A FAST, CONVENIENT PROGRAM FOR COMPUTATION OF SURFACE-WAVE DISPERSION CURVES IN MULTILAYERED MEDIA

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ABSTRACT
Surface wave analysis has become an important tool for exploration of crustal and mantle structure. The need exists for fast, convenient digital computer programs for computing theoretical dispersion curves and displacements for Rayleigh waves and Love waves. One such program for an IBM 7090 computer is described and made available to the scientific community. Among the conveniences are mail-order service, high speed, and choice of many options.

INTRODUCTION
In recent years, surface waves have been used extensively to determine crustal and upper mantle structure. More recently, surface waves have been used to infer some properties of the source. The introduction of electronic digital computers in determining dispersion curves in multilayered media is responsible for the renewed effort in this field (Stoneley, 1955; Dorman, Ewing, and Oliver, 1960). Whereas previously, tedious hand calculations for oversimplified models were necessary to interpret dispersion data, the use of computers permits rapid calculation of dispersion curves for realistic models. Dorman et al (1960) demonstrated that the accuracy of the multilayered approximation for continually varying elastic-depth functions is adequate when the number of homogeneous layers used to represent the function is sufficiently large (say, 20 layers to represent the outermost 300 km of the earth).

With the wider distribution of long period seismographs now taking place, the importance of surface wave dispersion analysis as a tool for geophysical exploration will increase further. Fast, convenient programs for computation of the dispersion curves will be needed. In this paper we make available our results for such a program based on our experience in coding the dispersion problem for three different computers.

We first programmed the dispersion computation for our small computer, the Bendix G-15D. Although this program gave the required precision, the speed was not high enough for large scale use (it required an overnight, unattended run for a typical 20 layer Rayleigh wave dispersion calculation). Much was learned from this initial attempt and a subsequent one written for an IBM 704 computer. Our final program is for the IBM 7090 computer.

Among the conveniences of the program are the following: speed, ability to specify one or more Love or Rayleigh modes, automatic layer reduction, optional determination of displacements at depth, optional insertion of approximate correction due to curvature of earth, root prediction for increasing speed, alternate group velocity computation methods, and ability to handle laminated plates and combinations of solid and liquid layers.

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Method

As for Dorman et al. (1960), the theoretical basis for this program is the Thomson-Haskell matrix formulation (Haskell, 1953) for the dispersion of Rayleigh and Love type surface waves on a plane layered elastic media. In the Thomson-Haskell formulation the fundamental relationship is the motion stress vector equation relating the vector at one side of an elastic solid layer to the vector at the other side of the same layer, or

\[ U_m = a_m^R U^*_{m-1} \quad V_m = a_m^L V^*_{m-1} \]  

(1)

where

\[ U_m = (\hat{u}_m / c, \hat{w}_m / c, \sigma_m, \tau_m) \quad V_m = (\hat{v}_m / c, \gamma_{tm}). \]

The quantities \( \hat{u}_m \) and \( \hat{w}_m \) are the horizontal and vertical particle velocities respectively, which are associated with Rayleigh type motion, and \( \hat{v}_m \) is the transverse horizontal velocity which is associated with Love type motion. The symbols \( \sigma_m \), \( \tau_m \), and \( \gamma_{tm} \) represent the normal stress, tangential stress in the direction of wave propagation, and the transverse tangential stress, respectively. The asterisk on the \( U^*_{m-1} \) and \( V^*_{m-1} \) vectors indicates that the vector refers to the side of the \( m \)th layer next to the \( m - 1 \)th layer. The \( a_m^R \) and \( a_m^L \) are the solid layer matrices for Rayleigh and Love type motion respectively. For the former this is a four by four matrix and for the latter a two by two. Each matrix is a function of phase velocity, period, layer thickness, and elastic constants of the layer.

Now at a welded contact, the motion stress vectors are continuous across the interfaces from one layer to the next, thus by repetitive substitution of eq. (1) and matrix multiplication, we have a relationship between two motion stress vectors separated by any number of welded elastic solid layers. With this relationship and the boundary conditions of the particular problem being considered, one obtains transcendental relationships which are functions only of the matrix elements: phase velocity, period, and the elastic constants of the media. The usual boundary conditions used are the vanishing of stress at the free surfaces and the vanishing of motion at infinite depth in the underlying half space.

Whenever liquid layers are involved, the matrix multiplication is performed for the solid layers above and below the liquid layer. Then the transcendental equation which relates the vertical stress to vertical particle velocity ratio at one side of the liquid layer to the other side is used as the basic phase velocity-period equation, since these quantities are continuous across liquid-solid interfaces.

A by-product of solving the period equation for Rayleigh waves is the horizontal to vertical particle velocity amplitude ratio at the free surface. Therefore, with this value and by iterative use of eq. (1) the corresponding vertical and horizontal particle velocity amplitude ratios with respect to the vertical free surface amplitude can be calculated at each interface. For Love waves the iterative use of eq. (1) gives the horizontal to transverse velocity amplitude ratio with respect to the free surface amplitude at each interface.

