We show that the irregular connection on $\mathbb{G}_m$ constructed by Frenkel-Gross ([FG]) and the one constructed by Heinloth-Ngô-Yun ([HNY]) are the same, which confirms the Conjecture 2.14 of [HNY].

The proof is simple, modulo the big machinery of quantization of Hitchin’s integrable systems as developed by Beilinson-Drinfeld ([BD]). The idea is as follows. Let $E$ be the irregular connection on $\mathbb{G}_m$ as constructed by Frenkel-Gross. It admits a natural oper form. We apply the machinery of Beilinson-Drinfeld to produce an automorphic D-module on the corresponding moduli space of $G$-bundles, with Hecke eigenvalue $E$. We show that this automorphic D-module is equivariant with respect to the unipotent group $I(1)/I(2)$ (see [HNY] for the notation) against the non-degenerate additive character $\Psi$. By the uniqueness of such D-modules on the moduli space, one knows that the automorphic D-module constructed using the Beilinson-Drinfeld machinery is the same as the automorphic D-module explicitly constructed by Heinloth-Ngô-Yun. Since the irregular connection on $\mathbb{G}_m$ constructed in [HNY] is by definition the Hecke-eigenvalue of this automorphic D-module, it is the same as $E$.

1. Recollection of [BD]

We begin with the review of the main results of Beilinson-Drinfeld ([BD]). We take the opportunity to describe a slightly generalized (and therefore weaker) version of [BD] in order to deal with the level structures.

Let $G$ be a simple, simply-connected complex Lie group, with Lie algebra $\mathfrak{g}$ and the Langlands dual Lie algebra $\mathfrak{g}^\vee$. Let $X$ be a smooth projective algebraic curve over $\mathbb{C}$. For every closed point $x \in X$, let $\mathcal{O}_x$ be the completed local ring of $X$ at $x$ and let $F_x$ be its fractional field. Let $D_x = \text{Spec} \mathcal{O}_x$ and $D_x^\infty = \text{Spec} F_x$. In what follows, for an affine (ind-)scheme $T$, we denote by $\text{Fun} T$ the (pro)-algebra of regular functions on $T$.

Let $G$ be an integral model of $G$ over $X$, i.e. $G$ is a (fiberwise) connected smooth affine group scheme over $X$ such that $G_{\mathbb{C}(X)} = G_{\mathbb{C}(X)}$, where $\mathbb{C}(X)$ is the function field of $X$. Let $\text{Bun}_G$ be the moduli stack of $G$-torsors on $X$. The canonical sheaf $\omega_{\text{Bun}_G}$ is a line bundle on $\text{Bun}_G$. As $G$ is assumed to be simply-connected, we have

**Lemma 1.** There is a unique line bundle $\omega_{\text{Bun}_G}^{1/2}$ over $\text{Bun}_G$, such that $(\omega_{\text{Bun}_G}^{1/2})^{\otimes 2} \simeq \omega_{\text{Bun}_G}$.

Now we assume that $\text{Bun}_G$ is “good” in the sense of Beilinson-Drinfeld, i.e.

$$\dim T^* \text{Bun}_G = 2 \dim \text{Bun}_G.$$}

In this case one can construct the D-module of the sheaf of critically twisted (a.k.a. $\omega_{\text{Bun}_G}^{1/2}$ twist) differential operators on the smooth site $(\text{Bun}_G)_{sm}$ of $\text{Bun}_G$, denoted by $D'$. Let $D' = (\text{End} D')^{op}$ be the sheaf of endomorphisms of $D'$ as a twisted D-module.
Then $D'$ is a sheaf of associative algebra on $(\text{Bun}_G)_{\text{sm}}$ and $D' \cong (D')^{op}$. For more details, we refer to [BD] §1.

