THEORETICAL CONSIDERATIONS ON TURBULENT DIFFUSION AND SEDIMENTATION

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THEORETICAL CONSIDERATIONS ON
TURBULENT DIFFUSION AND SEDIMENTATION

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Introduction

In many fluid flows of practical importance in engineering, agriculture, and meteorology, the presence in the flow of foreign matter which is transported by the flow is of great importance. Erosion by flood waters is one important example, the pollution of masses of water or atmosphere is another. The importance of the transported matter arises in various ways; in some cases principal interest centers on the quantity of an impurity removed by a flow, while in others a determination of the characteristics of the main flow itself depends to a large extent on the quantity and nature of solid matter entrained by the flow and carried along in it. The possibility of controlling or predicting such phenomena depends on a knowledge of the physical mechanisms of the processes which are involved. At the present time such knowledge is insufficient to cope with important practical problems.

When the foreign matter consists of solid particles, as in problems of erosion and sedimentation, it is natural to seek an analogy with molecular processes by supposing that the history of each particle or group of particles is characterized by a randomness of the same type present in molecular kinetics. A diffusion theory is thereby obtained, the practical value of which is, of course, determined solely by its ability to explain observed phenomena. Under some circumstances both qualitative and quantitative agreement is found to be good; in other important cases the agreement is less than satisfactory. Similar analogies are used when one considers processes near boundaries at which foreign matter is entrained into and leaves the flow. One then draws analogy with the recognized theories of turbulent transfer of momentum, heat, and vorticity. These attempts likewise meet with varying degrees of success.

Experiments in sedimentation (Vanoni, 1946) have shown not only limits of the theories which have so far been applied, but they have also indicated roughly the dependence of observed discrepancies on parameters in some cases where the analogy theories fail. These circumstances may facilitate attempts to obtain more refined theories. As a first

*See references and bibliography at end of report.
step toward obtaining such refinements, the investigation described by
the present report was conducted with a dual purpose: (1) to make an
examination of the fundamental physical and mathematical features of
classical molecular diffusion theory, with particular regard to the compli-
cations which are believed to be at the base of observed discrepancies
in analogy theories and (2) to survey the work which has been done in
allied fields, especially meteorology, in which similar problems have
been faced for many years by competent physicists and mathematicians,
in order to see which refinements may be carried over to the theory of
sedimentation and turbulent diffusion and to suggest new lines of attack
which may be fruitful.

As the difficulties with the theory are largely due to great mathe-
matical complications in any but the simplest problems, the discussion
which follows places emphasis on mathematical techniques. In order to
make the presentation more readable to workers in hydraulics who do
not make frequent use of modern advanced techniques of mathematical
physics, the discussion is confined to a moderately elementary level.
PART I

CLASSICAL THEORY

For simplicity the diffusion process will be considered first in a one-dimensional, unbounded region. Suppose that initially (at time $t=0$) there is a concentration of unit quantity of an impurity situated at $x=\xi$ and none elsewhere. At some later time $(t=\tau>0)$ part of this amount will have been transferred by an as yet unspecified process of diffusion to the point $x=\alpha$; this quantity is designated by $f(\xi,\alpha,\tau)d\alpha$.

At a still later instant $(t>\tau)$ part of $f(\xi,\alpha,\tau)d\alpha$ will be at $x$; if the processes are the same in both time intervals, this amount will be $f(\xi,\alpha,\tau)d\alpha f(\alpha,x,t-\tau)dx$ and the total concentration there per unit length is the combined effect of all points $\alpha$:

$$f(\xi,x,t) = \int_{-\infty}^{\infty} f(\xi,\alpha,\tau) f(\alpha,x,t-\tau) d\alpha$$

which can also be written

$$f(\xi,x,t+\tau) = \int_{-\infty}^{\infty} f(\xi,\alpha,t) f(\alpha,x,\tau) d\alpha \quad (1)$$

by interchanging $t$ and $\tau$, then replacing $\tau$ by $\tau+\tau$.

Eq. (1) is a quadratic integral equation for the concentration $f$.

Eq. (1) is too difficult to be solved, and this might have been anticipated since it describes a very general class of phenomena. By making assumptions about the mechanism of the diffusion process, it can be put into the form of a tractable differential equation of a well known type.

Denoting $f(\xi,x,t+\tau) \equiv U(x,t+\tau)$ Eq. (1) becomes

$$U(x,t+\tau) = \int_{-\infty}^{\infty} U(\alpha,t) f(\alpha,x,\tau) d\alpha$$

and by assuming that the process is the same at different points and depends not on the points $\alpha$ and $x$ separately but only on the distance $x-\alpha$
between them and on the time interval \( \tau \) which is considered, then we can write

\[
(f(x, \alpha, \tau) = p(x - \alpha, \tau)
\]

so that

\[
U(x, t + \tau) = \int_{-\infty}^{\infty} U(\alpha, t)p(x - \alpha, \tau)\,d\alpha
\]

(1')

The function \( p(x - \alpha, \tau) \) can now be interpreted as the probability that the net displacement of a particle by diffusion will be \( x - \alpha \) in a time interval \( \tau \). One easily deduces the differential equation of diffusion from the above integral equation. For this purpose the left-hand side is expanded in a Taylor series.

\[
U(x, t + \tau) = U(x, t) + \tau \frac{\partial U}{\partial t}(x, t) + \frac{\tau^2}{2} \frac{\partial^2 U}{\partial t^2}(x, t) + \ldots
\]

valid for small enough time intervals \( \tau \). Denoting the displacement \( x - \alpha \) by \( \xi \), the right-hand side can be expanded in powers of \( \xi \):

\[
\int_{-\infty}^{\infty} U(x, t)p(x - \alpha, \tau)\,d\alpha = \int_{-\infty}^{\infty} \left\{ U(x, t) - \frac{\partial U(x, t)}{\partial x} - \frac{\xi^2}{2} \frac{\partial^2 U(x, t)}{\partial x^2} + \ldots \right\} p(\xi, t)\,d\xi
\]

in which \( \frac{\partial^n U(x, t)}{\partial x^n} \) means \( \frac{\partial^n U(x - \alpha, t)}{\partial x^n} \) evaluated for \( \alpha = 0 \)

The first term in the last integral

\[
\int_{-\infty}^{\infty} U(x, t)p(\xi, t)\,d\xi = U(x, t)\int_{-\infty}^{\infty} p(\xi, t)\,d\xi = U(x, t)
\]

since the implied property of the function \( p \) is fundamental in its definition. The second involves

\[
\int_{-\infty}^{\infty} \xi^2 p(\xi, t)\,d\xi
\]

and vanishes if we assume that displacements to the right or to the left are of equal likelihood, i.e., if \( p(-\xi, t) = p(\xi, t) \)

Substituting the two expansions in (1), dividing by \( \tau \) and taking the limit as \( \tau \to 0 \), one finds that

\[
\frac{\partial U(x, t)}{\partial t} = -\frac{\partial}{\partial x} \left\{ U(x, t) \right\} + \frac{\partial^2}{\partial x^2} \left\{ \frac{\partial U(x, t)}{\partial x} \right\} - \frac{\partial^3}{\partial x^3} \left\{ \frac{\partial^2 U(x, t)}{\partial x^2} \right\} + \ldots \quad (2)
\]

where

\[
J_n = \lim_{\tau \to 0} \frac{1}{\tau} \int_{-\infty}^{\infty} \xi^n p(\xi, t)\,d\xi, \quad n = 1, 2, \ldots
\]

are assumed to exist. (It is clear that these functions exist if the "probability function" \( p \) vanishes rapidly enough for increasing \( |\xi| > 0 \).)

Eq. (2) is known in the mathematical theory of statistics as the
Fokker-Planck differential equation. (See, for example, Frank von Mises, Vol II; p. 595). On account of the fact that \( p(\mathbf{\xi}, \tau) \) is an even function of \( \mathbf{\xi} \) when displacements in one direction are assumed to be as probable as those in the opposite direction \( J_{2n+1} = 0 \) for \( n = 0, 1, 2, \ldots \). It can, moreover, be shown that in mechanical systems in which motions exist by virtue of external forces and thermal motions of molecules the probability function \( p \) is of such a form that

\[
J_{2n} = \begin{cases} \text{const.} \ 2k & n = 1 \\ 0 & n = 2, 3 \end{cases}
\]

so that for diffusion processes (2) reduces to

\[
\frac{\partial U(x,t)}{\partial t} = k \frac{\partial^2 U(x,t)}{\partial x^2}
\]

which is the diffusion equation for a homogeneous one-dimensional medium. The generalization of Eq. (3) to two or three dimensions is easily seen to be

\[
\frac{\partial U}{\partial t} = k \Delta U
\]

where

\[
\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \quad \text{or} \quad \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}
\]

in rectangular coordinates, respectively.

