Generating topological order: No speedup by dissipation

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Abstract: We consider the problem of converting a product state to a ground state of a topologically ordered system through a locally generated open-system dynamic. Employing quantum-information tools, we show that such a conversion takes an amount of time proportional to the diameter of the system. Our result applies to typical two- and four-dimensional topologically ordered systems as well as, for example, the three-dimensional and four-dimensional toric codes. It is tight for the toric code, giving a scaling with the linear system size. Our results have immediate operational implications for the preparation of topologically ordered states, a crucial ingredient for topological quantum computation. Dissipation cannot provide any significant speedup compared to unitary evolution.

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The classification of the phases of matter is one of the most fundamental problems in modern physics. A fruitful approach to this problem centers around the analysis of the time required to convert between states belonging to different phases. To accurately capture the nature of spatial correlations, one commonly considers conversion realized by unitary evolution generated by a local Hamiltonian. However, actual experimental systems are arguably more faithfully described by open-system dynamics rather than unitary evolution generated by a local Hamiltonian. Consequently, the question of interconversion times is modeled by a Markovian master equation: The proposal involves a time-independent local Liouvillian $L$ such that the state of the system relaxes towards a steady state variant envisions engineering system-bath couplings in such a way that the state of the system relaxes towards a steady state supported on the ground space of the toric code as shown in Ref. [1]. In other words, state preparation by locally generated unitary evolution is resource intensive to the extent of possibly being impractical.

More recently, inspired by the realization that dissipation can be useful for quantum-information processing, another technique for state preparation has been proposed [9]. This variant envisions engineering system-bath couplings in such a way that the state of the system relaxes towards a steady state supported on the ground space of the toric code. Physically, this means excitations are gradually eliminated by a transfer of entropy to the environment. Mathematically, the situation is modeled by a Markovian master equation: The proposal involves a time-independent local Liouvillian $L$ such that the corresponding evolution $e^{tL}$, for large enough $t$, maps product states to ground states of the toric code. What makes this proposal particularly attractive is the fact that, in contrast to, e.g., unitary circuits, it requires no active control. Here we show that such a dissipative evolution, although conceptually attractive, does not overcome the fundamental issue mentioned above: Preparation of any ground state of the toric code takes an amount of time which is at least proportional to the linear system size $L$. Therefore, the experimental toolbox involves all capabilities required for universal quantum computation in the circuit model, it is natural to consider the following process: Start with a product state (supposedly easy to prepare), and convert this to a ground state by a unitary circuit $U$ composed of local single- and two-qubit gates on the lattice. Bravyi et al. [1] have shown that any such circuit has a depth at least linear in $L$ [we denote this by $\Omega(L)$]. Their argument is quite general and applies to all locally generated unitary evolutions, i.e., evolutions under a (possibly time-dependent but) local Hamiltonian $H(s)$. More precisely, these are completely positive trace-preserving maps (CPTPMs) of the form

$$\mathcal{E}^{(T)}(\rho) = U(T) \rho U(T)^\dagger, U(T) = T \exp \left[ i \int_0^T H(s) ds \right] .$$

(2)

Such evolutions (including adiabatic preparation [8]) require time at least $T = \Omega(L)$ to turn any product state $|\Phi\rangle = \bigotimes_{j\in\mathbb{Z}^2} |\Phi_j\rangle$ into a ground state of the toric code as shown in Ref. [1]. In other words, state preparation by locally generated unitary evolution is resource intensive to the extent of possibly being impractical.

The prime example of a topologically ordered system is Kitaev’s toric code [5], a system of $n = 2L^2$ qubits arranged on the edges of a rectangular grid of size $L \times L$ with periodic boundary conditions (PBCs). The Hamiltonian $H = -\sum_a \Pi_a$ is composed of pairwise commuting projections $\Pi_a$ of each acting on four neighboring qubits on the lattice. In the language of quantum error-correcting codes (QECCs), the four-dimensional ground-state space of this Hamiltonian is identified with the code space of a QECC which encodes two logical qubits and can correct for errors in any topologically trivial region of the lattice. That is, a ground state is affected by an error acting exclusively within such a region, it is possible to perform a recovery operation that returns the system to its original state.