Recall the definition of opers on a curve (cf. [BD] §3]). Let $\text{Op} \otimes _G (D^x_\mathfrak{g})$ be the ind-scheme of $L_\mathfrak{g}$-opers on the punctured disc $D_x^\times$. Then there is a natural ring homomorphism

\begin{equation}
(1.1) \quad h_x : \text{Fun} \text{Op} \otimes _G (D^x_\mathfrak{g}) \rightarrow \Gamma(\text{Bun}_G, D').
\end{equation}

Let us briefly recall its definition. Let $\text{Gr}_{G,x}$ be the affine Grassmannian, which is an ind-scheme classifying pairs $(F, \beta)$, where $F$ is a $G$-torsor on $X$ and $\beta$ is a trivialization of $F$ away from $x$. Then we have $\text{Gr}_{G,x} \cong G(F_x)/K_x$, where $K_x = G(\mathcal{O}_x)$. Let $\mathcal{L}_{\text{crit}}$ be the pullback of the line bundle $\omega^1_{\text{Bun}_G}$ on $\text{Bun}_G$ to $\text{Gr}_{G,x}$, and let $\delta_e$ be the delta $D$-module on $\text{Gr}_{G,x}$ twisted by $\mathcal{L}_{\text{crit}}$. Let

$\text{Vac}_x := \Gamma(\text{Gr}_{G,x}, \delta_e)$

be the vacuum $\mathfrak{g}_{\text{crit},x}$-module at the critical level.

**Remark** 1.1. The module $\text{Vac}_x$ is not always isomorphic to $\text{Ind}^{\mathfrak{g}_{\text{crit},x}}_{\text{Lie}K_x + \mathbb{C}1}(\text{triv})$, due to the twist by $\mathcal{L}_{\text{crit}}$. For example, if $K_x$ is an Iwahori subgroup,

$\text{Vac}_x = \text{Ind}^{\mathfrak{g}_{\text{crit},x}}_{\text{Lie}K_x + \mathbb{C}1}(\mathbb{C}^- \rho)$,

is the Verma module of highest weight $-\rho$ ($-\rho$ is anti-dominant w.r.t. the chosen $K_x$).

Let $\text{Bun}_{G,x}$ be the scheme classifying pairs $(F, \beta)$, where $F$ is a $G$-torsor on $X$ and $\beta$ is a trivialization of $F$ on $D_x = \text{Spec} \mathcal{O}_x$. It admits a $(\mathfrak{g}_{\text{crit},x}, K_x)$ action, and $\text{Bun}_{G,x}/K_x \cong \text{Bun}_G$. Now applying the standard localization construction to the Harish-Chandra module $\text{Vac}_x$ (cf. [BD] §1]) gives rise to

$\text{Loc}(\text{Vac}_x) \cong D'$

as critically twisted $D$-modules on $\text{Bun}_G$. Recall that the center $Z_x$ of the category of smooth $\mathfrak{g}_{\text{crit},x}$-modules is isomorphic to $\text{Fun} \text{Op} \otimes _G (D^x_\mathfrak{g})$ by the Feigin-Frenkel isomorphism ([BD, §3.2], [F]). The mapping $h_x$ then is the composition

$\text{Fun} \text{Op} \otimes _G (D^x_\mathfrak{g}) \cong Z_x \rightarrow \text{End}(\text{Vac}_x) \rightarrow \text{End}(\text{Loc}(\text{Vac}_x)) \cong \Gamma(\text{Bun}_G, D')$.

If $G$ is unramified at $x$, then $h_x$ factors as

$h_x : \text{Fun} \text{Op} \otimes _G (D^x_\mathfrak{g}) \rightarrow \text{Fun} \text{Op} \otimes _G (D_x) \cong \text{End}(\text{Vac}_x) \rightarrow \Gamma(\text{Bun}_G, D')$,

where $\text{Op} \otimes _G (D_x)$ is the scheme (of infinite type) of $L_\mathfrak{g}$-opers on $D_x$.

The mappings $h_x$ can be organized into a horizontal morphism $h$ of $D_X$-algebras over $X$ (we refer to [BD] §2.6 for the generalities of $D_X$-algebras). Let us recall the construction. By varying $x$ on $X$, the affine Grassmannian $\text{Gr}_{G,x}$ form an ind-scheme $\text{Gr}_G$ formally smooth over $X$. Let $\pi : \text{Gr}_G \rightarrow X$ be the projection and $e : X \rightarrow \text{Gr}_G$ be the unital section given by the trivial $G$-torsor. Let $\delta_e$ be the delta $D$-module along the section $e$ twisted by $\mathcal{L}_{\text{crit}}$. Then we have a chiral algebra

$\text{Vac}_X := \pi_!(\delta_e)$.

over $X$ whose fiber over $x$ is $\text{Vac}_x$.

