

where c_0, c_1, c_2, \dots are polynomials in n , and are odd or even functions of n according as q is even or odd.

There is a recurrence relation

$$(n+1)(n+m+1)F_{n+1}^m(z) = -(2n+1)zF_n^m(z) + n(n-m)F_{n-1}^m(z). \quad (18)$$

It follows then from equations (16) and (4) that, if $q \geq 1$,

$$\bar{r}^q = \left(\frac{2q+1}{q+1}\right) \frac{n^2 a_0}{Z} \bar{r}^{q-1} - \frac{q(2l+1+q)(2l+1-q)}{q+1} \frac{n^2 a_0^2}{4Z^2} \bar{r}^{q-2} \quad (19)$$

and

$$\bar{r}^{-q-\delta} = \frac{1}{(l+1+q/2)(l-q/2)} \left\{ \left(\frac{2q+1}{q+1}\right) \frac{Z}{a_0} \bar{r}^{-q-2} - \frac{q}{q+1} \frac{Z^2}{n^2 a_0^2} \bar{r}^{-q-1} \right\}. \quad (20)$$

Formulas (4), (9) and (10) are the most useful for computational purposes.

¹ I. Waller, *Zeit. Physik*, **38**, 644 (1926).

² J. H. Van Vleck, *Proc. Roy. Soc. A*, **143**, 679 (1934). There is a slight error in his expression for $\bar{r}^{-\delta}$ for $l=2$, the coefficients being all 10 times too large.

³ δ may be equated to zero except in the case $q = 2l$.

⁴ W. N. Bailey, *Generalized Hypergeometric Series* (Cambridge Tract No. 32), p. 18.

⁵ The polynomial $F_n^0(z)$ is discussed by H. Bateman, *Tôkoku Math. J.*, **37**, 23 (1933). The general case will be treated by the author in a paper to be published soon.

⁶ W. N. Bailey, *Generalized Hypergeometric Series*, (Cambridge Tract No. 32), p. 22.

ON THE EQUATION OF DIFFUSION

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1. Under the title "The Analytical Theory of Probability," A. Kolmogoroff¹ published a few years ago a mathematically elegant account of what might be called the general theory of diffusion. As the main result of his analysis, he sets up two differential equations for transition probabilities (see below). The first of them contains as its independent variables the initial parameters of the transition probability and has an analytical form resembling the ordinary diffusion equation. Unlike it, the second differential equation (depending on the final parameters) has a different form unfamiliar to the physicist.

The puzzling asymmetry of the two equations, as to whose physical reasons the author gives no hint, was, in our opinion, the main cause why the work of Kolmogoroff did not receive more attention. Unless these reasons are fully understood, an intelligent application of the theory to practical problems is very difficult. We think it, therefore, worth while to present in the following lines a discussion of the mutual connection between the two equations of Kolmogoroff and of their relation to other forms of the diffusion equation used in physics.

2. The concept of the transition probability applies to systems composed of many identical elements. If one of these elements was at the time s in the state x , the probability of its being found at a later time t in the state y is called transition probability and is denoted by $v(x, s; y, t)$. The states x and y are in general defined by several geometrical and dynamical coördinates. Following Kolmogoroff, however, we shall restrict ourselves to the case of states depending on a single parameter, since the generalization for many parameters is obvious. Moreover, we shall take this single coördinate as continuous and deal with intervals between x and $x + \Delta x$, etc.

Kolmogoroff considers only the case of *independent* probabilities when $v(x, s; y, t)$ is not influenced by the presence or absence of other particles in the same or neighboring states. Under this restriction this function must satisfy the two fundamental requirements of the theory of probability

$$\int v(x, s; y, t) dy = 1, \quad (1)$$

$$\int v(x, s; y, t') v(y, t'; z, t) dy = v(x, s; z, t), \quad (2)$$

the integrals being extended over the whole region of variability of the parameter y .

