Stress Singularities Resulting From Various Boundary Conditions in Angular Corners of Plates in Extension

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As an analog to the bending case published in an earlier paper, the stress singularities in plates subjected to extension in their plane are discussed. Three sets of boundary conditions on the radial edges are investigated: free-free, clamped-clamped, and clamped-free. Providing the vertex angle is less than 180 degrees, it is found that unbounded stresses occur at the vertex only in the case of the mixed boundary condition with the strength of the singularity being somewhat stronger than for the similar bending case. For vertex angles between 180 and 360 degrees, all the cases considered may have stress singularities.

In amplification of some work of Southwell, it is shown that there are certain analogies between the characteristic equations governing the stresses in extension and bending, respectively, if ν, Poisson's ratio, is replaced by $-\nu$. Finally, the free-free extensional plate behaves locally at the origin exactly the same as a clamped-clamped plate in bending, independent of Poisson's ratio.

In conclusion, it is noted that the free-free case analysis may be applied to stress concentrations in V-shaped notches.

Introduction

In an earlier paper[2] the stress singularities at the vertex of a sector plate in bending were discussed, where it was shown that unbounded stresses may occur for certain vertex angles if the radial edges of the sector were (1) simply supported-simply supported, (2) clamped-free, (3) simply supported-free, or (4) clamped-simply supported. The analogous problem for a thin plate in a state of plane stress or strain is now considered for three cases, (*a*) clamped-clamped, (*b*) free-free, and (*c*) clamped-free, and it will be shown that of these three cases, only the last, namely, the clamped-free boundary condition, may have unbounded stresses at the corner for certain vertex angles.

Method of Solution

Following the extensional development for generalized plane stress in thin plates as given by Coker and Filon[3] the polar displacements in the radial and circumferential directions, U_r and U_θ, respectively, may be written in terms of the biharmonic stress function χ and the harmonic function ψ_1 in the form

$$2\mu U_r = -\frac{\partial\chi}{\partial r} + (1-\sigma)r\frac{\partial\psi_1}{\partial\theta} \quad [1]$$

$$2\mu U_\theta = -\frac{1}{r}\frac{\partial\chi}{\partial\theta} + (1-\sigma)r^2\frac{\partial\psi_1}{\partial r} \quad [2]$$

where μ is the shear modulus and $\sigma \equiv \nu/(1+\nu)$, ν being Poisson's ratio. Also, the stresses may be expressed as

$$\sigma_r = \frac{1}{r^2}\frac{\partial^2\chi}{\partial\theta^2} + \frac{1}{r}\frac{\partial\chi}{\partial r} \quad [3]$$

$$\sigma_\theta = \frac{\partial^2\chi}{\partial r^2} \quad [4]$$

$$\tau_{r\theta} = -\frac{1}{r}\frac{\partial^2\chi}{\partial r\partial\theta} + \frac{1}{r^2}\frac{\partial\chi}{\partial\theta} \quad [5]$$

χ is related to ψ_1 by

$$\nabla^2\chi = \frac{\partial}{\partial r}\left(r\frac{\partial\psi_1}{\partial\theta}\right) \quad [6]$$

The boundary conditions will be considered on two radial edges of a sector including a variable vertex angle α; the boundary conditions along the circumferential edge will be unspecified inasmuch as the stresses near the vertex will be locally determined solely by the boundary conditions along the radial edge providing the circumferential boundary is greater than several plate thicknesses from the vertex.

Let us assume solutions for χ and ψ_1 in the form

$$\chi = r^{\lambda+1}[b_1\sin(\lambda+1)\theta + b_2\cos(\lambda+1)\theta + b_3\sin(\lambda-1)\theta + b_4\cos(\lambda-1)\theta] \equiv r^{\lambda+1}F(\theta;\lambda) \quad [7]$$

$$\left.\begin{aligned}\psi_1 &= r^m(a_1\cos m\theta + a_2\sin m\theta)\\ &\equiv r^m G(\theta; m)\end{aligned}\right\} \quad [8]$$

On account of Equation [6], it must follow that $\lambda = m + 1$ in order that the powers of r be compatible and further

$$a_1 = -\frac{4}{\lambda-1}b_3;\quad a_2 = \frac{4}{\lambda-1}b_4 \quad [9]$$

by equating coefficients of like trigonometric terms.

Substitution of Equations [7] and [8] into [1], through Equation [5] utilizing Equation [9] gives

$$2\mu U_r = r^\lambda[-(\lambda+1)F(\theta) + (1-\sigma)G'(\theta)] \quad [10]$$

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[2] "Surface Singularities Resulting From Various Boundary Conditions in Angular Corners of Plates Under Bending," by M. L. Williams, U. S. National Congress of Applied Mechanics, Illinois Institute of Technology, Chicago, Ill., June, 1951.