Consider the problem of preparing any possibly mixed state $\rho$ supported entirely on the ground space of $H$. Assuming that the experimental toolbox involves all capabilities required for universal quantum computation in the circuit model, it is natural to consider the following process: Start with a product state (supposedly easy to prepare), and convert this to a ground state by a unitary circuit $U$ composed of local single- and two-qubit gates on the lattice. Bravyi et al. [1] have shown that any such circuit has a depth at least linear in $L$ [we denote this by $\Omega(L)$]. Their argument is quite general and applies to all locally generated unitary evolutions, i.e., evolutions under a (possibly time-dependent but) local Hamiltonian $H(s)$. More precisely, these are completely positive trace-preserving maps (CPTPMs) of the form

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More recently, inspired by the realization that dissipation can be useful for quantum-information processing, another technique for state preparation has been proposed [9]. This variant envisions engineering system-bath couplings in such a way that the state of the system relaxes towards a steady state supported on the ground space of the toric code. Physically, this means excitations are gradually eliminated by a transfer of entropy to the environment. Mathematically, the situation is modeled by a Markovian master equation: The proposal involves a time-independent local Liouvillian $L$ such that the corresponding evolution $e^{tL}$, for large enough $t$, maps product states to ground states of the toric code. What makes this proposal particularly attractive is the fact that, in contrast to, e.g., unitary circuits, it requires no active control. Here we show that such a dissipative evolution, although conceptually attractive, does not overcome the fundamental issue mentioned above: Preparation of any ground state of the toric code takes an amount of time which is at least proportional to the linear system size $L$. Therefore, the experimental toolbox involves all capabilities required for universal quantum computation in the circuit model, it is natural to consider the following process: Start with a product state (supposedly easy to prepare), and convert this to a ground state by a unitary circuit $U$ composed of local single- and two-qubit gates on the lattice. Bravyi et al. [1] have shown that any such circuit has a depth at least linear in $L$ [we denote this by $\Omega(L)$]. Their argument is quite general and applies to all locally generated unitary evolutions, i.e., evolutions under a (possibly time-dependent but) local Hamiltonian $H(s)$. More precisely, these are completely positive trace-preserving maps (CPTPMs) of the form

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Our argument relies on two basic ingredients. The first is the fact that generating long-range correlations between two regions from an initial product state requires a time proportional to the distance between the regions.

This conceptually straightforward observation was already made in Ref. [1] and follows from Lieb-Robinson bounds. The second ingredient is system specific and consists in proving the existence of two distant observables \( f \) and \( g \) that are correlated, that is, satisfy \( \langle fg \rangle_\rho > \langle f \rangle_\rho \langle g \rangle_\rho \) for topologically ordered target states \( \rho \). Together, these two statements imply a bound on the preparation times of TQO states via open-system dynamics.

For concreteness, let us consider how our argument specializes to Kitaev’s toric code. For this case, our conclusion reads:

**Theorem 1.** Consider Kitaev’s toric code on a PBC \( L \times L \) lattice. Let \( \mathcal{L}(s) \) be a time-dependent local Liouvillian with bounded-strength, constant-range, or exponentially decaying interactions. Consider evolution for time \( t \) under this Liouvillian, i.e., the completely positive trace-preserving map,

\[
\mathcal{E}(t) = T \exp \left[ \int_0^t \mathcal{L}(s) ds \right]. \tag{3}
\]

If \( \mathcal{E}(t) \) turns a product state \( |\Phi⟩ = \bigotimes_{j=1}^{2L} |\Phi_j⟩ \) into a state \( \rho \) supported on the ground state of the toric code up to a constant error \( \epsilon \ll 1 \) in a trace norm [10] (i.e., \( \|\mathcal{E}(t) - \rho\|_1 \leq \epsilon \)), then \( t \geq \Omega(L) \).

Theorem 1 shows that preparation of TQO states is challenging even when allowing dissipative processes: There is no advantage in terms of preparation time compared to unitary dynamics. Several remarks are in order: Since our interest is in assessing the viability of proposals involving a Markovian master equation, we have formulated our result in terms of dissipative evolution maps of the form (3). However, our derivation applies to a much larger class of locality-preserving evolutions (as defined below). In particular, our results provide a strict generalization of the case of unitary dynamics considered in Ref. [1]. Namely, the evolutions are not assumed to be invertible (as are unitaries) nor are they assumed to be encoders (transferring information localized on a qubit into the ground-state space). As a consequence, Theorem 1 provides a stronger no-go result and requires a novel proof approach. The interested reader is referred to the Supplemental Material [11] for a discussion of nonunitary encoders, including a general bound on the encoding time \( t \).