**Lemma 2.** The sheaf $\text{Vac}_X$ is flat as an $\mathcal{O}_X$-module.
For any chiral algebra $\mathcal{A}$ over a curve, one can associate the algebra of its endomorphisms, denoted by $\mathcal{E}nd(\mathcal{A})$. As sheaves on $X$,

$$\mathcal{E}nd(\mathcal{A}) = \text{Hom}_\mathcal{A}(\mathcal{A}, \mathcal{A}),$$

where $\text{Hom}$ is taken in the category of chiral $\mathcal{A}$-modules. Obviously, $\mathcal{E}nd(\mathcal{A})$ is an algebra by composition. Less obviously, there is a natural chiral algebra structure on $\mathcal{E}nd(\mathcal{A}) \otimes \omega_X$ which is compatible with the algebra structure. Therefore, $\mathcal{E}nd(\mathcal{A})$ is a commutative $\mathcal{D}_X$-algebra. If $\mathcal{A}$ is $\mathcal{O}_X$-flat, there is a natural injective mapping $\mathcal{E}nd(\mathcal{A})_x \to \text{End}(\mathcal{A}_x)$ which is not necessarily an isomorphism in general, where $\text{End}(\mathcal{A}_x)$ is the endomorphism algebra $\mathcal{A}_x$ as a chiral $\mathcal{A}$-module. However, this is an isomorphism if there is some open neighborhood $U$ containing $x$ such that $\mathcal{A}|_U$ is constructed from a vertex algebra. We refer to $[R]$ for details of the above discussion.

Let $U \subset X$ be an open subscheme such that $\mathcal{G}|_U \simeq G \times U$, then by the above generality, the Feigin-Frenkel isomorphism gives rise to

$$\text{Spec} \mathcal{E}nd(\mathcal{V}ac_U) \simeq \text{Op}_{\mathcal{L}}|_U,$$

where $\text{Op}_{\mathcal{L}}$ is the $\mathcal{D}_X$-scheme over $X$, whose fiber over $x \in X$ is the scheme of $L^g$-opers on $D_x$. Recall that for a commutative $\mathcal{D}_X$-algebra $\mathcal{B}$, we can take the algebra of its horizontal sections $H_{\mathcal{V}}(U, \mathcal{B})$ (or so-called conformal blocks) $[BD] \S 2.6$, which is usually a topological commutative algebra. For example,

$$\text{Spec} H_{\mathcal{V}}(U, \text{Op}_{\mathcal{L}}(\mathcal{G})) \simeq \text{Op}_{\mathcal{L}}(\mathcal{U})$$

is the ind-scheme of $L^g$-opers on $U$ ($[BD] \S 3.3$). As $H_{\mathcal{V}}(U, \mathcal{E}nd(\mathcal{V}ac_U)) \to H_{\mathcal{V}}(X, \mathcal{E}nd(\mathcal{V}ac_X))$ is surjective, we have a closed embedding

$$\text{Spec} H_{\mathcal{V}}(X, \mathcal{E}nd(\mathcal{V}ac_X)) \to \text{Op}_{\mathcal{L}}(\mathcal{U}).$$

Let $\text{Op}_{\mathcal{L}}(\mathcal{X})_{\mathcal{G}}$ denote the image of this closed embedding. This is a subscheme (rather than an ind-scheme) of $\text{Op}_{\mathcal{L}}(\mathcal{U})$.