A simpler but less fundamental deduction of Eq. (3) or (4) is obtained by making the explicit assumption that the rate of diffusive transfer of particles is proportional to the gradient of their concentration \( U \). Considering the simplest one-dimensional case again, the increase of concentration \( U \) in an element of extent \( dx \) and cross section \( A \) in time \( dt \) is \( \frac{\partial U}{\partial x} Adt \). The total flow into the elementary area at \( x \) is \( -k \frac{\partial U}{\partial x} Adt \) under the above assumption, while that into the face at \( x + dx \) is

\[
+ \left[ k \frac{\partial U}{\partial x} + \frac{\partial}{\partial x} (k \frac{\partial U}{\partial x}) \delta x \right] Adt
\]

Hence in the absence of sources or sinks

\[
\frac{\partial U}{\partial t} \delta x Adt = \frac{\partial}{\partial x} (k \frac{\partial U}{\partial x} \delta x) Adt
\]

\[
-k \frac{\partial U}{\partial x} Adt \rightarrow \begin{bmatrix} \text{rectangle} \end{bmatrix} \rightarrow - \left[ k \frac{\partial U}{\partial x} + \frac{\partial}{\partial x} (k \frac{\partial U}{\partial x}) \delta x \right] Adt
\]
so that since homogeneity implies $\frac{\partial k}{\partial x} = 0$, one again obtains

$$\frac{\partial U}{\partial t} = k \frac{\partial^2 U}{\partial x^2}$$

(3)

or its generalization Eq. (4). The first derivation of Eq. (3) has the advantage of displaying the deeper physical significance of the assumption that diffusive transfer is quantitatively proportional to the gradient of concentration. Pursuing similar considerations Einstein showed in 1905 that

$$k = \frac{RT}{6\pi \eta a}$$

(4)

where

- $R$ = gas constant per mole = $1.372 \times 10^{-6}$ erg/deg.
- $T$ = absolute temperature
- $\eta$ = viscosity coefficient of the medium (solvent)
- $a$ = particle diameter

Many physically important generalizations of Eqs. (3) and (4) are encountered in engineering and science. These follow from the necessity of relaxing assumptions of homogeneity and isotropy of the medium so that the diffusion coefficient is variable on the one hand and from the existence of preferred directions of motion for the diffusing particles on the other hand. The latter situation arises, for example, when the density of the particles differs sufficiently from that of the medium so that a gravitational force field has a significant effect, as in sedimentation processes. An example of the former situation is in diffusion in a stably stratified atmosphere in which case coefficients of diffusion depend on elevation above sea level.

Such generalizations of the simplest diffusion processes present formidable complications of the mathematics involved in obtaining solutions of the resulting partial differential equations, and consequently few solutions have been obtained, compared with the number which have been found for the simpler processes governed by Eqs. (3) and (4).

It is known that many of the fundamental features of the solutions of Eqs. (3) and (4) also characterize the more complicated processes. This ceases to be true, however, for those cases in which the coefficient $k$ depends on the concentration itself, or those in which the equation is non-linear for other reasons.

Insight into the physical mechanism of the diffusion processes can be obtained by finding solutions of Eqs. (3) and (4) in particular cases. In the remainder of the present chapter typical and otherwise significant examples will be analyzed. Emphasis will be placed on the features which are found to characterize all diffusion processes. At the same time various mathematical techniques suitable for the solution of diffusion
problems will be employed and relative advantages indicated. Particular attention will also be given to elucidating those experiments which are suggested by the theoretical analysis as suitable for dealing with the practically important problem of evaluating diffusion coefficients numerically. Connections with related fields of mathematical physics will also be briefly indicated.

Applications of classical diffusion theory to meteorology and hydraulics, as well as modern refinements of mathematical technique will be discussed in the second chapter.

In the simplest problem of diffusion the concentration of the diffusing substance or property varies in one direction only, and the initial distribution is known. If the diffusion coefficient does not vary from one point to another or with time, the distribution at later instants of time is easily determined. The resulting initial value problem is then expressed by the differential Eq. (3)

\[ \frac{\partial U}{\partial t} = k \frac{\partial^2 U}{\partial x^2} \]  

combined with the initial condition, at \( t = 0 \),

\[ U(x,0) = \Phi(x) \]

Assuming the medium to be of infinite extent, no boundaries are present at which further conditions must be satisfied.

In this case a product-type solution is obtained by assuming that \( U \) is the product of one factor which depends on the coordinate \( x \) alone, multiplied by a factor which depends only on the time, \( t \), i.e.,

\[ U(x,t) = X(x)T(t) \]

Then Eq. (3) becomes

\[ k \frac{X''}{X} = \frac{T'}{T} \]

where primes denote (ordinary) differentiation.

The left-hand side of this equation is a function of \( x \) alone and by reason of this equation equals a function of \( t \) alone; hence each is equal to the same constant, \( \alpha = -\lambda^2 k \)

Then

\[ \begin{align*}
\lambda'' + \lambda^2 \lambda &= 0 \\
T' + \lambda^2 kT &= 0
\end{align*} \]

which has the two particular solutions \( \sin \lambda x \) and \( \cos \lambda x \). The time dependence is then found from

\[ T' + \lambda^2 kT = 0 \]

which has the particular solution \( e^{-\lambda^2 kt} \) which is finite for all \( t \geq 0 \) when \( \lambda^2 \) is real and positive.

Summation of particular solutions for \( X \) and \( T \) leads to the general solution
\[ U(x,t) = \int_0^\infty [A(\lambda) \cos \lambda x + B(\lambda) \sin \lambda x] e^{-\lambda^2 kt} d\lambda \]

which involves two functions \( A(\lambda) \) and \( B(\lambda) \) which can be determined so that the initial condition \( U(x,0) = \Phi(x) \) for \( t = 0 \) is satisfied.

If \( \Phi(x) \) is defined for all real values of \( x \), sectionally continuous over each finite interval of the \( x \)-axis, and becomes small enough as \( x \) becomes large so that the integral is absolutely convergent, and if, finally, we agree to define \( \Phi(x) \) at a point of discontinuity \( x_0 \) by the mean value of its limits from the right and left of \( x_0 \); that is,

\[ \Phi(x_0) = \frac{1}{2} [ \Phi(x_0+0) + \Phi(x_0-0) ] \]

then Fourier's Integral Theorem states that any such function can be represented as

\[ \Phi(x) = \frac{1}{\pi} \int_0^\infty d\lambda \int_0^\infty \Phi(\lambda) \cos \lambda(x-x) d\lambda \]

Comparison with the form taken by \( U(x,t) \) for \( t = 0 \) and expansion of \( \cos \lambda(x-x) \) shows that

\[ A(\lambda) = \frac{1}{\pi} \int_{-\infty}^{\infty} \Phi(\lambda) \cos \lambda d\lambda, \quad B(\lambda) = \frac{1}{\pi} \int_{-\infty}^{\infty} \Phi(\lambda) \sin \lambda d\lambda \]

and hence

\[ U(x,t) = \frac{1}{\pi} \int_0^\infty e^{-\lambda^2 kt} d\lambda \int_0^\infty \Phi(\lambda) \cos \lambda(x-x) d\lambda \]

which can be simplified by interchanging the order of integrations and carrying out the integration with respect to \( \lambda \). Since

\[ \int_0^\infty e^{-\lambda^2 kt} \cos \lambda(x-x) d\lambda = \text{Re} \int_0^\infty e^{-\lambda^2 kt} e^{i\lambda(x-x)} d\lambda \]

and

\[ -\lambda^2 kt + i\lambda(x-x) = -\left( \lambda \sqrt{k t} - \frac{i(x-x)}{2\sqrt{kt}} \right) - \frac{(x-x)^2}{4kt} \]

then

\[ \int_0^\infty e^{-\lambda^2 kt + i\lambda(x-x)} d\lambda = \frac{1}{\sqrt{kt}} e^{-\frac{(x-x)^2}{4kt}} \int_0^\infty e^{-\xi^2} d\xi = \frac{1}{2\sqrt{\pi kt}} e^{-\frac{(x-x)^2}{4kt}} \]

since

\[ \int_0^\infty e^{-\xi^2} d\xi = \frac{\sqrt{\pi}}{2} \]

is easily proved.

Finally

\[ U(x,t) = \frac{1}{2\sqrt{\pi kt}} \int_{-\infty}^{\infty} \Phi(\lambda) e^{-\frac{(x-x)^2}{4kt}} d\lambda \quad (5) \]
That Eq. (5) is actually the solution to the problem must be establis hed, since there was no a priori assurance that the assumed product-type solution exists. To do this one observes first that

$$\frac{i}{\sqrt{t}} e^{-\frac{(\alpha-x)^2}{4kt}}$$

is a solution of Eq. (3), for any value of $\alpha$, by direct differentiation and substitution in Eq. (3). A simple limiting procedure then shows that the initial condition is satisfied.

Letting $\xi = \frac{\alpha-x}{2\sqrt{kt}}$ in Eq. (5)

it becomes

$$U(x,t) = \frac{1}{\sqrt{\pi}} \int_0^\infty \phi(2\xi \sqrt{kt} + x) e^{-\xi^2} d\xi$$

which, in the limit as $t \to 0$ clearly becomes

$$U(x,0) = \frac{1}{\sqrt{\pi}} \int_0^\infty \phi(x) e^{-\xi^2} d\xi = \phi(x)$$

A physical interpretation of the solution (5) as the superposition of effects of a distribution of fundamental solutions is apparent. One such solution is

$$U_{\gamma\alpha}(x,t) = \frac{\gamma}{2\sqrt{rt}} e^{-\frac{(\alpha-x)^2}{4kt}}$$

(6)

and it is evident that as $t \to 0$

$$U_{\gamma\alpha}(x,0) = \begin{cases} 0 & \alpha \neq \alpha \\ \infty & \alpha = \alpha \end{cases}$$

whereas

$$\int_{-\infty}^\infty \frac{\gamma}{2\sqrt{rt}} e^{-\frac{(\alpha-x)^2}{4kt}} dx = \gamma$$

These last two properties of $U_{\gamma\alpha}(x,t)$ show that (6) represents an instantaneous "diffusion pole" of strength $\gamma$ situated at $x=\alpha$ at time $t=0$, so that (6) gives the distribution in space and time $t>0$ resulting from the release at time $t=0$ at location $x=\alpha$ of quantity $\gamma$ per unit area of the $yz$ plane of an impurity into an otherwise pure substance, when the physical process is one of diffusion. It is to be noted that the speed of propagation is infinite since $U_{\gamma\alpha}(x,t) > 0$ for all $|x|<\infty$ and $t > 0$.

The solution (5) therefore represents the superposition of the effects of such instantaneous poles of strength $\phi(x) d\alpha$ at $x=\alpha$, each such pole acting independently of all the others.