Theorem 1 is tight, i.e., cannot be improved: In Ref. [12], an explicitly constructed local Liouvillian is shown to prepare arbitrary ground states of the toric code starting from initial product states in a time linear in \( L \).

A Liouvillian \( \mathcal{L} \) is said to be local if it can be written in the form

\[
\mathcal{L} = \sum_{X \subset A} \mathcal{L}_X, \tag{4}
\]

with each term \( \mathcal{L}_X \) acting exclusively on a bounded-diameter subset \( X \) of the full lattice denoted by \( A \). One of the key facts we use is that a local Liouvillian (whether time dependent or not), generates a *locality-preserving* dissipative evolution \( \mathcal{E}(t) \) according to (3). This property is shared with a unitary dynamics (2) generated by a local (or quasilocal) Hamiltonian. It has various different manifestations: Physically, it means that information propagates only at a constant speed during the evolution. Mathematically, this can be expressed by so-called Lieb-Robinson bounds [13]. The version which is most useful in our context [14], however, states that any such evolution \( \mathcal{E}^{\rho} \) may essentially be substituted by a localized evolution map \( \mathcal{E}^{\rho}_{B(r)} \) when only local observables \( O_B \) supported on a region \( B \) are considered. That is, the map \( \mathcal{E}^{\rho}_{B(r)} \) is supported on the \( \rho \) neighborhood \( B(r) := \{ x \in \Lambda | d(x, B) \leq r \} \) of \( B \) and approximates the Heisenberg evolution of \( \mathcal{E} \) in operator norm [15],

\[
\| (\mathcal{E}(t))^{\rho} (O_B) - \mathcal{E}^{\rho}_{B(r)}(O_B) \|_\infty \leq \| O_B \|_\infty |B| G(r). \tag{5}
\]

A Lieb-Robinson bound guarantees the existence of such a map \( \mathcal{E}^{\rho}_{B(r)} \) with an approximation error of the form \( G(r) = C e^{-t - r} \) for some non-negative constants \( C, v, \gamma \). In the case of dissipative evolution (3), \( \mathcal{E}^{\rho}_{B(r)} \) is the evolution for time \( t \) under the Liouvillian \( \mathcal{L}_{B(r)} = \sum_{x \in B(r)} \mathcal{L}_x \) obtained by neglecting terms with support outside \( B(r) \) in the sum (4). Our results apply generally to any evolution \( \mathcal{E}(t) \) satisfying such a bound: Beyond (3) and (2), this includes, e.g., any family \( \{ \mathcal{E}(t) \}_{t \geq 0} \) of (unitary or nonunitary) circuits where each \( \mathcal{E}(t) \) has a circuit depth upper bounded by \( t \).

Let us now sketch the proof of Theorem 1, which is illustrated in Fig. 1. We focus on a pair of anticommuting logical operators \((X, \bar{Z})\) associated with an encoded logical qubit [16]. These can be chosen as tensor products of Pauli \( X \) or \( Z \) operators along cycles winding around the torus in the horizontal and vertical directions, respectively. For any state \( \rho \) we know that the expectation values satisfy \( \langle \bar{Z} \rangle^2 + \langle \bar{X} \rangle^2 \leq 1 \). Without loss of generality we can assume \( \langle \bar{Z} \rangle^2 \leq 1/2 \). Let us now consider two incubations \( \bar{Z}^{(1)} \) and \( \bar{Z}^{(2)} \) of \( \bar{Z} \) such that their support is separated by a distance \( L/2 \) (concretely, \( \bar{Z}^{(1)} \) is supported on a vertical strip, whereas \( \bar{Z}^{(2)} \) is its translation in the horizontal direction). Since \( \rho \) is in the code space, we have that \( \bar{Z}^{(1)} \) and \( \bar{Z}^{(2)} \) coincide on \( \rho \) and \( \langle \bar{Z}^{(1)} \bar{Z}^{(2)} \rangle_\rho = 1 \).