On the other hand, as argued in $[BD] \S 2.8$, the mapping $h_x$ is of crystalline nature so that it induces a mapping of $\mathcal{D}_X$-algebras

$$h : \mathcal{E}nd(\mathcal{V}ac_X) \to \Gamma(\text{Bun}_{\mathcal{G}}, \mathcal{D}') \otimes \mathcal{O}_X,$$

which induces a mapping of horizontal sections

$$h_{\mathcal{V}} : H_{\mathcal{V}}(X, \mathcal{E}nd(\mathcal{V}ac_X)) \to \Gamma(\text{Bun}_{\mathcal{G}}, \mathcal{D}').$$

Therefore, (1.3) can be rewrite as a mapping

$$h_{\mathcal{V}} : \text{Fun} \text{Op}_{\mathcal{L}}(\mathcal{X})_{\mathcal{G}} \to \Gamma(\text{Bun}_{\mathcal{G}}, \mathcal{D}').$$

We recall the characterization $\text{Op}_{\mathcal{L}}(\mathcal{X})_{\mathcal{G}}$.

**Lemma 3.** Let $X \setminus U = \{x_1, \ldots, x_n\}$. Assume that the support of $\mathcal{V}ac_{x_i}$ (as an $\mathcal{D}_{x_i}$-module) is $Z_{x_i} \subset \text{Op}_{\mathcal{L}}(\mathcal{D}_{x_i}^{+})$ (i.e. $\text{Fun}(Z_{x_i}) = \text{Im}(\text{Op}_{\mathcal{L}}(\mathcal{D}_{x_i}^{+}) \to \text{End}(\mathcal{V}ac_{x_i})))$. Then

$$\text{Op}_{\mathcal{L}}(\mathcal{X})_{\mathcal{G}} \simeq \text{Op}_{\mathcal{L}}(U) \times \prod_i \text{Op}_{\mathcal{L}}(\mathcal{D}_{x_i}^{+}) \prod_i \mathcal{Z}_{x_i}. $$

The mapping (1.4) is a quantization of a classical Hitchin system. Namely, there is a natural filtration ($[BD] \S 3.1$) on the algebra $\text{Fun} \text{Op}_{\mathcal{L}}(U)$ whose associated graded is the algebra of functions on the classical Hitchin space

$$\text{Hitch}(U) = \bigoplus_i \Gamma(U, \Omega^{d_i+1})$$

where $d_i$s are the exponent of $\mathcal{G}$ and $\Omega$ is the canonical sheaf of $X$. On the other hand, there is a natural filtration on $\Gamma(\text{Bun}_{\mathcal{G}}, \mathcal{D}')$ coming from the order of the
Let \( \chi \in \text{Op}_{Lg}(X)_G \subset \text{Op}_{Lg}(U) \) be a closed point, which gives rise to a \( Lg \)-oper \( \mathcal{E} \) on \( U \). Let \( \varphi_\chi : \text{Fun Op}_{Lg}(X)_G \to \mathbb{C} \) be the corresponding homomorphism of \( \mathbb{C} \)-algebras. Then

\[
\text{Aut}_\mathcal{E} := (\mathcal{D}' \otimes \text{Fun Op}_{Lg}(X)_G, \varphi_\chi) \otimes \omega_{\text{Bun}_G}^{-1/2}
\]

is a Hecke-eigensheaf on \( \text{Bun}_G \) with respect to \( \mathcal{E} \) (regarded as a \( LG \)-local system).

Remark 1.3. The statement of the above theorem is weaker than the main theorem in [BD] in two aspects: (i) if \( G \) is the constant group scheme (the unramified case), then \( \text{Op}_{Lg}(X)_G = \text{Op}_{Lg}(X) \) is the space of \( Lg \)-opers on \( X \). In this case, Beilinson and Drinfeld proved that

\[
\text{Fun Op}_{Lg}(X) \simeq \Gamma(\text{Bun}_G, \mathcal{D}')
\]

and therefore \( \text{Aut}_\mathcal{E} \) is always non-zero in this case; (ii) in the unramified case, the automorphic D-module \( \text{Aut}_\mathcal{E} \) is holonomic.

The proofs of both assertions are based on the fact that the classical Hitchin map is a complete integrable system. If the level structure of \( G \) is not deeper than the Iwahori level structure (or even the pro-unipotent radical of the Iwahori group), then by the same arguments, the above two assertions still hold. However, it is not obvious from the construction that \( \text{Aut}_\mathcal{E} \) is non-zero for the general deeper level structure, although we do conjecture that this is always the case. In addition, for arbitrary \( G \), the automorphic D-modules constructed as above will in general not be holonomic. This is the reason that we need to use a group scheme different from [HNY] in what follows.