The nature of the fundamental solution (6) will be clarified further by considering simple particular distributions $\phi(x)$. 
The connection between diffusion and probability is also suggested by the form of Eq. (6) and merits brief discussion before considering special cases. We compare (6) with the Gaussian law of error
\[ dE = \sqrt{\beta / \pi} e^{-\beta x^2} dx \]
where \( dE \) is the probability of an error in a measuring process whose precision is given a "precision factor" \( \beta \). In the case of (6) this factor is \( \gamma^2(4kt)^{-1} \); infinite precision at \( t=0 \) which corresponds to absolute concentration of impurity in the plane \( x=\alpha \) and decreasing precision (concentration) at later time \( t>0 \).

The solution (6) is plotted in Fig. 1, from which important geometrical properties are evident.

For successive instants of time \( 0 < t_1 < t_2 < t_3 < \cdots \), the curves are all symmetrical about \( x=\alpha \), have steadily decreasing maxima and constant area enclosed between each and the x-axis, and ultimately approach \( \phi(x,\infty) = 0 \) as the quantity \( \gamma \) of substance is distributed over an ever widening range.

As an application of the formula (5) consider a very long cylindrical container which is divided by a partition normal to the axis of the cylinder; on one side of the partition there is a "pure" substance, i.e., vanishing concentration of impurity, on the other side there is a uniform concentration \( \phi = \text{const} \). At time \( t=0 \) the partition is removed and a pollution of the pure substance by a diffusion process ensues.

The analytical formulation of the problem then consists of the differential Eq. (3) and initial condition in which
\[ \phi(x) = \begin{cases} 0 & x < 0 \\ \phi = \text{const} & x > 0 \end{cases} \]
so that the solution (5) becomes
\[ \phi(x,t) = \frac{\phi}{2\sqrt{\pi kt}} \int_{0}^{\infty} e^{-\frac{(x-\alpha)^2}{4kt}} d\alpha = \frac{\phi}{2} \left[ 1 + \text{erf} \frac{x}{2\sqrt{kt}} \right] \]
where the error function is, by definition,

$$\text{erf} \left( \frac{X}{2 \sqrt{kt}} \right) = \frac{2}{\sqrt{\pi}} \int_0^X e^{-y^2} \, dy$$

This function is tabulated in many convenient references (e.g. see ref. 4).

The concentration for successive instants of time $t > 0$ is then found from tables of values of the error function; the distribution for successive instants of time $0 = t_0 < t_1 < t_2 < \cdots < t_\infty = \infty$, is indicated qualitatively in Fig. 2. The solution is seen to

represent a continuous diffusion from the region $X > 0$ to the region $X < 0$. On both sides of $X = 0$ the value $\frac{U_0}{2}$ is approached asymptotically.

The total quantity which has passed the plane $X = 0$ up to time $T$ is easily computed. Denoting the quantity per unit area by $Q(T)$

$$Q(T) = \int_0^T k \frac{\partial U(0,t)}{\partial x} \, dt = \int_0^T \frac{U_0 \sqrt{kT}}{4 \sqrt{t}} \, dt = \frac{U_0}{2 \sqrt{\pi}} \sqrt{\frac{kT}{\pi}}$$

which suggests that if the quantity $Q$ is measurable the diffusion coefficient is obtained immediately from it as

$$\kappa = \frac{\pi T}{Q(T)} \frac{Q(T)}{U_0^2}$$

It is sometimes also of importance to obtain an idea of how rapidly the diffusion process progresses through a medium and upon what parameters this depends. In the present problem one can determine the diffusion time which will lapse before a given concentration $U < U_0/2$ is developed at some point $X < 0$.

This is easily obtained from Eq. (8) since

$$x = 2\sqrt{kt} \, \text{erf}^{-1} \left\{ \frac{2U}{U_0} - 1 \right\}$$ (9)
gives the coordinate \( \chi \) at which the concentration will be precisely \( \mathcal{U} \) at time \( t \), where the notation \( \text{erf}^{-1} \) is the function inverse to \( \text{erf} \), i.e., if

\[
Z = \text{erf} \chi = \frac{2}{\sqrt{\pi}} \int_{\chi}^{\infty} e^{-\xi^2} d\xi
\]

then

\[
\text{erf}^{-1}\{Z\} = \chi
\]

For a given concentration \( \mathcal{U} \), then

\[
\frac{dx}{dt} = \sqrt{\frac{k}{t}} \text{erf}^{-1}\left\{\frac{2\mathcal{U}}{\mathcal{U}_0} - 1\right\}
\]

so that the propagation speed for a given concentration \( \mathcal{U} \) decreases with time and is proportional to the square root of the coefficient \( k \).

In the same manner one can determine, in the region \( \chi > 0 \), how much time is required for the concentration to decrease from \( \mathcal{U}_0 \) to some value \( \mathcal{U} > \frac{\mathcal{U}_0}{2} \). This problem may be of importance in some applications.

It is clear that Eqs. (9) or (10) can be used to determine \( k \), the diffusion coefficient from observations of concentration at a measured position and instant of time.

Another example of the simplest diffusion process results when the initial distribution of a diffusing substance consists of a layer of uniform concentration embedded in an initially undiluted field. Specifically if we take

\[
\Phi(\chi) = \begin{cases} 
\mathcal{U} & |\chi| < 1 \\
0 & |\chi| > 1
\end{cases}
\]

where \( \mathcal{U} \) = constant, then Eq. (5) becomes

\[
\mathcal{U}(\chi, t) = \frac{1}{2\sqrt{\pi kt}} \int_{-1}^{+1} \mathcal{U} e^{-\frac{(\chi - \alpha)^2}{4kt}} d\alpha
\]

\[
= \frac{\mathcal{U}}{2} \left[ \text{erf} \frac{1 + \chi}{2\sqrt{kt}} + \text{erf} \frac{1 - \chi}{2\sqrt{kt}} \right]
\]

which has the general form of the curves in Fig. 2, from which several characteristics of the diffusion process can readily be seen. It is clear that the concentration decreases most rapidly initially in the region which originally contained the diffusing substance (or property), subsequently decreases there more slowly. In the immediately adjacent region \((|\chi| = |1 + \varepsilon|)\) the concentration increases at first then decreases. At large distances \(|\chi| >> 1\) there is a slower increase of concentration followed by a decrease ultimately approaching zero as the diffused matter is spread over an
effectively ever-increasing interval.

The exponential behavior of the solution (11), as well as that of Eqs. (5) and (8), will be seen to be typical of diffusion processes.

Although the solution (5) represents a highly idealized problem, there are problems of practical importance which are elucidated by it. In practical problems of diffusion, more generally, the physical circumstances are often too complicated to permit complete analytical solution. In such cases it is often worthwhile to investigate the usefulness of the solution to an appropriately chosen idealized problem such as (5), for example. This will be done for Eq. (11).

As the problem was formulated, the assumption was made of dependence on only one space variable $X$; this corresponds to assuming a plane layer of infinite extent (and thickness two units, $-1 < X < 1$). In a practical problem in which an adulterant is initially concentrated in a plane layer of finite thickness and of finite extent solution (5) does not apply, strictly speaking. One may expect, however, that for points much closer to the layer than to other boundaries, and at relatively small values of the time $t$, the solution will be useful, since the influence of a boundary will not have had sufficient time to make its influence felt. To make these ideas more precise, consider a point with coordinate $X = 1 + \xi$, $0 < \xi << 1$; from (11) it is clear that the first term will predominate, i.e.,

$$\text{erf} \frac{2 + \xi}{2 \sqrt{kt}} \gg \text{erf} \frac{-\xi}{2 \sqrt{kt}}$$

and "small $t$" means that for

$$\frac{2 + \xi}{2 \sqrt{kt}} \gg 1$$

or

$$t \ll \left(\frac{2 + \xi}{2 \sqrt{k}}\right)^2$$
the solution, which then becomes:

$$U(x,t) = \frac{1}{2} \operatorname{erf} \left( \frac{1 + \frac{x}{2\sqrt{kt}}}{2\sqrt{kt}} \right)$$

may be expected to represent a real situation fairly closely. This shows that the larger the diffusion coefficient (compared with the quantity $\frac{2 + x}{2\sqrt{t}}$ of the same dimensions, of course), the longer will be the time $t$ during which the solution (11) is useful for points near the initial layer. The limits on this distance are likewise easily obtained quantitatively in terms of the physical parameter $k$ and time $t$ by means of inequalities.

In the examples considered up to this point the diffusion coefficient $k$ has been assumed to be independent of the concentration $U$ but in some cases this assumption may be untenable. In this event (or to determine if this is so), the form of $k(U)$ can be found by integrating the results of measurements of the concentration in a process corresponding to simple but special initial boundary conditions.

By relaxing the assumption $k = \text{constant}$ in the derivation of Eq. (3), the diffusion equation becomes

$$\frac{\partial U}{\partial t} = k(U) \frac{\partial^2 U}{\partial x^2} + k'(U) \left( \frac{\partial U}{\partial x} \right)^2$$

(12)

where $k'(U) = \frac{\partial k}{\partial U}$.

The highly non-linear character of Eq. (12) precludes the possibility of integrating the equation for a significant physical problem, but its peculiar form permits reduction to an ordinary differential equation, so that $k(U)$ can be found.