The program is written in FAP for the IBM 7090 and can be used to find the dis-
Dispersion curves in multilayered media

Dispersion (phase velocity versus period) and displacement or velocity ratios at each interface for Rayleigh or Love type motion in the following problems: elastic layers over an elastic half space, where up to two liquid layers are possible; laminated plates, or elastic layers as above, bounded on both sides by free surfaces; and finally, a curvature approximation for the above arrays. If two liquids are used, one must be the surface layer. The half space can be either solid or liquid, but if it is a liquid, it must be counted as one of the two possible liquids.

The general computational procedure is to find the zeros of a function $F$ (the characteristic or frequency function), which is a function of phase velocity, wave number, and the elastic constants of the layers. This is done by initially specifying the phase velocity and a trial value of the wave number $k$. The elements of the matrix in eq. (1) are formed for each layer and then multiplied by the matrix for the layer below it (increasing $m$ index), starting with the solid layer nearest the free surface. The numerical value of the $F$ function is then calculated from elements of the final product matrix and stored. The $F$ function is formed so that its value is positive for $k$'s less than the root in the gravest mode. Thus new trial values of $k$ (of increasing or decreasing size depending on whether the initial $F$ is positive or negative respectively) are used until a change in sign of $F$ is detected. Linear interpolation and extrapolation are then repeatedly used to find smaller $F$ values until $k$'s corresponding to $F$'s of different sign are within the precision interval desired. The root is then designated as the resulting interpolated value of $k$.

After each root has been found for Rayleigh waves, the elements used to calculate $F$ are recomputed using the root $k$. These new elements are then combined to form the horizontal to vertical particle velocity amplitude ratio at the free surface. Then the group velocity is computed by numerical differentiation of the phase velocity values. This is accomplished by perturbing the phase velocity $c$ by a small amount and then finding a new $k$ root to be used in the difference relation. If the displacement option has been selected, the program will now calculate the particle velocity ratios at each interface by iterative use of eq. (1).

The phase velocity is decremented by a specified amount, and then a new $k$ and group velocity are calculated by repeating the above procedure. At first the initial trial $k$ is the root of the previous phase velocity, but after three roots have been found, the trial $k$ for the next phase velocity is determined from the three previous roots by Newton’s formula for backward extrapolation. The program selects the normal mode order by counting the number of zero crossings of the $F$ function for a given phase velocity as $k$ increases from zero. The fundamental mode being the first zero crossing. In this way any particular mode can be specified.

Some Programming Problems

In most problems phase velocity decreases as $k$ increases, and since numbers of the order of $\exp \left( k \sum_{i=1}^{n} d_i \right)$, where $d_i$ is the layer thickness are involved in calculating $F$, the larger root values of $k$ will lead to machine overflow (even though floating point arithmetic is used) if the total number of layers remains constant. Now Haskell showed in the case of solid layers overlying a solid half space, that whenever the phase velocity became less than the shear velocity of a layer, say the $i^{th}$ layer, and also less than the shear velocities of the layers between that layer
and the half space, that the $F$ function in the form he used can be factored at large 
k. Only one of the factors is a function of $k$, the rest of the factors represent the 
Stoneley wave velocity equations for the solid-solid interfaces between the layer 
and the half space when individually equated to zero. The $k$ dependent factor, 
when equated to zero, is equivalent to an $F$ function where the half space and the 
layers between it and the $i^{\text{th}}$ layer have been replaced by a half space with the 
same elastic constants as the $i^{\text{th}}$ layer. This also holds for problems where liquid 
layers are included provided the $F$ function is arranged properly. Physically this 
means that the larger the $k$ value the shorter the wave length and thus the shallower 
the depth of penetration for surface waves and the smaller the influence of the 
dereper layers. We use this feature of surface waves in our program to prevent 
overflow and reduce computing time. Thus when the $F$ factor becomes larger than a 
specified amount, the program will reduce the number of layers, starting with the 
half space and will continue to eliminate layers with shear velocity greater than the 
phase velocity until the $F$ function is small again. Then before proceeding to calcu-
late the new root, the program will recalculate the root for the preceding phase 
velocity in order to verify that no loss in the precision of $k$ was caused by the layer 
reduction.

Precision problems occur in another way. Most operations are executed in the 
single precision mode. There are two exceptions to this which rarely occur but 
evertheless arise the machine automatically shifts to double precision calcula-
tions. The first case is when the computed value of $F$ (including its sign) has lost 
significance in single precision for values of $k$ which are separated by more than the 
required root precision. The second case occurs whenever the $k$ root for the per-
turbed phase velocity is identical in single precision with the unperturbed phase 
velocity root in calculating the group velocity. If the perturbation of $c$ is too small, 
especially when $k$ is a slowly varying function of $c$, the shift to double precision is 
sometimes not enough to give a significant difference in $k$. Therefore the group 
velocity will be in error. As a check against this, group velocities are also calculated 
from successive roots, using the $c$'s for which the $k$ roots have been calculated. 
These group velocities are printed out at the end of each problem.