2.

Now we specialize the group scheme \( G \). Let \( G \) be a simple, simply-connected complex Lie group, of rank \( \ell \). Let us fix \( B \subset G \) a Borel subgroup and \( B^- \) an opposite Borel subgroup. The unipotent radical of \( B \) (resp. \( B^- \)) is denoted by \( U \) (resp. \( U^- \)). Following [HNY], we denote by \( G(0, 1) \) the group scheme on \( \mathbb{P}^1 \) obtained from the dilatation of \( G \times \mathbb{P}^1 \) along \( B^- \times \{0\} \subset G \times \{0\} \) and \( U \times \{\infty\} \subset G \times \{\infty\} \).

Following loc. cit., we denote \( I(1) = G(0, 1)(\mathcal{O}_\infty) \).

Let \( G(0, 2) \to G(0, 1) \) be the group scheme over \( \mathbb{P}^1 \) so that they are isomorphic away from \( \infty \) and \( \mathcal{O}(0, 2)(\mathcal{O}_\infty) = I(2) := [I(1), I(1)] \). Then \( I(1)/I(2) \simeq \prod_{i=0}^\ell U_{\alpha_i} \), where \( \alpha_i \) are simple affine roots, and \( U_{\alpha_i} \) are the corresponding root groups. Let us choose for each \( \alpha_i \) an isomorphism \( \Psi_i : U_{\alpha_i} \simeq \mathbb{G}_a \). Then we obtain a well-defined morphism

\[
\Psi : I(1) \to I(1)/I(2) \simeq \prod_{i=0}^\ell U_{\alpha_i} \simeq \prod_{i=0}^\ell \mathbb{G}_a \to \mathbb{G}_a.
\]

Let \( I_\theta := \ker \Psi \subset I(1) \).

As explained in loc. cit., there is an open substack of \( \text{Bun}_{G(0,2)} \), which is isomorphic to \( \mathbb{G}_a^{\ell+1} \). For the application of Beilinson-Drinfeld’s construction, it is
convenient to consider $\text{Bun}_{\mathcal{G}(0, \psi)}$, where $\mathcal{G}(0, \psi) \to \mathcal{G}(0, 1)$ is a isomorphism away from $\infty$ and $\mathcal{G}(0, \psi)(\mathcal{O}_\infty) = I_\psi \subset I(1) = \mathcal{G}(0, 1)$. Then $\text{Bun}_{\mathcal{G}(0, \psi)}$ is a torsor over $\text{Bun}_{\mathcal{G}(0, \psi)}$ under the group $I_\psi / I(2) \cong \mathbb{G}_a$ and there is an open substack of $\text{Bun}_{\mathcal{G}(0, \psi)}$ isomorphic to $\mathbb{G}_a$.

**Lemma 5.** The stack $\text{Bun}_{\mathcal{G}(0, \psi)}$ is good in the sense of [BD] §1.1.1.

**Proof.** Since $\text{Bun}_{\mathcal{G}(0, \psi)}$ is a principal bundle over $\text{Bun}_{\mathcal{G}(0, 1)}$ under the group $I(1)/I_\psi \cong \mathbb{G}_a$, it is enough to show that $\text{Bun}_{\mathcal{G}(0, 1)}$ is good. It is well-known in this case $\text{Bun}_{\mathcal{G}(0, 1)}$ has a stratification by elements in the affine Weyl group of $G$ and the stratum corresponding to $w$ has codimension $\ell(w)$ and the stabilizer group has dimension $\ell(w)$. Therefore $\text{Bun}_{\mathcal{G}(0, 1)}$ is good.

Let $S_w$ denote the preimage in $\text{Bun}_{\mathcal{G}(0, \psi)}$ of the stratum in $\text{Bun}_{\mathcal{G}(0, 1)}$ corresponding to $w$. Then $S_1 \cong \mathbb{A}^1$, and for a simple reflection $s$, $S_1 \cup S_s \cong \mathbb{P}^1$. In particular, any regular function on $\text{Bun}_{\mathcal{G}(0, \psi)}$ is constant.