This will be done for the diffusion process with initial conditions

$$U(x,0) = \begin{cases} 0 & \text{if } x < 0 \\ U_0 = \text{const.} & \text{if } x > 0 \end{cases}$$

To obtain the ordinary differential equation it is tentatively assumed that the concentration $U$ depends only on one new variable $\eta = xt^\lambda$ where the exponent $\lambda$ is to be determined. Under this assumption Eq. (12) becomes

$$\lambda \eta \frac{dU}{d\eta} = \left\{ k(U) \frac{d^2 U}{d\eta^2} + k'(U) \left( \frac{dU}{d\eta} \right)^2 \right\} t^{2\lambda + 1}$$

(12')

which clearly will depend on $\eta$ but not $x$ or $t$ separately, if the condition

$$2\lambda + 1 = 0$$

is satisfied, i.e., if $\eta = \sqrt{\frac{x}{t}}$. For $x > 0, t \to 0, \eta \to +\infty$, while for $x < 0, t \to 0, \eta \to -\infty$ hence the new equation is to be satisfied subject to the two "boundary" conditions:

$$U(\eta) \bigg|_{\eta = \infty} = U(0) = U_0 = \text{const.}$$

$$U(-\infty) = 0$$
Rewriting (12') as
\[ \frac{d^2 U}{d\eta^2} + \frac{\eta}{2k(U)} \frac{dU}{d\eta} + \frac{k'(U)(dU)^2}{k(U)d\eta} = 0 \]
it is evident, since
\[ k(U) \frac{d^2 U}{d\eta^2} + k(U) \frac{dU}{d\eta} = \frac{d}{d\eta} \left[ k(U) \frac{dU}{d\eta} \right] \]
that
\[ k(U) \frac{dU}{d\eta} = -\int \eta \frac{dU}{d\eta} \, d\eta \]
Hence if one determines \( U = U(\eta) \) (by measurements of a controlled experiment, or otherwise), then the dependence of diffusion coefficient \( k \) on concentration \( U \) is obtained by quadrature from the relation
\[ k(U) = -\frac{d}{dU} \int \eta \frac{dU}{d\eta} \, d\eta = -\frac{d}{dU} \int_0^U \frac{\eta}{2} \, dU \]  \hspace{1cm} (13)
In most cases in which the properties of the medium are not greatly influenced by the diffusion process itself, e.g., those in which diffusion occurs in weak mixtures, the diffusion coefficient may not be expected to depend sensitively on the concentration. When these conditions are not fulfilled, however, or when for other reasons it appears that this dependence is not negligible, formula (13) in conjunction with suitable measurements will furnish the form of the dependence and quantitative values of \( k \).

**Diffusion in a Bounded Medium with Uniform Concentration at Boundary**

If there is a wall at \( x = 0 \) at which diffusing matter is completely removed so that the concentration there is zero, while the distribution of \( U \) is known at \( t = 0 \) for \( x > 0 \), then the problem is given by the differential equation
\[ U_t = k U_{xx} \]  \hspace{1cm} (3)
one initial condition

**I. C.**
\[ U(x,0) = \Phi(x), \quad x > 0 \]
and one boundary condition

**B. C.**
\[ U(0,t) = 0 \]
This combined initial- and boundary- value problem can be immediately reduced to the pure initial value problem
\[ U_t = k U_{xx} \]
\[ U(x,0) = \Phi(x) \quad x > 0 \quad \text{and} \quad \Phi(-x) = -\Phi(x) \quad x < 0 \]
which amounts to an odd reflection in the plane \( x = 0 \)

From (5), then

\[
U(x,t) = \frac{1}{2\sqrt{\pi kt}} \left[ \int_{-\infty}^{0} \Phi(x) e^{-\frac{(x-x)^2}{4kt}} \, dx + \int_{0}^{\infty} \Phi(x) e^{-\frac{(x+x)^2}{4kt}} \, dx \right]
\]

which is the required solution obtained by simple reflection in the plane \( x = 0 \).

**Diffusion in Two Dimensions**

In the examples considered above it was assumed that diffusion occurred in a medium in which all physical quantities depend on only one space coordinate \( x \). A somewhat less special situation occurs when diffusion is no longer assumed to be identical in every plane \( x = \text{const.} \), but also depends on another coordinate, \( y \), say, but remains independent of \( z \). Such a diffusion process might exist if the medium is bounded by walls whose generators are all parallel to the \( z \) axis.

If the medium is again assumed to be homogeneous and isotropic, the differential equation for the diffusion process Eq. (4) becomes

\[
\frac{\partial U}{\partial t} = k \left( \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} \right)
\]

and general solutions of this equation will be obtained by first considering the special case in which \( U \) depends on time \( t \) and the distance \( r \) of a point from the \( z \)-axis.

Since

\[
\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = \frac{\partial^2 U}{\partial r^2} + \frac{1}{r} \frac{\partial U}{\partial r}
\]

in this case, then by assuming that a solution can be of the form \( U(r,t) = R(r) \bar{T}(t) \) one obtains from the differential equation

\[
\frac{1}{R} \left[ R'' + \frac{R'}{r} \right] = \frac{\bar{T}'}{k \bar{T}} = -\lambda^2
\]

in which \( \lambda \) is a real constant, as previously. Then the two resulting ordinary differential equations

\[
\bar{T}' + k \lambda^2 \bar{T} = 0
\]

\[
R'' + \frac{R'}{r} + \lambda^2 R = 0
\]
have solutions \( T = e^{-k\xi t} \) and \( R = A J_0(\lambda r) + B Y_0(\lambda r) \) where \( J_0 \) and \( Y_0 \) are the symbols for two independent solutions of Bessel's differential equation. Considering only solutions which are finite at \( r = 0 \) one has the general solution
\[
U(r, t) = \int_{0}^{\infty} A(\lambda) e^{-k\xi^2 t} J_0(\lambda r) d\lambda
\]
in which \( A(\lambda) \) is so far an arbitrary function of \( \lambda \) which can be so chosen that initial conditions are satisfied for a specific problem.

To determine the form \( A(\lambda) \) one uses the theorem from the theory of Bessel's functions for the representation of a function as a double integral over Bessel's functions (see, for example, Whittaker and Watson, p. 385)
\[
\Phi(\xi) = \int_{0}^{\infty} J_0(\lambda r) \lambda d\lambda \int_{0}^{\infty} \Phi(\alpha) J_0(\lambda \alpha) d\alpha.
\]
(15)

This representation is analogous to the use of Fourier's Integral Theorem which was made in obtaining Eq. (5) above; Eq. (15) is called the Fourier-Bessel integral.

Since at \( t = 0 \), Eq. (14) becomes
\[
U(r, 0) = \Phi(r) = \int_{0}^{\infty} A(\lambda) J_0(\lambda r) d\lambda
\]
comparison with (15) shows that
\[
A(\lambda) = \Phi(\alpha) J_0(\lambda \alpha) \alpha d\alpha
\]
and hence the solution of Eq. (4') satisfying the initial condition \( U(r, 0) = \Phi(\alpha) \) is
\[
U(r, t) = \int_{0}^{\infty} \Phi(\alpha) J_0(\lambda \alpha) \alpha e^{-k\xi^2 t} J_0(\lambda r) d\alpha d\lambda
\]
(16)

One integration of the double integral (16) can be effected by considering a unit source at the origin, i.e., \( \Phi(\alpha) = 0 \) for \( \alpha = 0 \), but
\[
\int_{0}^{\infty} \Phi(\alpha) \alpha d\alpha = \frac{1}{2}
\]
for then (16) becomes
\[
U_0(r, t) = \int_{0}^{\infty} e^{-k\xi^2 t} J_0(\lambda r) \lambda d\lambda
\]
(17)
which can be evaluated. Substituting
\[ J_0(\lambda r) = \sum_{n=0}^\infty \frac{(-1)^n (\lambda r)^{2n}}{2^{2n} (n!)^2} \]
in the integral (17), integrating each term by parts and summing, one easily obtains
\[ U_0 = \frac{e^{-\frac{r^2}{4kt}}}{2\sqrt{\pi kt}} ; \]
introducing a constant factor one may then write
\[ U_0(r,t) = \frac{\gamma e^{-\frac{r^2}{4kt}}}{4\pi kt} \]
(18)
as the expression for a two-dimensional instantaneous diffusion pole of strength \( \gamma \) situated at \( r=0 \) (compare with Eq. (6). The correctness of the choice of the constant factor is easily verified by a simple integration. That Eq. (18) is actually a solution of (4) is verified directly by differentiation or by observing that the series for \( J_0 \) is uniformly and absolutely convergent. The generalization corresponding to a distribution of such poles over the x-y plane, with strength \( \Phi(x,r) \) at \( x=\xi, y=r \) clearly is
\[ U(x,y,t) = \frac{1}{2\pi kt} \int \Phi(\xi,\eta) e^{-\frac{(x-\xi)^2+(y-\eta)^2}{4kt}} d\xi d\eta \]
(19)

**Diffusion in Three Dimensions**

With a slight but suggestive change of notation one may observe that the one-dimensional diffusion pole (6) at \( i=0 \) and of unit strength is
\[ \frac{1}{2\sqrt{\pi kt}} e^{-\frac{r^2}{4kt}} \]
and that in two dimensions the analogous solution for a two-dimensional diffusion pole was found to be
\[ \frac{1}{2\sqrt{\pi^{1.5}kt}} e^{-\frac{r^2}{4kt}} \]
(18)
One may, therefore, inquire whether in three dimensions the unit pole at the origin would be (with \( r^2 = x^2+y^2+z^2 \))
\[ \frac{1}{2\sqrt{\pi^{1.5}kt}} e^{-\frac{r^2}{4kt}} \]
(20)
That this conjecture is correct can be determined by proceeding in the same manner as previously by finding product solutions of the three-dimensional diffusion equation. Or it can be verified directly by
differentiation of (20) and by showing that the value of the integral

$$\int_0^\infty \frac{4\pi r^2 dr}{[2(\pi kt)]^{3/2}} e^{-\frac{r^2}{4\pi kt}} = 1$$

On account of the importance of this solution and the methods by which it will be obtained, the solution (20) will be established by different methods than those mentioned above. Starting from the equation for diffusion in three dimensions with spherical symmetry

$$\frac{\partial U}{\partial t} = k \left( \frac{\partial^2 U}{\partial r^2} + \frac{2}{r} \frac{\partial U}{\partial r} \right) \quad (4'')$$

it is clear that $V = U$ satisfies the equation

$$\frac{\partial V}{\partial t} = k \frac{\partial^2 V}{\partial r^2} \quad (21)$$

of one-dimensional diffusion. If we are interested in obtaining solutions $U$ which are regular at the origin $r=0$, we will want to find solutions of Eq. (21) for which $V=0$ at $r=0$.