As mentioned earlier, it is also possible to reduce the number of layers used in 
computation as $k$ increases for the liquid layer cases, but care must be taken that 
the terms in the $F$ function are properly arranged so that mode jumping does not 
occur at high frequencies. This is because for values of $c$ less than the compres-
sional velocity of the liquid and for $kd_i$ large ($d_i$ = liquid thickness), the $F$ func-
tion can be factored into two terms, both of which are functions of $k$. One term 
when equated to zero is equivalent to the period relationship or $F$ function for solid 
layers between a solid half space on one side and a liquid half space on the other. 
The other factor when equated to zero corresponds to the case for solid layers 
between a liquid half space and a free surface. At still larger $k$ the latter term 
factors asymptotically into two more terms. The one which is not a function of $k$ 
when equated to zero is equivalent to the Stoneley velocity equation for the liquid-
solid interface at the liquid half space. The other is the $F$ function where the liquid 
half space has been replaced by a solid half space with the same elastic properties 
as the solid layer above the liquid. Thus since the liquid case factors into two terms 
which are both functions of $k$, it is possible to make the unfortunate arrangement
of terms of the $F$ function so that one of these factors appears in the denominator of the expression. Then at large $k$ it will be possible to find only the root which corresponds to the other factor. This was accidentally done in the first liquid case programs and the factor which represented the solid layers above the liquid layer near the free surface was lost at large $k$. Thus instead of approaching the curve for solid layers without a liquid layer at short wavelengths, the dispersion curve asymptotically approached the Stoneley velocity for the solid-liquid interface below the liquid layer.

![Flow diagram of dispersion computation](image)

**Fig. 1.** Simplified flow diagram of dispersion computation.

**Results**

Our program was designed for “mail order” service on an IBM 7090 computer. The user fills out an order sheet specifying the layer constants (up to 500 layers), the particular modes desired, the initial phase velocity, the final phase velocity, and the number of roots. An initial guess of wave number is desirable but unnecessary. Any of the following options may also be specified:

1) Love or Rayleigh wave dispersion or both in layered medium with or without underlying solid or liquid half space.

2) One or two liquid layers for Rayleigh wave dispersion.

3) Displacement $s$ at each interface and at selected depths in half space normalized to surface displacement.

4) Approximate allowance for curvature\(^1\) valid for periods up to 200–300 sec.

\(^1\) Alterman, Jarosch and Pekeris (1961) showed that the curvature of the Earth affects phase velocities for periods greater than 50 sec, and group velocity for periods greater than 250 sec. They further showed that a linear modification of layer velocities was sufficient to yield phase velocities correct to 1% for periods up to 300 sec. The linear increase is given by the factor $(1 + z/a)$ where $z$ is depth from the surface and $a$ is the earth radius.
5) Printout of details of "F" value calculation to determine root regions in the c, k plane.

The computation speed for Rayleigh waves is such that a 20 layer dispersion curve with 20 values of phase and group velocity can be obtained in 2 minutes. Love wave calculations are approximately eight times faster.

A simplified flow diagram of the program appears in fig. 1. The sequence of computation and the flags for the various options are shown. Sample results for typical calculations appear in figs. 2 and 3.
The notation is as follows:

D  Layer thickness
ALPHA  Compressional wave velocity in layer
BETA  Shear wave velocity in layer
RHO  Density in layer
KD  Dimensionless wave number with respect to top layer
T  Period in sec.
C, U  Phase and group velocity in km/sec. Asterisk on U indicates automatic use of double precision
Ratio of vertical to horizontal displacement at free surface
Number of layers used in computation. Change in N indicates automatic layer reduction is operative
Interface at which displacement is calculated starting at free surface; repeated number with asterisk indicates liquid-solid interface
Horizontal and vertical displacement at M\textsuperscript{th} interface with respect to vertical displacement at free surface (floating point notation)

Fig. 2 shows output format (printed off-line) for an 18 layer calculation of Rayleigh wave dispersion between periods of 65 to 33 sec. The layer constants are printed in the upper part and the results for the Rayleigh mode (M 1, 1) appear in the lower part. Group velocity was computed by both methods described earlier. Automatic layer reduction and switching to double precision was used. Fig. 3 illustrates results for displacements at depth for Rayleigh waves in a 24 layer structure with an intermediate liquid layer. Note continuity of vertical displacement and discontinuity in horizontal displacement at liquid-solid interfaces.

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References

Alterman, Z., H. Jarosch, and C. L. Pekeris

Dorman, J., M. Ewing, and J. Oliver

Haskell, N. A.

Stoneley, R.

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