Let us describe $\text{Op}_{\mathcal{G}(X)}$ in this case.

At $0 \in \mathbb{P}^1$, $K_0 = \mathcal{G}(0, \psi)(\mathcal{O}_0)$ is the the Iwahori subgroup $I_\mathcal{G}$ of $G(F_0)$, which is $ev^{-1}(B^-)$ under the evaluation map $ev: G(\mathcal{O}) \to G$, and

$$\text{Vac}_0 = \text{Ind}_{\text{Lie}(\mathcal{G})}^{\text{Lie}(\mathcal{G}) + C_1}(\mathbb{C} - \rho).$$

is just the Verma module $M_{-\rho}$ of highest weight $-\rho$ ($-\rho$ is anti-dominant w.r.t. $B^-$), and it is known ([F] Chap. 9) that $\text{Fun Op}_{\mathcal{G}}(D^\times_0) \to \text{End}(\mathcal{V}_{-\rho})$ induces an isomorphism

$$\text{Fun Op}_{\mathcal{G}}(D_0) \cong \text{End}(\mathcal{V}_{-\rho}),$$

where $\text{Op}_{\mathcal{G}}(D_0) \cong \text{End}(\mathcal{V}_{-\rho})$ is the scheme of $\mathcal{G}$ opers on $D_0$ with regular singularities and zero residue. Let us describe this space in concrete terms. Let $f = \sum_i X_{-\alpha_i}$ be the sum of root vectors $X_{-\alpha_i}$ corresponding negative simple roots $-\alpha_i$ of $L^\times$. After choosing a uniformizer $z$ of the disc $D_0$, $\text{Op}_{\mathcal{G}}(D_0) \cong \text{End}(\mathcal{V}_{-\rho})$ is the space of operators of the form

$$\partial_z + f + L^\times b[z]$$

up to $\mathcal{U}(\mathcal{O})$-gauge equivalence.

At $\infty \in \mathbb{P}^1$, $K_\infty = \mathcal{G}(0, \psi)(\mathcal{O}_\infty) = I_\psi$. Denote

$$\mathbb{W}_{\text{univ}} = \text{Vac}_\infty = \text{Ind}_{\text{Lie}(\mathcal{G})}^{\text{Lie}(\mathcal{G}) + C_1}(\text{triv}).$$

It is known ([FF] Lemma 5) that $\text{Fun Op}_{\mathcal{G}}(D^\times_\infty) \to \text{End}(\mathbb{W}_{\text{univ}})$ factors as

$$\text{Fun Op}_{\mathcal{G}}(D^\times_\infty) \to \text{Fun Op}_{\mathcal{G}}(D^\times_\infty)_{1/h} \to \text{End}(\mathbb{W}_{\text{univ}}),$$

where $\text{Op}_{\mathcal{G}}(D^\times_\infty)_{1/h}$ is the scheme of opers with slopes $\leq 1/h$ (as $\mathcal{G}$-local systems) and $h$ is the Coxeter number of $L^\times$. To give a concrete description of this space, let us complete $f$ to an $\mathfrak{sl}_2$-triple $\{e, \gamma, f\}$ with $e \in L^\times B$. Let $L^\times e$ be the centralizer of $e$ in $L^\times$. Then $d_i = \deg L^\times g_i^e$. Then after choosing a uniformizer $z$ on the disc $D_\infty$, $\text{Op}_{\mathcal{G}}(D^\times_\infty)_{1/h}$ is the space of operators of the form

$$\partial_z + f + \sum_{i=1}^{\ell-1} + z^{-d_i-1}(L^\times g_i^e)[z] + z^{-d_i-2}(L^\times g_i^e)[z]$$