The fundamental solution of Eq. (21) was found above to be

$$V = \frac{1}{2\sqrt{\pi kt}} e^{-\frac{r^2}{4\pi kt}} \quad (22)$$

which corresponds to

$$U = \frac{1}{2r/\sqrt{\pi kt}} e^{-\frac{r^2}{4\pi kt}}$$

which, however, is clearly not regular at $r=0$ and is, therefore, not a suitable solution. A more useful solution is obtained from (22) by differentiation with respect to $r$, since

$$\frac{\partial V}{\partial r} = \frac{-re^{-\frac{r^2}{4\pi kt}}}{4\sqrt{\pi kt}}$$

is clearly also a solution of (21); the corresponding solution of (4'') is

$$U = \frac{\gamma e^{-\frac{r^2}{4\pi kt}}}{[2\sqrt{\pi kt}]^3} \quad (23)$$

within a constant multiplicative factor; (23) is regular at $r=0$ and indeed represents a diffusion pole of strength $\gamma$ at time $t=0$ situated at $r=0$.

Comparison of the diffusion poles for one, two, and three dimensions, (6), (18), and (20) indicate that two- and three-dimensional effects result solely in increasingly rapid diffusion of concentration $U$, that is at a point in the medium the concentration approaches zero more rapidly; the spatial distribution remains the same -- $e^{-\frac{r^2}{4\pi kt}}$, in all cases.
On the Integral Equation of Diffusion

It is of interest to note in connection the the integral equation (1') that its solutions also satisfy the partial differential Eq. (3) of diffusion, when the "probability function" \( p(x-\alpha,\tau) \) is even in \( x-\alpha \). With a slight change of notation, it is

\[
U(x,t) = \int_{-\infty}^{\infty} U(\alpha,\tau) p(x-\alpha, t-\tau) d\alpha
\]  

(24)

and according to the meaning of the function \( p \) and its relation to the fundamental solution (6), which can be written now

\[
U_{t,x}(x, t-\tau) = \frac{\gamma}{2\sqrt{\pi k(t-\tau)}} e^{-\frac{(x-\alpha)^2}{4k(t-\tau)}}
\]  

(6')

it is seen that the solutions of (1') are linear superpositions of fundamental solutions of Eq. (3); consequently the solutions of Eq. (1') also possess this property.

By making use of the fact that the function

\[
V_n(x,t) = (-t)^{\frac{n}{2}} H_n\left(\frac{x}{2\sqrt{k t}}\right)
\]  

(25)

is a solution of the diffusion Eq. (3), where \( H_n \) is the Hermite polynomial of degree \( n \), a linear integral equation with symmetric kernel can be obtained from Eq. (1'). On account of the completeness of the functions \( H_n \), the initial concentration as a function of the coordinate \( x \) can be expanded in a series of Hermite polynomials and the resulting integral equation can be solved by standard methods.

An alternative method of obtaining the integral equation, equally applicable in the case of diffusion in several dimensions, is to proceed from the differential equation by seeking product-type solutions. This leads to ordinary differential equations which can be transformed into integral equations by elementary considerations. This approach may be especially useful when initial values are not most easily expressed in closed form, or when for other reasons an approximate solution is sought.

Turning from highly idealized diffusion problems to more realistic situations, the mathematical treatments become increasingly complex. The techniques of Laplace Transform and Green's Functions are found in some such problems to be quite useful. Particularly to indicate the application of Laplace Transform methods, consider next the one-dimensional diffusion process in a semi-infinite medium in which the concentration at the wall \( U(0,t) \) is permitted to vary with time (as in the
presence of absorbers, or when the diffusing substance is emitted into the medium intermittently). If the initial concentration on \( U(x,0) = \Phi(x) \), say, then the mixed initial and boundary value problem becomes

\[
P. D. E. \quad U_t = k U_{xx}, \quad U = U(x,t) \quad (26)
\]

I. C. \quad U(x,0) = \Phi(x) \quad (27)

B. C. \quad U(0,t) = \varphi(t) \quad (28)

and by introduction of the Laplace Transform of \( U(x,t) \) as

\[
V(x,s) = \int_0^\infty e^{-st} U(x,t) dt = \mathcal{L}\{U,s\} \quad (29)
\]

(see, for example, Churchill's book on Laplace Transform Techniques listed in the bibliography), the problem is at once reduced to an ordinary differential equation

\[
-\Phi(x) + sV(x,s) = k V_{xx}(x,s) \quad (26')
\]

of the second order and two boundary conditions

\[
V(0,s) = f(s) = \int_0^\infty e^{-st} \varphi(t) dt \quad (28')
\]

\[
V(\infty,s) < \infty
\]

in which \( s \) appears as a parameter. The incorporation of the initial condition (27) into the differential equation (26') is an essential feature of the method and is obtained at once from the definition of the Laplace Transform:

\[
\mathcal{L}\{U_t,s\} = \int_0^\infty e^{-st} U_t dt = e^{-st} U(x,t)|_{t=0}^\infty + sV(x,s) = -\Phi(x) + sV(x,s)
\]

The solution to the problem consisting of (26') and (28') is easily expressed in any of several forms, the most suitable one for present purposes being

\[
V(x,s) = f(s) e^{-\frac{\sqrt{s}x}{k}} - \int_0^\infty \frac{\Phi(\xi)}{k} G(x,\xi) d\xi \quad (30)
\]

where the Green's Function

\[
G(x,\xi) = \begin{cases} 
-\frac{\xi m e^{-mx}}{m} & \text{for } x < \xi \\
\frac{\xi m e^{-mx}}{m} & \text{for } x > \xi 
\end{cases} \quad m = \sqrt{\frac{k}{s}}
\]
The correctness of the solution (30) is easily established by straightforward differentiation; the preference for the form (30) is connected with its suitability to generalization and the manner in which the initial value \( \xi(x) \) is exhibited as the coefficient of the term \( G(x, \xi) \) which does not depend on initial conditions. More systematic introduction of the Green's Function for diffusion processes will be considered later and its further properties will be discussed.

To express the solution in terms of the concentration \( U(x, t) \) rather than its Laplace Transform \( V(x, \xi) \) one uses the convolution theorem and tables of transforms, for greatest simplicity. After some minor manipulations (30) then becomes

\[
U(x, t) = \frac{x^2}{2\sqrt{\pi k t}} \int_0^t \frac{e^{-\frac{x^2}{4k(t-\tau)}}}{(t-\tau)^{3/2}} \phi(\tau) d\tau + \int_0^\infty \frac{-\phi(\xi)}{2\sqrt{\pi k t}} \left\{ e^{\frac{(\xi-x)^2}{4kt}} - e^{\frac{(\xi+x)^2}{4kt}} \right\} d\xi
\]

which is readily interpreted in terms of diffusion poles distributed in time for \( t > 0 \) and of strength proportional to the value at the wall \( x = 0 \) plus a spatial distribution of instantaneous sources (at \( t = 0 \)) which satisfy the initial condition (27).

**Diffusion in Convection Flow**

The differential equation (4) of the diffusion process was derived by considering the case of zero relative mean velocity between the diffusing substance and its surrounding medium. The presence of a superimposed mean flow introduces a new term into the differential equation. Considering a three-dimensional element of dimensions \( \delta x, \delta y, \delta z \) and the mean flow velocities of the particles by \( \nabla (V_x, V_y, V_z) \) the net rate of efflux of concentration per unit time is clearly

\[
\left\{ \frac{\partial}{\partial x} \left[ -k \frac{\partial U}{\partial x} + UV_x \right] + \frac{\partial}{\partial y} \left[ -k \frac{\partial U}{\partial y} + UV_y \right] + \frac{\partial}{\partial z} \left[ -k \frac{\partial U}{\partial z} + UV_z \right] \right\} \delta x \delta y \delta z
\]

so that the diffusion equation becomes

\[
\frac{\partial U}{\partial t} = \text{div}(k \text{grad} \, U) - \text{div}(U \nabla V)
\]
in vector notation when the diffusion coefficient is assumed to be constant, or

$$\frac{\partial U}{\partial t} = k \Delta U - \text{div}(U \nabla)$$  \hspace{1cm} (4')

If axes are oriented so that \( \chi \) is in the direction of the gravity force and if, moreover, all particles are assumed to be of the same size so that each has the same (constant) velocity of free-fall \( \nabla_x = -V, \nabla_y = \nabla_z = 0 \), then (4') becomes, for one-dimensional diffusion

$$\frac{\partial U}{\partial t} = k \frac{\partial^2 U}{\partial x^2} + \nabla \frac{\partial U}{\partial x}$$  \hspace{1cm} (32)

An important simplification of (32) is made by a simple change of variables. Introducing

$$W(x,t) = U(x,t) e^{\frac{V}{2k}(x + \frac{V}{2}t)}$$  \hspace{1cm} (33)

(32) becomes

$$\frac{\partial W}{\partial t} = k \frac{\partial^2 W}{\partial x^2}$$  \hspace{1cm} (3)

the differential equation of one-dimensional diffusion in the absence of an external force field.

The transformation (33) can be deduced in the following manner.

One first observes that the first-derivation term \( \nabla \frac{\partial U}{\partial x} \) can be eliminated since the equation contains terms with only constant coefficients and has two independent variables \( x \) and \( t \). If there were no time dependence an exponential transformation would replace the first derivative term by a term proportional to \( U \) itself; the presence of the time variable provides just sufficient additional freedom to permit the reduction to equation (3). The precise form of equation (33) is obtained by considering the transformation

$$W(x,t) = U(x,t) e^{ax + bt}$$

and determining the constants \( a \) and \( b \) by the condition that the coefficients of \( W \) and \( W_x \) vanish in the resulting equation.