By mere mathematical transformations, these conditions lead directly to the equations of diffusion in Kolmogoroff's form

$$-\frac{\partial v(x, s; y, t)}{\partial s} = a(x, s) \frac{\partial}{\partial x} v(x, s; y, t) + b(x, s) \frac{\partial^2}{\partial x^2} v(x, s; y, t), \quad (3)$$

$$\frac{\partial v(x, s; y, t)}{\partial t} = -\frac{\partial}{\partial y} [a(y, t)v(x, s; y, t)] + \frac{\partial^2}{\partial y^2} [b(y, t)v(x, s; y, t)]. \quad (4)$$

The meaning of the coefficients is the *mean transition velocity* of an element at the time t and in the state y

$$a(y, t) = \lim_{\Delta \rightarrow 0} \frac{1}{\Delta} \int (\eta - y)v(y, t; \eta, t + \Delta) d\eta, \quad (5)$$

and the mean spread of this velocity

$$b(y, t) = \lim_{\Delta \rightarrow 0} \frac{1}{2\Delta} \int (\eta - y)^2 v(y, t; \eta, t + \Delta) d\eta. \quad (6)$$

Another type of probability which also has considerable physical interest is the distribution probability or density of distribution $D(s, t; y)$ which is defined as the chance of finding an element in the state y at the time t , no matter where it came from. It is clear from the definition of the function $v(x, s; y, t)$ that the density of distribution is connected with it in the following way

$$D(s, t; y) = \int D(s, s; x) v(x, s; y, t) dx, \quad (7)$$

provided $D(s, s; x)$ is the known density at the time s . Hence $D(s, t; y)$ must also satisfy the equation (4).

$$\frac{\partial D}{\partial t} = -\frac{\partial}{\partial y} [a(y, t)D] + \frac{\partial^2}{\partial y^2} [b(y, t)D]. \quad (8)$$

3. The most important case is the one in which the coefficients $a(y)$ and $b(y)$ are independent of time. The system is then called "homogeneous in time" and it is apparent from (5) and (6) that the transition probability can be represented as $v(x, y, t-s)$, depending only on the elapsed time $t-s$. It will be sufficient for our purpose to consider this simpler case. In fact when we meet in the applications probabilities that are not homogeneous in time they usually are also not independent and, therefore, outside the pale of Kolmogoroff's theory. As the equation (8) shows, the main physical characteristic of these systems is that in them the density of distribution D tends toward a state of equilibrium with the definite value $D_0(y)$. Supposing that the system has reached its equilibrium state, we can apply thermodynamical reasoning: For any interval of time the number of systems which move from the state x into the state y must be equal to the number of those which move in the opposite direction. This leads to the formula

$$D_0(x)v(x, y, t-s) = D_0(y)v(y, x, t-s), \quad (9)$$

as a direct expression of the second law of thermodynamics.

For the function $v(y, x, t-s)$, the first equation of diffusion (3) takes the simplified form

$$\frac{\partial v(y, x, t-s)}{\partial t} = a(y) \frac{\partial}{\partial y} v(y, x, t-s) + b(y) \frac{\partial^2}{\partial y^2} v(y, x, t-s). \quad (10)$$

Substituting from (9) $v(y, x, t-s) = D_0(x)v(x, y, t-s)/D_0(y)$, we find

$$\frac{\partial}{\partial t} \frac{v(x, y, t-s)}{D_0(y)} = a(y) \frac{\partial}{\partial y} \left[\frac{v(x, y, t-s)}{D_0(y)} \right] + b(y) \frac{\partial^2}{\partial y^2} \left[\frac{v(x, y, t-s)}{D_0(y)} \right].$$