up to $\mathcal{U}(\mathcal{O}_\infty)$-gauge equivalence.
Therefore, $\text{Op}_{\mathfrak{g}}(X)\mathcal{G}(0,\Psi)$ is isomorphic to
\[
\text{Op}_{\mathfrak{g}}(X)(0,\text{RS}),_{(\infty,1/\hbar)} := \text{Op}_{\mathfrak{g}}(D\infty)_{1/\hbar} \times \text{Op}_{\mathfrak{g}}(D\infty) \times \text{Op}_{\mathfrak{g}}(D_{\hbar}) \text{Op}_{\mathfrak{g}}(D_{\hbar}).
\]
As observed in [FG], $\text{Op}_{\mathfrak{g}}(X)(0,\text{RS}),_{(\infty,1/\hbar)} \simeq \mathbb{A}^1$. Indeed, let $z$ be the global coordinate on $\mathbb{A}^1 = \mathbb{P}^1 - \{\infty\}$. Then the space of such opers are of the form
\[
\nabla = \partial_z + \frac{f}{z} + \lambda e_{\theta},
\]
where $f$ is the sum of root vectors corresponding to negative simple roots and $e_{\theta}$ is a root vector corresponding to the highest root $\theta$.

According to [1], there is a ring homomorphism
\[
h_\nabla : \mathbb{C}[\lambda] \to \Gamma(\text{Bun}_{\mathfrak{g}(0,\Psi)}, D').
\]

Let us describe this mapping more explicitly. Recall that there is an action of $I(1)/I_\Psi \simeq \mathbb{G}_a$ on $\text{Bun}_{\mathfrak{g}(0,\Psi)}$, and therefore the action of $\mathbb{G}_a$ induces an algebra homomorphism
\[
a : U(Lie I(1)/I_\Psi) \to \Gamma(\text{Bun}_{\mathfrak{g}(0,\Psi)}, D').
\]

**Lemma 6.** We have $h_\nabla(\lambda) = a(\xi)$ for some non-zero element $\xi \in Lie I(1)/I_\Psi \simeq \mathbb{C}$.

**Proof.** Consider the associated graded $h^{cl} : \text{gr} \mathbb{C}[\lambda] \to \Gamma(T^*\text{Bun}_{\mathfrak{g}(0,\Psi)}, \mathcal{O})$, which is the classical Hitchin map. Recall that the filtration on $\mathbb{C}[\lambda]$ comes from the existence of $b$-opers, and therefore the symbol of $\lambda$ is identified with a coordinate function on
\[
\text{Hitch}(X)\mathcal{G}(0,\Psi)
\]
\[
\simeq \bigoplus_{i=1}^{\ell-1} \Gamma(\mathbb{P}^1, \Omega^{d_i+1}((d_i) \cdot 0 + (d_i + 1) \cdot \infty) \bigoplus \Gamma(\mathbb{P}^1, \Omega^{d_\ell+1}((d_\ell) \cdot 0 + (d_\ell + 2) \cdot \infty))
\]
\[
\simeq \mathbb{A}^1.
\]

On the other hand, it is easy to identify the Hitchin map with the moment map associated to the action of $I(1)/I_\Psi$ on $\text{Bun}_{\mathfrak{g}(0,\Psi)}$. Therefore, $h_\nabla(\lambda) = a(\xi) - c$ for some constant $c$. Up to normalization, we can assume that $d_\Psi(\xi) = 1$. We show that $c = 0$. Indeed, consider the automorphic D-module $\text{Aut} = D'/D'\lambda$ on $\text{Bun}_{\mathfrak{g}(0,\Psi)}$. It is $I(1)/I_\Psi$-equivariant against $\mathcal{E}$, with eigenvalue the local system on $\mathbb{G}_m$ represented by the connection $\partial_z + \frac{\xi}{z}$ by Theorem [4] which is regular singular. However, if $c \neq 0$, by [HNY] Theorem 4(1), the eigenvalue for this $\text{Aut}$ should be irregular at $\infty$. Contradiction.

Finally, for any $\chi \in \text{Op}_{\mathfrak{g}}(X)(0,\text{RS}),_{(\infty,1/\hbar)}$ given by $\lambda = c$, $\text{Aut}_\xi = D'/D'(\lambda - c)$ is a D-module on $\text{Bun}_{\mathfrak{g}(0,\Psi)}$, equivariant against $(I(1)/I_\Psi, c\Psi)$. By the uniqueness of such D-modules on $\text{Bun}_{\mathfrak{g}(0,\Psi)}$ (same argument as in [HNY] Lemma 2.3), this must be the same as the automorphic D-module as constructed in [HNY]. We are done.

**References**