Several general observations about the reduction of (32) to (3) are in order. The elimination of the first-order term \( \frac{\partial U}{\partial x} \) does not change the fundamental characteristics of the solutions, since it is
not a highest-order term, whereas both of the other terms are:

\[ \frac{\partial u}{\partial t} \] is the highest order term with respect to the variable \( t \), \( k \frac{\partial^2 u}{\partial x^2} \)

with respect to \( x \); the removal of either of these terms is accompanied by drastic changes in the properties of solutions of the equation, including the loss of one initial or boundary condition. The reduction to equation (3) is nonetheless of great importance since it identifies each sedimentation problem described by (32) and suitable auxiliary conditions, with a simple diffusion problem with altered auxiliary conditions. Consequently the fundamental properties of solutions of the equation of diffusion (the existence and significance of diffusion poles, for example) apply to the more general physical phenomenon in which an external force field is present.

The distribution of sediment has been determined by Chandrasekhar for the case in which a unit quantity of sedimentary particles, per unit area in the \( y-z \) plane, is initially (at \( t = 0 \)) concentrated in the plane \( x = x_0 \). The problem is then to find the concentration \( u(x,t) \) which satisfies the partial differential equation (P.D.E.) and the initial condition (I.C.)

\[
P.D.E. \quad \frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} + v \frac{\partial u}{\partial x}
\]

I.C. \[ u \rightarrow \delta(x-x_0) = \begin{cases} 1 & x = x_0 \\ 0 & x \neq x_0 \end{cases} \]

and also a boundary condition (B.C.) expressing the fact that all particles are reflected at the floor of the vessel,

\[
B.C. \quad k \frac{\partial u}{\partial x} + v u = 0 \quad \text{at } x = 0 \quad \text{for all } t > 0
\]

Letting

\[ W = U e^{\frac{\sqrt{v}(x-x_0)}{2k} + \frac{v^2}{4k^2} t} \] (33')

the P.D.E. becomes

\[ \frac{\partial w}{\partial t} = k \frac{\partial^2 w}{\partial x^2} \]
the I. C. and B. C. become

I. C. \[ W \rightarrow \delta(x-x_0) \quad \text{as} \quad t \rightarrow 0 \]

B. C. \[ k \frac{\partial W}{\partial x} + \frac{V W}{2} = 0 \quad \text{at} \quad x = 0 \quad \text{for all} \quad t > 0 \]

The problem is therefore reduced to a standard one in the theory of heat conduction, and the solution is given by

\[
W = \frac{1}{\sqrt{\pi}kt} \left\{ \exp\left[ -\frac{(x-x_0)^2}{4kt} \right] + \exp\left[ -\frac{(x+x_0)^2}{4kt} \right] \right\} \frac{\sqrt{\pi}}{k} \exp\left[ \frac{V^2t}{4k} - \frac{V(x+x_0)}{2k} \right] \int_{-\infty}^{\infty} \exp(-x^2)dx
\]

which can be written at once in terms of the concentration \( \bar{U} \) by means of (33)!

**The Use of Green's Functions in Connection with Diffusion Problems**

The presence of boundaries which confine a diffusion medium presents difficulty in obtaining a representation of the concentration distribution in time and space. A technique which is of great value in dealing with such problems involves the concept of Green's Functions. These functions furnish insight into the manner in which general solutions are obtained from distributions of the fundamental diffusion poles, and, therefore, merit a somewhat detailed discussion.

To orient ideas our consideration returns first to the diffusion process in a one-dimensional homogeneous medium of unlimited extent in both directions. In this case we have previously found that the concentration is given by

\[
U(x,t) = \frac{1}{2\sqrt{\pi kt}} \int_{-\infty}^{\infty} \bar{\Phi}(\alpha) e^{-\frac{(\alpha-x)^2}{4kt}} d\alpha
\]  

which satisfies the diffusion equation (3) and initial condition

\[ U(x,0) = \bar{\Phi}(x) \]

To obtain the solution when there is one boundary situated at \( x=0 \), say, the simplest "reflection" yields the solution. If there is an absorber at \( x=0 \) which removes the dilutant entirely so that \( U(0,t)=0 \), then it is clear that this condition is satisfied by putting, for \( x<0 \), \( \bar{\Phi}(x) = -\bar{\Phi}(-x) \) in (5), for if \( \bar{\Phi} \) is continuous then it must vanish at \( x=0 \).
Hence (5) becomes

\[ U(x,t) = \int_0^\infty \Phi(\alpha) G \, d\alpha \]

where

\[ G = \frac{i}{2\sqrt{\pi kt}} \left[ e^{-\frac{(\alpha-x)^2}{4kt}} - e^{-\frac{(\alpha+x)^2}{4kt}} \right] \]

This Green's Function \( G \) may be interpreted physically. It plainly represents the sum of the effects of a diffusion pole (or source) of unit strength situated at \( \chi = \alpha \) and acting at \( t = 0 \), plus the effect of a second pole of opposite sign situated at \( \chi = -\alpha \). The sum is therefore an odd function of \( \chi \) which consequently vanishes at \( \chi = 0 \). The solution is then the resultant of a distribution along the \( x \) axis of such poles, of strength \( \Phi(\alpha) \) at \( \chi = \alpha \) on the positive \( x \)-axis and of such strength for negative \( \chi \) that the condition of vanishing at \( \chi = 0 \), i.e., \( U(0,t) = 0 \), is identically satisfied. A similar procedure suggests itself for the process in which the boundary is an opaque wall through which dilutant may not pass. In this case it is seen from previous considerations that the condition to be satisfied is that \( \frac{\partial U(0,t)}{\partial x} = 0 \).

This means that \( \frac{\partial U}{\partial x} \) should be an odd function of \( x \) (in contrast with the case just discussed in which \( U \) itself is odd in \( x \)), or \( U \) is even in \( x \). For a pole at \( \chi = \alpha \), therefore, the corresponding reflected pole at \( \chi = -\alpha \) is of the same sign. The process in which there is partial absorption at the boundary is slightly more complicated in detail; in this case the boundary condition is of the form \( \frac{\partial U(0,t)}{\partial x} + kU(0,t) = 0 \) which may be regarded as intermediate between the two limiting cases just discussed which correspond to \( h = \infty \) and \( h = 0 \), respectively.

The great value of the concept of reflections of fundamental solutions and their combinations into Green's Functions is the suitability to generalization. The one-dimensional diffusion process may be treated when the medium is of finite extent, this is, when there are two boundaries present. It will be seen that in this case each diffusion pole must be reflected not once, but infinitely many times in order to satisfy boundary conditions. The method is also useful for diffusion in two and three dimensions.

A particularly elegant form is assumed by the Green's Function for the one-dimensional bounded diffusion process. Since important physical ideas are also elucidated thereby, the analytical treatment
will be considered next.

The starting point is the seemingly special and artificial process in which there is periodicity in the space coordinate; i.e., \( U(x+1, t) = U(x, t) \) for all \(-\infty < x < \infty \), and the initial distribution is symmetric about \( x=0 \). In this case

\[
\Phi(x) = \sum_{n=0}^{\infty} A_n \cos(2\pi nx)
\]

\[
A_0 = \frac{1}{\sqrt{2}} \int_{-1/2}^{1/2} f(x) \, dx, \quad A_n = 2 \int_{-1/2}^{1/2} f(x) \cos 2\pi nx \, dx
\]

since, on account of the assumed symmetry, all sine terms are absent. Clearly the solution is obtainable by multiplying each term in the sum by \( e^{-(2\pi n)^2 kt} \), since

\[
U_n = \cos(2\pi nx) e^{-(2\pi n)^2 kt}
\]

clearly satisfies equation (3). Specializing the initial distribution so that a unit pole is situated at \( x=0 \) (and hence also at \( x=\pm 1, \pm 2, \text{ etc.} \)), that is, so that

\[
\Phi(x) = 0 \quad \text{for} \quad x \neq 0 \quad \text{but} \quad \int_{-s}^{+s} \Phi(x) \, dx = 1, \quad s>0
\]

then \( A_0 = 1, A_1 = A_2 = \cdots = 2 \) and the solution is

\[
U(x, t) = 1 + 2 \sum_{n=1}^{\infty} e^{-4\pi^2 n^2 kt} \cos(2\pi nx)
\]

which is the theta function of Jacoby. In the more usual notation, letting \( \tau = 4\pi^2 k t \), this becomes

\[
\mathcal{G}(x| \tau) = 1 + 2 \sum_{n=1}^{\infty} e^{i n^2 \pi \tau} \cos(2\pi nx)
\]

It will be seen that this notation is convenient for expressing the Green's Function in the example which is considered next.

It is now possible to obtain in simple form the solution of the problem of diffusion in a medium bounded at both ends, extending from \( x=0 \) to \( x=+l \), say. For simplicity consider that absorbers are situated at both walls, so that \( U(0, t) = U(+l, t) = 0 \).
The Green's Function can now be constructed by following a purely intuitive argument. The solution

\[ U(x,t) = \int_0^l \Phi(x) G \, d\xi \]

is written in terms of the Green's Function \( G \) which is interpreted in terms of a diffusion pole and its reflections in the following manner:

A unit positive diffusion pole, \( A \), situated at \( x = \xi \) must clearly be reflected by another, \( B \), with negative sign at \( x = -\xi \) in order that \( U(0,t) = 0 \); \( B \) requires a reflection, \( C \), with opposite (positive) sign at \( 2l - \xi \) in order to satisfy the condition \( U(l,t) = 0 \); reflections are likewise required by \( C \). The pole \( A \) also requires a negative reflection \( K \) at \( 2l - \xi \) in order to satisfy \( U(-\xi,t) = 0 \); \( K \), in turn, requires further reflections but it is already clear that the distribution in the interval \(-l < x < l\) is continued in the interval \(-l < x < 3l\); it

is easily shown that the periodic continuation extends indefinitely on both sides of the interval \(-l < x < l\). This permits the Green's Function to be written at once as

\[ G = \mathcal{G}\left( \frac{x - \xi}{2l} \right) - \mathcal{G}\left( \frac{x + \xi}{2l} \right) \]

so that

\[ U(x,t) = \int_0^l \Phi(x) \left[ \mathcal{G}\left( \frac{x - \xi}{2l} \right) - \mathcal{G}\left( \frac{x + \xi}{2l} \right) \right] d\xi \]

It can be shown that Fourier's Method yields the same result in this case. That (28) is the correct solution to the mixed initial-boundary value problem can be established by means of showing its identity with any other representation and by observing the uniqueness of such solutions.
The equivalence of all representations of the solution notwithstanding, one or another representation may be of advantage if one is particularly interested in the earlier stages of the diffusion process, or the later, for example. The rapid convergence of the theta function for large values of $\gamma$ (or of $kt$) makes the representation (28) most suitable for the later stages of the process.