This constitutes the system-specific ingredient of the proof: We have identified two distant normalized observables \( \bar{Z}^{(1)} \)

![FIG. 1. The figure on the right illustrates the support of the logical operator incarnations \( \bar{Z}^{(1)} \) and \( \bar{Z}^{(2)} \) in a square with a PBC. On the left is a qualitative illustration of the support for the Heisenberg evolved observable \( \mathcal{E}(t) \bar{Z}^{(1)} \bar{Z}^{(2)} \). The light gray areas correspond to regions which may be excluded from the support of the operator while still allowing for a good approximation using the locality-preserving property.](Image)
and \( \hat{Z}^{(2)} \) with covariance lower bounded by a constant in any ground state \( \rho \), i.e.,
\[
(\hat{Z}^{(1)} \hat{Z}^{(2)})_{\rho} - (\hat{Z}^{(1)})_{\rho} (\hat{Z}^{(2)})_{\rho} \geq 1/2. \tag{6}
\]

The operators \( \hat{Z}^{(1)} \) and \( \hat{Z}^{(2)} \) witness long-range correlations of \( \rho \). Formally, let us define the covariance correlation \( C_{\rho}(B_{1};B_{2}) \) of regions \( B_{1} \) and \( B_{2} \) with respect to a state \( \rho \) by
\[
C_{\rho}(B_{1};B_{2}) := \sup_{\|f\|_{1} = \|g\|_{1} = 1} |\langle f|g \rangle_{\rho} - \langle f \rangle_{\rho} \langle g \rangle_{\rho}|,
\]
where \( \text{supp}(\cdot) \) denotes the region providing nontrivial support to an operator. The presence of nonzero covariance correlations guarantees the existence of correlated operators without making them explicit. The following lemma then completes the proof of Theorem. It also allows obtaining analogous results for a large family of TQO states as well as any other target density operators with long-range covariance correlations.

**Lemma 1.** Consider two regions \( B_{1} \) and \( B_{2} \) satisfying \( d(B_{1},B_{2}) = \Theta(L) \) and \( \mathcal{B}_{1} \cup \mathcal{B}_{2} = \Theta(\text{poly}(L)) \). Let \( \mathcal{E}(\cdot) \) be a family of CPTPMs obeying a Lieb-Robinson bound of the form (5). If \( \|\mathcal{E}(\Phi)_{\rho} - \rho\|_{1} \leq \epsilon \) for some initial product state \( \Phi \) and a state \( \rho \) satisfying \( 0 < \epsilon \ll C_{\rho}(B_{1};B_{2}) < 1 \), then \( t = \Theta(L) \).

Suppose an initial product state \( \Phi = \bigotimes_{\mathcal{J}} \Phi_{\mathcal{J}} \) evolves into an \( \epsilon \) approximation of \( \rho \), i.e., \( \|\mathcal{E}(\cdot)_{\rho} - \rho\|_{1} \leq \epsilon \). By hypothesis, there exist observables \( f \) and \( g \) witnessing \( \beta := C_{\rho}(B_{1};B_{2}) \) correlations between regions \( B_{1} \) and \( B_{2} \) of \( \rho \). We may express the correlations in \( \mathcal{E}(\cdot)_{\Phi} \) in the Heisenberg picture as
\[
(\mathcal{E}(\cdot)_{\Phi})(f g)_{\Phi} - \langle(\mathcal{E}(\cdot)_{\Phi}(f))_{\Phi} (\mathcal{E}(\cdot)_{\Phi}(g))_{\Phi} \geq \beta - 3\epsilon. \tag{7}
\]

The locality-preserving nature of the evolution \( \mathcal{E} \) implies that it may be approximated by a spatially truncated evolution operator with support on two disjoint regions \( B(r) = B_{1}(r) \cup B_{2}(r) \). Under such a truncation, the left-hand side of Eq. (7) is approximately zero as correlations in a product state vanish. \( C_{\rho}(B_{1}(r);B_{2}(r)) \) is \( 0 \). More precisely, the approximation error from Eq. (5) upper bounds the left-hand side of (7), hence,
\[
3|B|G(r) + |B|^{2}G^{2}(r) \geq \beta - 3\epsilon, \tag{8}
\]
where \( B = B_{1} \cup B_{2} \). Choosing \( r \) proportional to \( L \), we conclude that \( t = \Theta(L \gamma / v) \) is necessary in order to fulfill this inequality.

The conclusion reached here is generic for topologically ordered states: Dissipative processes cannot create such states from product states in a time independent of the system size. Our techniques rigorously establish this impossibility for typical two-dimensional (2D) topologically ordered systems, including, e.g., the Levin-Wen model [17], Bombin’s color codes [18], any 2D system described by a topological quantum-field theory, the three-dimensional and four-dimensional (4D) toric codes [5] as well as any scale and translation-invariant stabilizer (STS) code [19] in \( D \leq 3 \). Indeed, the key feature, the existence of observables analogous to those presented for the toric code and ensuring the conditions of Lemma 1 (i.e., nonzero covariance correlations), holds for a large family of TQO states. The strategy is to take \( B_{1} \) and \( B_{2} \) to be the regions supporting two distant incarnations of the same uncertain operator (i.e., an operator for which a deterministic outcome is not expected on \( \rho \)).