[3] "A Treatise on Photoelasticity," by E. G. Coker and L. G. N. Filon, Cambridge University Press, 1931, p. 163.

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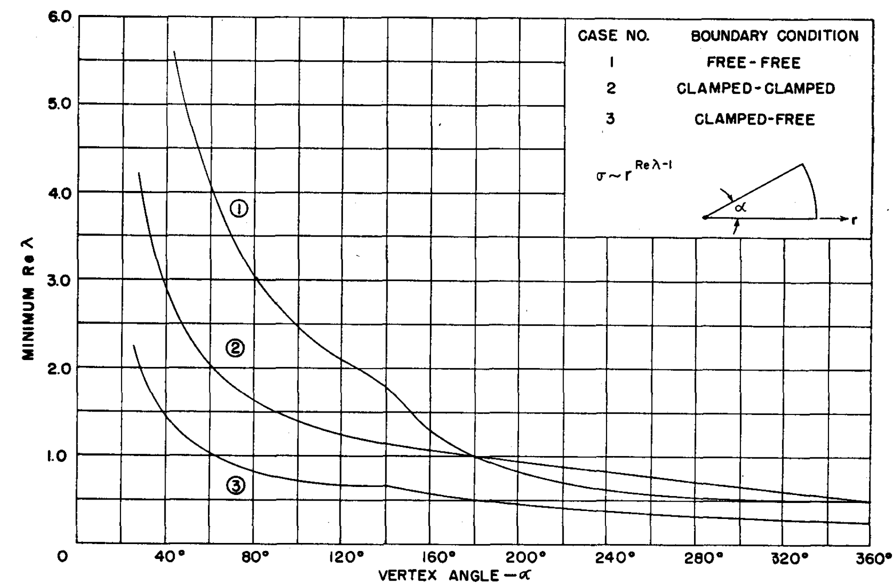

Fig. 1 Variation of Minimum Real Part of Eigen Value With Vertex Angle

$$2\mu U_\theta = r^\lambda[-F'(\theta) + (1 - \sigma)(\lambda - 1)G(\theta)] \ldots\ldots [11]$$

$$\sigma_r = r^{\lambda-1}[F''(\theta) + (\lambda + 1)F(\theta)] \ldots\ldots [12]$$

$$\sigma_\theta = r^{\lambda-1}[\lambda(\lambda + 1)F(\theta)] \ldots\ldots [13]$$

$$\tau_{r\theta} = r^{\lambda-1}[-\lambda F'(\theta)] \ldots\ldots [14]$$

Having therefore a representation of the stresses and displacements for the sector plate in extension, the effects of different homogeneous boundary conditions may be investigated.

Free-Free Edges

If both radial edges are prescribed to be free, $\sigma_\theta(0) = \tau_{r\theta}(0) = 0$, $\sigma_\theta(\alpha) = \tau_{r\theta}(\alpha) = 0$ which by Equations [13] and [14] require $F(0) = F'(0) = F(\alpha) = F'(\alpha) = 0$ where the primes indicate differentiation with respect to θ. While $F(\theta)$ contains four undetermined constants $b_i (i = 1, 2, 3, 4)$, and four boundary conditions are prescribed, the constants cannot be determined uniquely because the four equations are homogeneous. For there to be a solution, the determinant must vanish. In the case of the plate with free-free edges, this eigen equation is

$$\sin z = C_1 z \ldots\ldots [15]$$

where $z = \lambda\alpha$ and $C_1 = \pm(\sin \alpha)/\alpha$. Equation [15] determines the value or values of the eigen value λ as a function of α.

Inasmuch as the continuity of displacements requires $\lambda > 0$, any value of Equation [15] satisfying this restriction will give an admissible solution for the extensional problem. Furthermore, as may be seen from Equations [12] to [14], a value of λ such that $0 < \lambda < 1$ will give unbounded stresses near the vertex of the sector. The purpose of this paper is to investigate the occurrence of any values of λ (or its real part if λ is complex) which lie in this region. Stated alternatively, it is desired to find the minimum root of Equation [15] with the restriction that $\lambda > 0$.

The results, showing min Re λ as a function of α for Poisson's ratio $\nu = 0.3$, ($\sigma = {}^3/_{13}$) are presented in Fig. 1.