A remarkable relation between certain solutions of the wave equation and solutions of the equation of diffusion has been pointed out by Bateman. It states that if $u(x,y,z,t)$ is a solution of the wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

which satisfies the initial conditions at $t = 0$:

$$u = G(x,y,z)$$
$$\frac{\partial u}{\partial t} = 0$$

then

$$v = \frac{2}{\sqrt{\pi}} \int_0^\infty u(x,y,z,2s\sqrt{\gamma}) e^{-s^2} ds$$

is "usually" a solution of the diffusion equation

$$\frac{\partial V}{\partial t} = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

in three dimensions and satisfies the condition, when $\gamma = 0$:

$$V = G(x,y,z)$$

The diffusion equation (4) in which the coefficient $k$ appears can easily be put into the form (4'') by a simple transformation performed on the time variable. A rigorous statement has not been given of the conditions under which the solutions are related in the above manner. The special nature of the conditions under which the solutions of (4'') and (34) are related restricts the usefulness of the theorem considerably; the advisability of investigating the possibility of generalizing it suggests itself.

It is to be noted that the theorem is of especially timely interest to those who are concerned with solutions of the diffusion equation, in view of the great progress made in recent years in developing new methods for finding solutions of the wave equation. It thus appears that the powerful methods of Hadamard and Riesz may be brought to bear on the diffusion equation. For a discussion of these methods, see, for example, Webster and Baker and Copson, p 54.
PART II

TURBULENT DIFFUSION

The simplest application of the equation of molecular heat conduction to problems of turbulent diffusion is obtained by formally replacing the coefficient in Eq. (3) of Part I by a different one which is more appropriate for a turbulent process. The suggestion for taking this step is the belief that the random behavior of a turbulence eddy is in some sense analogous to the behavior of a gaseous molecule, and the ultimate justification is the suitability of the resulting theory for describing and predicting diffusion phenomena. The choice of a diffusion coefficient in this approach is of course determined as the value which best suits the theory for a given set of measurements.

In many cases satisfactory results are obtained in this manner, and it is no weakness of the theory which requires different coefficients for different diffused quantities (momentum, heat, particles, etc.) and coefficients which may depend also on space coordinates, time, and even on the concentration. One may attribute this necessity to the fact that different "mixing lengths" are appropriate in each case, or otherwise. In some important cases, however, a theory cannot be formulated in this manner, no matter how complex a form is ascribed to the diffusion coefficient, and doubt is then cast on the fundamental hypothesis that the mechanism of turbulent eddies is at all analogous to a collection of molecules.

In order to examine the significant features of the entire theory-by-analogy, and to point out refinements which have been proposed, it is convenient first to retrace the usual steps in the derivation of the diffusion equation, examining each crucial step critically later.

For present purposes it is suitable to consider turbulent diffusion processes occurring in air; specialization to the case of liquids is then accomplished more readily than the reverse step. For further simplicity the discussion will be confined to the diffusion of heat; many of the principles which are involved could, of course, be illustrated by considering other quantities instead.

The equation of atmospheric turbulent heat conduction expresses a heat budget for an element of air at a given level of vertical coordinate, $Z$, say, measured positive upward: the local temperature increase is determined by the surplus of heat entering the element compared to that which is being removed by eddies. More precisely it is supposed that an element of air which is initially a normal and average sample of its surroundings is displaced adiabatically by the action of turbulence. When it comes to rest, generally at a different elevation, heat transfer occurs as a process at constant pressure. Denoting the
vertical eddy velocity by \( w \), the density by \( \rho \), the temperature by \( T \),
the upward flux of heat across unit horizontal area in unit time is

\[
\overline{\rho w c_p T}
\]

where the mean value, indicated by the bar, is required because of the fluctuating character of the quantities which are involved and represents a time- and space-averaging over a horizontal plane.

The actual fluctuations can be exhibited by writing \( T = T_0 + T' \),
where the quantity with subscript is the "steady" value of temperature and for simplicity will be assumed to depend on the vertical coordinate only; the quantity with the prime superscript is the local temperature anomaly. Adopting a similar convention with regard to the density, and taking the specific heat at constant pressure, \( c_p \), to be a constant, the above expression becomes

\[
\overline{\rho w c_p T} = \overline{\rho w c_p (T_0 + T')} = \overline{\rho w c_p T'} = \rho_0 c_p \overline{wT'}
\]

The vanishing of the term containing \( T_0 \) is a result of the fact that is assumed to depend on \( z \) only, and the mean vertical transport of mass \( \bar{\rho} w \) must vanish at a given level. The approximation involved in writing \( \rho_0 \) for \( \rho \) is that the variations \( \rho' \) are assumed very small compared to the density itself, and the existence of an appreciable flux depends ultimately on the existence of a correlation between \( W \) and \( T' \).

The net flux into a layer of thickness \( dz \) is then

\[
-\frac{\partial}{\partial z}(\rho_0 c_p \overline{wT'})dz
\]

and the heating accomplished thereby is

\[
\frac{d}{dt}(\rho_0 c_p T_0 dz)
\]

so that

\[
\rho_0 c_p \frac{dT_0}{dt} = -\frac{\partial}{\partial z}(\rho_0 c_p \overline{wT'})
\]

\[(1)\]

under the present assumptions.

The most difficult step of relating the fluctuating component to known quantities, which is in fact the central problem of all work in turbulence, is usually accomplished by making the assumptions described above. More explicitly, if the vertical distance travelled by an eddy before it comes to rest at \( z \) is denoted by \( \ell \), then the temperature at \( z - \ell \) is

\[
T_0(z - \ell) = T_0(z) - \ell \frac{\partial T_0(z)}{\partial z} + \frac{\ell^2}{2} \frac{\partial^2 T_0(z)}{\partial z^2} + \ldots
\]

if the
temperature distribution is a smooth enough one. If, moreover, the length \( l \) is sufficiently small, the third and subsequent terms are negligible. The adiabatic cooling caused by expansion of the rising element is \( l \frac{\partial T_0}{\partial z} \) where \( \Gamma \) denotes the (constant) adiabatic lapse rate, equal to 1°C. per hundred meters in the atmosphere, approximately. Consequently the eddy, when it reaches elevation \( z \) it has temperature

\[
T_0(z) + T' = T_0(z) - l \frac{\partial T_0}{\partial z} - \Gamma l \quad \text{and} \quad T' = -l \left( \frac{\partial T_0}{\partial z} + \Gamma \right)
\]

so that Eq. (1) becomes

\[
\frac{dT_0}{dz} = \frac{\partial}{\partial z} \left[ \frac{l}{\Gamma} \left( \frac{\partial T_0}{\partial z} + \Gamma \right) \right] = \frac{\partial}{\partial z} \left[ K \frac{\partial T_0}{\partial z} \right]
\]

when the coefficient of turbulent heat conduction is defined as

\[
K = \frac{l}{\Gamma} W
\]

When \( K \) is considered to be a constant, and the derivative

\[
\frac{dT_0}{dt} = \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + w \frac{\partial}{\partial z}
\]

is approximated by the first term, one obtains

\[
\frac{\partial T_0}{\partial z} = K \frac{\partial^2 T_0}{\partial z^2}
\]

which is identical to the classical equation of heat conduction in its simplest form. The last approximation is strictly true when the component velocities of the medium, \( U, V, W \) all vanish.

The precise significance of the length \( l \) is somewhat obscured by the idealizations which have been made (by neglecting variations of density, for example) and at this point it may be regarded empirically as that length which, combined with the vertical component of velocity \( W \), yields the appropriate value of the coefficient \( K \).

Numerous objections have been raised with regard to the equation (3) when it is applied to turbulent phenomena, and some of these criticisms will be briefly indicated at this point.

The model of a turbulence eddy which has been visualized in the above derivation introduces a characteristic or mixing length through which it is supposed that the eddy first moves without exchanging its surplus or defect of heat, and only then does mixing start. This very greatly idealized model demands first consideration by reason of the resulting mathematical simplicity of the differential equation (3). But it is clear that, even if the history of an eddy is of the kind assumed, there is no reason to expect that a single length \( l \) describes all eddies equally well. The nature of external disturbances, the proximity of boundaries, the extent of energy dissipation, and other factors may, moreover, be expected greatly to influence the distance traversed by such an eddy. All of these factors and others are effectively lumped together when the value of \( K \) is assigned by properly interpreting
measurements of concentration in a controlled experiment. Under similar circumstances the value previously determined can then be used to predict concentrations when the experiment is repeated. It is of course not useful under sufficiently different circumstances and one weakness of the theory is that it gives no indication of the tolerable variations of physical parameters, much less the form of dependence of the resultant change on parameter variation.