As a distribution density, $D_0(y)$ must be an integral (independent of time) of the same equation which is satisfied by $v(x, y, t-s)$. We shall see that this requirement is fulfilled if we put

$$D_0(y) = \frac{C}{b} \exp \int \frac{a}{b} dy, \quad (11)$$

where C is a constant. In fact, the equation for the transition probability $v(x, y, t-s)$ becomes then

$$\frac{\partial v}{\partial t} = -\frac{\partial}{\partial y} [a(y)v] + \frac{\partial^2}{\partial y^2} [b(y)v], \quad (12)$$

while that for the equilibrium density can be written in the form

$$0 = -\frac{\partial}{\partial y} [a(y)D_0] + \frac{\partial^2}{\partial y^2} [b(y)D_0]. \quad (13)$$

These equations are of the second Kolmogoroff type which is thus obtained from the first. The asymmetry between the two types is, therefore, shown to be an immediate consequence of the second law of thermodynamics.

4. It is important to understand the physical nature of the coefficients a and b of the diffusion equation. According to the equation (5), $a(y)$ is the mean transition velocity of all elements starting from the state y . At first sight, one should be inclined to think that systems in which this mean velocity vanishes must tend toward an equilibrium with uniform density of distribution. This is, however, not the case: Applying the second law of thermodynamics we found, in the preceding section,² that the equilibrium distribution $D_0(y)$ is not given by the general integral of the equation (13) but by the special integral (11). The condition for D_0 being independent of y is, therefore, $\partial \left[\exp \left(\int \frac{a}{b} dy \right) / b \right] / \partial y = 0$ or $a - \frac{db}{dy} = 0$. In other words, the mean transition velocity a contains a part (equal to db/dy) which has no tendency to change an existing uniformity of distribution but the remainder $a - db/dy$ is operative in disturbing it. This fact is in complete harmony with the considerations by which the equation of diffusion (8) is usually derived in physics. The argument starts from calculating the number of elements which move across a plane, $y = \text{const}$, per unit area and unit time. There are two classes of causes for this motion to be considered. Such causes as *external forces or temperature gradients* belong to the first class and produce an effect of the form $c_1 D$ simply proportional to the density. (For instance,

in the case of an external force F the coefficient $c_1 = MF$, where M is called the mobility). The causes of the second type are called *forces of diffusion* and produce an effect proportional to the negative gradient of the density $-c_2 \partial D / \partial y$. The accumulation of the elements in any point is, therefore, determined by the (negative) divergence of the total effect. This leads to the equation²

$$\frac{\partial D}{\partial t} = -\frac{\partial}{\partial y} (c_1 D) + \frac{\partial}{\partial y} \left(c_2 \frac{\partial D}{\partial y} \right). \quad (14)$$

Comparing this with the form (8) we find

$$b = c_2, \quad a - db/dy = c_1. \quad (15)$$

The expression for c_1 is in agreement with the conclusions arrived at above. It shows that the form (14) of the equation of diffusion is a good guide to the physicist as it brings in evidence the quantities significant from his point of view. On the other hand, Kolmogoroff's equations (3) and (4) are more general since they include the case of transition probabilities which are inhomogeneous in time.

¹ A. Kolmogoroff, *Mathematische Annalen*, 104, 415 (1931).

² The substitution of the general integral of equation (13) for D_0 in (9) and (10) does not lead to the equation (12). Kolmogoroff claims the same results by imposing upon D_0 the condition $\lim D_0 = 0$ for $y = \pm \infty$. That this argument is unconvincing, appears from the fact that it would exclude the uniform density. It is inconvenient to restrict oneself to finite systems.

³ E.g., P. S. Epstein, *Gerlands Beitr. Geophysik*, 35, 154 (1932).

ON THE STATISTICAL THEORY OF TURBULENCE

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G. I. Taylor¹ gave an important impetus to the statistical theory of turbulence by introducing the concept of "isotropic" turbulence, defined by the feature that the mean squares and mean products of the velocity components and of their derivatives are invariant with respect to rotation and reflection of the coördinate axes. Taylor found that under the assumption of the isotropy, the squares and double products of the first derivatives of the velocity components can be expressed by one single correlation function $R(y)$. This function is defined by the ratio $\frac{u_1 u_2}{u^2}$ where u_1 and