Clamped-Clamped Edges

Fixed edges at $\theta = 0$ and $\theta = \alpha$ require that $U_r(0) = U_\theta(0) = 0$, $U_r(\alpha) = U_\theta(\alpha) = 0$. Upon using Equations [10] and [11] and again formulating the four simultaneous homogeneous equations, the eigen equation in this case again becomes

$$\sin z = C_2 z \ldots\ldots [16]$$

where $z = \lambda\alpha$ and

$$C_2 = \pm \frac{\sin \alpha}{(3 - 4\sigma)\alpha}$$

The min Re$\lambda(\alpha)$ for this case are also shown in Fig. 1 for $\nu = 0.3$.

Clamped-Free

The mixed boundary condition of a clamped-free plate in extension requires $\sigma_\theta(0) = \tau_{r\theta}(0) = U_r(\alpha) = U_\theta(\alpha) = 0$. Using Equations [10], [11], [13], and [14] the eigen equation is found to be of somewhat different form, namely

$$\sin^2 z = \left[\frac{4(1 - \sigma)^2}{3 - 4\sigma}\right] - \left[\frac{\sin^2 \alpha}{(3 - 4\sigma)\alpha^2}\right] z^2 \ldots\ldots [17]$$

The minimum values of the eigen parameter for this case are also shown in Fig. 1 as a function of vertex angle for $\nu = 0.3$.

Comparison of Results

A study of Fig. 1 reveals that of the three cases considered, only the plate with mixed boundary conditions, namely, clamped-free, may have a singularity if the vertex angle lies between 63 and 180 degrees.

As discussed elsewhere,[2] this does not mean that a singularity must occur, because other higher eigenvalues from Equation [17], at 90 deg, for example, will not introduce singular behavior at the origin. On the other hand, an arbitrary loading along the circumferential boundary formed by a linear combination of the eigen functions in the manner suggested by the author[4] will contain the eigen function giving rise to the singularity and hence locally in the vicinity of the origin introduce unbounded stresses.

For vertex angles between 180 and 360 degrees, all the cases considered may have stress singularities.

[4] "Theoretical and Experimental Effect of Sweep Upon the Stress and Deflection Distribution in Aircraft Wings of High Solidity. Part 6—The Plate Problem for a Cantilever Sector of Uniform Thickness," by M. L. Williams, AFTR 5761, November, 1950.

TABLE 1 EIGEN EQUATIONS FOR BENDING AND EXTENSION

Case no.	Boundary condition	Eigen equation	Loading	Constants
1	Free-free	$\sin z = C_1 z$	Bending	$C_1 = \pm \frac{1-\nu}{3+\nu}\frac{\sin\alpha}{\alpha}$
			Extension	$C_1 = \pm \frac{\sin\alpha}{\alpha}$
2	Clamped-clamped	$\sin z = C_2 z$	Bending	$C_2 = \pm \frac{\sin\alpha}{\alpha}$
			Extension	$C_2 = \pm \frac{1+\nu}{3-\nu}\frac{\sin\alpha}{\alpha}$
3	Clamped-free	$\sin^2 z = K_1^2 - K_2^2 z^2$	Bending	$K_1 = \pm \left[\frac{4}{(3+\nu)(1-\nu)}\right]^{1/2}$
				$K_2 = \pm \left[\frac{1-\nu}{3+\nu}\right]^{1/2}\frac{\sin\alpha}{\alpha}$
			Extension	$K_1 = \pm \left[\frac{4}{(3-\nu)(1+\nu)}\right]^{1/2}$
				$K_2 = \pm \left[\frac{1+\nu}{3-\nu}\right]^{1/2}\frac{\sin\alpha}{\alpha}$

Another conclusion can be drawn from a comparison of the extensional and bending results. For easy reference a table giving the eigen equations for both extension and bending is herein as Table 1. It is seen that the eigen equations for the free-free extensional case are exactly the same as for the clamped-clamped bending case which means that locally a clamped bent plate behaves as a free extensional plate.

For the cases of generalized plane stress the eigen equations for the free-free bending and clamped-clamped extension cases, and for the clamped-free bending and extension cases are the same if ν is replaced by $-\nu$. Southwell noted this association of Poisson's ratio between the bending and extension cases[5] and, furthermore, derived the correspondence of the boundary conditions. The point to be observed here is that the characteristic equation for the eigen values of the stress and deformation behavior are also analogous in the two cases.

In conclusion it is suggested that the analysis for the free-free case be applied to determine the type of stress variation at the base of V-shaped notches in plates or rods.

[5] "On the Analogues Relating Flexure and Extension of Flat Plates," by R. V. Southwell, *Quarterly Journal of Mechanics and Applied Mathematics*, vol. 3, part 3, September, 1950, pp. 257–270.