As an example, consider the class of STS codes [19] in \( D \leq 3 \) dimensions which can be assumed to have a 2D-dimensional code space. Two facts imply the existence of distant incarnations \( \mathcal{E}_{[1]}^{(1)}, \mathcal{E}_{[2]}^{(2)} \) of an uncertain observable \( \mathcal{E} \) in this case: The first is dimensional duality [19, Theorems 4 and 5]: In \( D \leq 3 \) dimensions, there exist sets of logical Pauli operators \( \{X_{1}, \ldots, X_{k}\} \) and \( \{\hat{Z}_{1}, \ldots, \hat{Z}_{k}\} \) with commutation relations \( \hat{X}_{j} \hat{Z}_{j} \hat{X}_{j} \hat{Z}_{j} = (-1)^{j} \). The content of the referred theorem states that the logical operators may be taken to have complementary geometric dimensionality in their support. In three dimensions, operator pairs are supported in one stringlike and the other in brane(plane-like) regions, or alternatively, one is supported in the full volumelike region where the complementary operator will be pointlike (corresponding to a nontopological degree of freedom which can be observed on a local region). For any ground state \( \rho \) there is a constant amount of uncertainty associated with at least one observable from each pair \( \{\hat{X}_{j}^{\pm}, \hat{Z}_{j}^{\pm}\} \) \((\mathcal{E}_{[1]}^{(1)} = \mathcal{L}_{j}^{(1)}, \mathcal{E}_{[2]}^{(2)} = \mathcal{E}_{[2]}^{(2)} \)

In particular, this implies that there is an index \( j \in [1,2] \) such that the measurement outcome \( \mathcal{M}_{L_{j}}(\rho) \) when measuring \( \rho \) is not deterministic: Its Shannon entropy \( H(\mathcal{M}_{L_{j}}(\rho)) \) is lower bounded by a constant independent of the system size. We can now proceed as before for the toric code, taking \( \mathcal{E}_{[1]}^{(1)} = \mathcal{L}_{j}^{(1)} \) and \( \mathcal{E}_{[2]}^{(2)} \) as the observable obtained by translating \( \mathcal{E}_{[1]}^{(1)} \) by a distance \( L / 2 \). This gives two distant correlated observables for \( \rho \), showing that Lemma 1 applies.
Importantly, the fact that we are considering open-system dynamics allows us to study the preparation of arbitrary mixed states. An example is the Gibbs state of the 4D toric code, which is expected to be topologically ordered even at finite temperatures [4,21]. Thermal fluctuations will exponentially suppress the expectation values of any bare incarnation of logical observables and simple products thereof. However, one may construct error-corrected logical observables having a width growing with $L$ which provide nontrivial correlations. Such error-corrected logical observables satisfying the conditions of Lemma 1 can also be constructed for noisy code states of a topological QECC (i.e., codes states on which local errors are randomly applied with a density below a code-dependent threshold).

There are, however, important states for which proving a distinction from the trivial phase remains outstanding. Two salient examples are (a) the toric code or TQFTs on the sphere and (b) Haah-type codes [22]. In the first case the absence of logical operators impedes applying the arguments presented. In the second case, the logical operators are not low dimensional, and distant incarnations cannot be found.

To summarize, our work establishes fundamental limits on the preparation of topologically ordered states by possibly nonunitary local processes. One may view these results in the general context of classifying different phases: Here the notion of local unitary circuits plays a crucial role in defining equivalence [2]. By considering open-system dynamics, we open the door to classifying phases of mixed states, such as Gibbs states of local Hamiltonians. A distinguishing feature of locally generated open-system dynamics is that both long-range classical and quantum correlations may be removed in sublinear time yet may require an amount of time growing at least linearly with the diameter in order to be produced. Hence the picture of phases corresponding to locally unitarily equivalent states is enriched by a partial order among such phases, corresponding to the possibility of a locality-preserving open-system evolution to take states in one phase into another.

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[10] The trace norm is defined as $\|A\|_1 = \text{tr}(\sqrt{A^*A})$.
[15] The operator norm for a Hermitian operator is given by $\|A\|_\infty = \max_{\langle \psi \rangle} |\langle \psi |A|\psi \rangle|$. Logical operators preserve the ground-state space.