gives a separable and hence (completely) solvable system. As the solution of separable equations,

or of a system of such equations, is now straightforward routine, it will suffice to state the final result. To condense the formulas, write

$$
\begin{aligned}
&a \equiv a_1a_2a_3a_4a_5, \; b \equiv b_1b_2b_3b_4b_5, \; c \equiv c_1c_2c_3c_4c_5,\\
&f \equiv f_1f_2f_3f_4f_5, \; g \equiv g_1g_2g_3g_4g_5, \; h \equiv h_1h_2h_3h_4h_5,\\
&\alpha \equiv b_1c_1f_1g_1h_1, \; \beta \equiv a_1c_2f_2g_2h_2, \; \gamma \equiv a_2b_2f_3g_3h_3,\\
&\theta \equiv a_3b_3c_3g_4h_4, \; \phi \equiv a_4b_4c_4f_4h_5, \; \psi \equiv a_5b_5c_5f_5g_5;\\
&\pi \equiv abcfgh, \; m \equiv pm_1m_2, \; n \equiv tn_1n_2.
\end{aligned}
$$

Thus p, m_1, m_2 are bound parameters whose product is m; similarly for t, n_1, n_2 and n. The $a_i, \ldots, g_i$ are independent parameters. Define

$$
\begin{aligned}
A &\equiv m_1afg^2 - n_1\alpha\theta\phi^2, \; B \equiv m_2bfh^2 - n_2\beta\theta\psi^2,\\
C &\equiv m_2n_1bh^2\alpha\phi^2 - m_1n_2ag^2\beta\psi^2,
\end{aligned}
$$

introduce the parameters x, y, z and define e, for assigned values of all the parameters, as an arbitrary integer multiple of the reciprocal of the greatest common divisor of

$$xy^2A, \; xz^2B, \; y^2z^2C.$$

(If e is merely an arbitrary integer, the values of r, s, X, Y, Z, W stated presently, with p, m_1, m_2, t, n_1, n_2 as above, satisfy the equations identically, but this does not exhaust the possibilities. The stated definition of e is necessary.) Introduce the parameter u. The required values of r, s are

$$r = e^2pu^2x^2y^2z^2abc^2AB, \; s = e^2tu^2x^2y^2z^2\alpha\beta\gamma^2AB.$$

To state the corresponding values of X, Y, Z, W define

$$
\begin{aligned}
F &\equiv m_2n_1bh^2\alpha\phi^2 - ag^2\beta\psi^2\\
G &\equiv 2m_1m_2abfg^2h^2 - m_1n_2ag^2\beta\theta\psi^2 - m_2n_1bh^2\alpha\theta\phi^2,\\
H &\equiv m_1n_2afg^2\beta\psi^2 + m_2n_1bfh^2\alpha\phi^2 - 2n_1n_2\alpha\beta\theta\phi^2\psi^2.
\end{aligned}
$$

Introduce a parameter k. Then

$$
\begin{aligned}
2X &= ekx^2y^2z^2f\theta F, \; Y = e^2kux^2y^2z^2\pi AB,\\
2Z &= ekx^2y^2z^2fG, \; 2W = ekx^2y^2z^2\theta H.
\end{aligned}
$$

Including the bound parameters p, m_1, m_2, t, n_1, n_2 there are in all 41. Each of m, n is of degree 3; each of r, s, is of degree 71; each of X, Z, W is of degree 49, and Y is of degree 83. The degree of each of the identities giving the solution of the cubic system is thus 169.