A Note on the Ricatti Differential Equation

The general Riccati differential equation

\[ \frac{dy}{dx} + A(x)y^2 + B(x)y = C(x) \]  

(1)

is important in many problems of physical and technical research. It cannot be solved by exact integration, therefore an appropriate representation of the solution by a convergent sum is desirable.

Eq. (1) can be transformed by

\[ y = \frac{a_1(x)}{a_2(x)} \]  

(2)

into a system of two homogeneous differential equations of first order

\[ \frac{da_1}{dx} = K(x)a_2 \]  

(3)

with

\[ a(x) = \begin{bmatrix} a_1(x) \\ a_2(x) \end{bmatrix}, \quad K(x) = \begin{bmatrix} 0 & C \\ A & B \end{bmatrix}. \]

The normalized solution of (3) is the matrix: \( I \) (unit matrix)

\[ \Omega_1 = \begin{bmatrix} \omega_1 & \omega_2 \\ \omega_2 & \omega_1 \end{bmatrix} = 1 + \int x K(x) dx \]

\[ + \int x \int x_1 K(x_1) K(x) dx dx_1 + \cdots \]  

(4)

The solution of (1) is therefore

\[ y(x) = \frac{y_0}{\omega_2(x_1)} + \omega_1(x_1) \omega_2(x) \]  

(5)

If \( A \) and \( B \) are constants then \( \omega_1 \) can be written in the form

\[ \omega_1 = 1 + \sum_{n=1}^{\infty} A_n \int x_1 x_2 \cdots \]

\[ \int x_{n+1} C(x_1) C(x_2) \cdots C(x_{n+1}) \]

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Metal Contact Double Injection in GaAs

Recent studies on the GaAs injection laser have indicated certain problems associated with the localization of the region of population inversion to the immediate vicinity of the \( p-n \) junction. It has also been pointed out that very heavy doping is necessary for laser action. However, in heavily doped material the absorption of light is large, raising the threshold current for laser action. These problems can be overcome simultaneously by using a metal-GaAs-metal structure operating in the double injection mode described by Lampert. A metal of low work function is used to inject electrons at one surface and one of high work function to inject holes at the opposite surface. When sufficient double injection occurs to neutralize the bulk between the electrodes, the voltage required to sustain the current drops to a low value.

Experiments were conducted on semi-insulating GaAs faces. The smooth etched surface was prepared with a Bromine-methanol polish and coated with barium by vacuum evaporation. The opposite surface was mechanically polished, cleaned quickly in CP-4 etch and coated with platinum. Volt-ampere curves were taken at room temperature with a single sweep display. A typical I-V trace is shown in Fig. 1. The negative resistance was not obtained with the reverse polarity.

It has been shown that the barrier heights of contacts to GaAs made with metals of widely different work functions are very nearly the same. This effect has been attributed to surface states. It appears that in the present experiments the application of bias can supply on a dynamic basis the charge needed to shift the quasi-fermi level to the band edges. In any case, the present author has been able to find no reasonable explanation for the observed negative resistance characteristics other than double injection.

In addition to avoiding highly doped material in or near the optically active region, double injection structures allow the injection mechanisms to be separated from the recombination and radiative mech

\[ \text{Fig. 1.-I-V curve trace of Ba-GaAs-Pt double injection diode. Vertical scale 10 v/cm. Horizontal scale 20 ma/cm. Diode was 0.01-inch diameter and approximately 9.005-inch thick.} \]
and
\[ p_\theta(\theta) = \frac{1}{2\pi}, \quad \text{for } 0 < \theta < 2\pi \]
\[ = 0 \text{ elsewhere}. \]

The probability density of \( y \), given that \( \omega \) is true, \( p[y/\omega] \) has a chi-squared distribution with \( M \) degrees of freedom; the probability density of \( y \), given that \( \omega \) is true, \( p[y/\omega] \), has not been determined. Application of the central limit theorem allows us to approximate both of these conditional probability functions by the normal probability density having mean and variance given by

\[ p_y(y) = \frac{1}{\sqrt{2\pi \sigma_y^2}} \exp \left( -\frac{y^2}{2\sigma_y^2} \right), \]

\[ MV = M\sigma_y^2 \]

If we make the substitutions,
\[ h^2 = \left[ \frac{A^4}{16} + A^2\sigma_a^2 + \sigma_a^4 \right] \]
\[ S = \frac{A^4}{2} \]
\[ N = \sigma_a^2, \quad (4) \]

the likelihood ratio expressed in terms of \( y \) becomes

\[ l(y) = \frac{N}{h} \exp \left( -\frac{1}{2} \frac{\left[ y - M(S + N) \right]^2}{2Mh^2} \right) \]
\[ \exp \left( -\frac{1}{2} \frac{\left[ y - MN \right]^2}{2Mh^2} \right) \]

For small signal-to-noise ratios, \( h \approx N \), we may approximate \( l(y) \) by

\[ l(y) \approx \exp \left( -\frac{1}{2} \frac{-2MSy-M(S^2+2SN)}{2MN} \right) \]

We wish to compare the likelihood ratio \( l(y) \) to a threshold \( \beta \). For the purposes of the present discussion we choose \( \beta = 1 \). This corresponds to the maximum likelihood criteria, minimizes the total error probability and yields a particularly simple result. We say that a signal is present if

\[ y \geq MS \left( \frac{1}{2} + \frac{N}{S} \right) \]

The relation between \( p(y/\omega_a) \), \( p(y/\omega) \) and \( \beta \) for this choice of \( \beta \) is shown in Fig. 1.

If the variable \( y \) is normalized with respect to the standard deviation, \( \sqrt{2MN} \), i.e., \( \tilde{y} = y/[2MN] \) and \( k_1 \), set equal to \( MS/2 \), \( k \) provides an index of the reliability of detection where

\[ k = \frac{MS}{\sqrt{2MN}} \]
\[ \frac{N}{2\sqrt{2}} \]

Solving for \( T \), using (8) and (1),

\[ T = \frac{4k^2}{BL} \left( \frac{S}{N} \right)^2 \left( \frac{S}{N} \right)^2 \]

\[ \left( 9 \right) \]

Suppose now that the original bandwidth \( B \) is divided into \( L \) equal-width subchannels making the individual channel bandwidth \( B/L \) cps and signal-to-noise ratio \( L/S \), (9) becomes

\[ T = \frac{4k^2}{BL} \left( \frac{S}{N} \right)^2 \]
\[ \left( 10 \right) \]

For a fixed value of \( T \), the value of \( k \) for each of the \( L \) subchannels is

\[ k_L = k_1 \sqrt{L} \]
\[ \left( 11 \right) \]

and

\[ P_\beta = P \left\{ \text{error in a bank of } L \text{ sub-channels} \right\} \]
\[ = 1 - (1 - P_\epsilon)^L \]
\[ \approx LP_\epsilon \approx \frac{\sqrt{L}}{\sqrt{2k_1}} \exp \left( -\frac{Lk_1^2}{2} \right) \]

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The Influence of Terrain Shielding on Radio Wave Propagation at 8000 Mc

Terrain shielding measurements were made at the MIT Lincoln Laboratory, Millstone Hill, Mass. communication terminal during June 20–22, 1962. The transmitting terminal consisted of a 60-foot diameter X-band parabolic antenna mounted on an azimuth elevation mount. A crystal controlled transmitter was employed.

A truck was used to make measurements in the field. The receiving antenna con-