A direct consequence of the geometry of the assumed mechanism is that the coefficient $K$ is positive: an upward velocity (positive $w$) carries an element through a (positive) change of elevation $L$, and a downward-moving eddy corresponds to a negative $L$. In both cases $w$ and $L$ are of the same sign and consequently have a non-negative product.

The values of $K$ determined experimentally are found to be positive and furnish a theory which is in many respects satisfactory. In the realm of atmospheric phenomena, however, even in processes which are dominated by turbulent effects, there are great difficulties in application of the theory. One is the almost universal increase in potential temperature* with elevation. Such a stratification should be destroyed by the effects of turbulence within a few hours, according to the present theory, yet it is observed to be persistent for much longer periods of time over large areas: it is in fact characteristic of the greatest part of the atmosphere at all times.

This paradox and others led Priestley and Swinbank to a modification of the theory outlined above, based on one point of fallacious logic in the classical theory. The assumption is made that the eddy originated at a level $Z - L$ where it was a normal sample of its environment possessing the average value for that level of the transferred quantity, which is heat in this case. The contradiction is that if the eddy is a normal sample of its environment it is not more likely to have the average temperature than some other value; the other values are indeed present because of the very existence of turbulence. Alternatively, while the assumed normal sample could be expected to possess average values of physical properties in the earliest stages of the development of turbulence, the same is not to be expected at later times. The consequence of ignoring this contradiction is that the theory considers only indiscriminate and random mixing (analogous to diffusion of molecules) but fails to account for an actual discrimination in the direction of motion on the part of a moving eddy: an

*The potential temperature is a convenient reference value, defined as the temperature which would be assumed by a parcel of air when it is carried by adiabatic processes to a standard pressure.
ascending eddy which is warmer than its surroundings is more likely to continue rising.

These ideas are made more exact by supposing that the normal sample of its environment has a temperature anomaly $T''$, so that its temperature is $T_0(z-\xi)+T''$ when it is at rest with respect to its surroundings. Then its temperature at level $z$ is

$$T_0(z) + T' = T_0(z-\xi) + T'' - |\xi|$$

so that according to previous assumptions about $T_0(z)$

$$T' = -Q\left(\frac{\partial T_0}{\partial z} + \Gamma\right) + T''$$

and the equation corresponding to (2) will be

$$\frac{dT_0}{dz} = \frac{\partial}{\partial z} \left\{ K\left(\frac{\partial T_0}{\partial z} + \Gamma\right) - wT''\right\}$$

(4)

The first term in brackets is the usual one which represents transfer of heat from high potential temperature to regions of low potential temperature. The second term is the result of the additional consideration that an eddy may originate with a temperature different from the average of its surroundings. If an element warmer than its surroundings (so that $T''>0$) tends to rise, so that $w>0$, the product is positive; it is likewise positive for a cooler element. Hence the term $wT''$ is essentially positive. It represents the correlation of the vertical velocity at level $z$ with the excess temperature of the element when it was last at rest. Therefore, the second term in brackets in equation (4) tends to cancel the effect of the first. It has been conjectured that the establishment of this balance may be the reason why turbulence does not succeed in destroying the upwardly increasing potential temperature. It has been suggested by Priestley and Swinbank that the new effect, which they label the buoyancy effect, may not only decrease the mechanical effects represented by the first term in brackets in (4), but may be greater than it; this would correspond to negative values of $K$. Such values are considered unlikely by other workers in the field of diffusion problems, and in this connection it has been pointed out by Dr. H. J. Stewart* that in the measurements which are thought to support the conclusion that $K<0$ is possible, distinction is not made between coefficients for momentum and those for heat. These experimental difficulties are recognized by Priestley and Swinbank who point out, in fact, that all values of $K$ previously obtained may need correction before they can be used for comparison with the new theory.

* Private communication.
An equation similar to (4) has been given independently by Ertel, representing fundamentally similar mechanism, but with different conclusions from those indicated above.

To consider other aspects of the theory-by-analogy, the buoyancy effect just discussed will be neglected, and attention will be focused on the arguments which led to the derivation of equation (3) for the diffusion of heat. It was found that the temperature anomaly $T'$ is proportional to the temperature gradient, and the proportionality factor was taken to be a simple length. If one considers other diffused quantities, however, similar assumptions and correspondingly simple results appear less plausible in some instances. For, proceeding more generally, suppose $Z$ denotes the quantity whose diffusion is being studied, and let $\chi$ be the amount of $Z$ per unit mass of air. The diffusion equation was seen from first principles to be

$$\frac{\partial}{\partial t}(\rho \chi) = -\frac{\partial}{\partial z}(\text{upward flux});$$

further progress requires an assumption about the form of the flux. When an eddy travels from one level to another carrying with it the amount $\chi$ corresponding to the original level, the new level acquires a quantity proportional to the difference at the two levels. In particular, if $\chi$ is the same at both levels, the net flux at the second level is zero. It thus seems reasonable to assume

$$\text{upward flux} = -C \frac{\partial \chi}{\partial z},$$

where $C$ may depend on the amount of air crossing the surface across which the flux is measured. It may also depend on $Z$, on $\chi$, and on $\frac{\partial \chi}{\partial z}$, but must remain finite when $\frac{\partial \chi}{\partial z} = 0$ since the net flux must then be zero. Dependence on $\chi$ and its derivatives must be ruled out of practical consideration at once since if it were retained, the resulting equation would be non-linear in the unknown quantity $\chi$ and consequently cannot be treated except in very special cases. In this way one is compelled in formulating a theory to consider only mathematically tractable problems; these of course may have little relation to the physical processes which occur in turbulent eddy diffusion. Consequently it seems that to whatever extent the diffusion coefficient depends on concentration, mathematical difficulties are insuperable at present.

But even when the dependence of $C$ on $\chi$ is ignored, assuring linearity of the partial differential equation, great practical difficulties remain. In general the resulting equations are understood only to the extent that
circumstances are known under which solutions exist and possess familiar properties (the same is not true of non-linear equations). But only the simplest functional forms of $C(z)$ lead to equations which are tractable. The solutions, taken from classical heat conduction theory correspond to $C = \text{const}$, for example. In the case of turbulent diffusion near boundaries it is thought that the dependence should be logarithmic. The practical difficulty of solving the diffusion equation in this form is so great that no solutions are known to exist. W.G.L. Sutton, Jaeger, and others, using Laplace Transform and other methods, have succeeded in obtaining solutions in some cases where $C(z)$ contains only powers of $z$. These solutions involve great mathematical complication and, when the logarithmic functions are approximated by powers, represent the highest point in the development of the theory. The solutions found include two- and three-dimensional cases in which the diffusion coefficient depends on the vertical coordinate only.

The proximity of a boundary and the nature of its surface are known to exert a strong influence on both the qualitative and quantitative features of turbulent diffusion processes. The lower layers of the earth's atmosphere have been studied most thoroughly in this regard. Extensive research centered in England in recent years has greatly improved the understanding of physical processes in the atmosphere near solid and liquid plane surfaces. The theory developed by O.G. Sutton, Rossby and Montgomery, Sheppard, Calder and others, guided and confirmed by extensive measurements made by Deacon, Pasquill, and others has clarified the nature of turbulent diffusion throughout the friction layer of the atmosphere and its implications for diffusion. The occurrence of a relatively thick turbulent friction layer may likewise be anticipated in sediment-laden flows; in these flows the adaption of diffusion theory to conditions of "aerodynamically rough" flows is of interest.

A formulation by O.G. Sutton of the theory of turbulent diffusion of matter, consisted of a mathematical treatment which was found to be satisfactory in many respects, but failed to provide results of the correct absolute magnitude. It was subsequently conjectured that the theory, which was based on earlier work of Sutton's, was strictly applicable only to flow over smooth surfaces. The smoothness referred to will be defined more precisely presently, and its implications for diffusion will also be seen. Two rather different approaches to an improved theory have subsequently been developed by Calder and by Sutton. Calder's treatment is a direct application of a theory proposed by Nikuradse and Schlichting for flow in pipes and wind tunnels, and his work has led to solutions of important two-dimensional diffusion problems which are in excellent agreement with experiment. Sutton has modified his version of the statistical theory of turbulence, taking account of roughness. This has permitted solutions of both two-and
three-dimensional problems which are in entirely satisfactory agreement with experiments in conditions of small temperature gradient. Sutton's treatment has clarified the fundamental concepts involved by a critical examination of the basic relations employed in the theory of rough flow. It suits present purposes to review the main features of this work, starting from first principles.

Flow near smooth surfaces will be briefly considered first in a manner which permits direct comparison with the case of rough-surface flow. By choosing an \( x \)-axis in the local direction of a steady mean flow, the mean velocity \( \overline{u} \) is in the direction of this axis. If we consider, moreover, a flow which is two-dimensional, for simplicity, then \( \overline{u} \) depends only on the distance \( z \) above the surface. When the pressure gradient in the direction of flow \( \frac{\partial p}{\partial x} \) is a small quantity compared with other flow quantities of the same dimensions, then the shearing stress \( \tau \) per unit area does not vary with height. This is generally true in shallow layers near the surface, at least, and is easily seen by considering the forces acting on an element fluid. The equality of forces in the \( x \)-direction yields at once

\[
p dz - (p + \frac{\partial p}{\partial x} dx) dz + (\tau + \frac{\partial \tau}{\partial z} dz) dx - \tau dx = 0
\]

or

\[
\frac{\partial \tau}{\partial z} = \frac{\partial p}{\partial x}
\]

Hence, if the pressure gradient vanishes the shearing stress does not depend on \( z \) and so has the same value at all heights, equal to its value \( \tau_0 \) at the surface. The shearing stress is made up chiefly of the turbulent or Reynolds stress \( -\rho u'w' \) where \( \rho \) is the fluid density, \( u', w' \), are the turbulent velocity components. Prandtl related the turbulent velocity components to the mean flow by putting

\[
\tau = \rho \ell^2 \left( \frac{d \overline{u}}{dz} \right)^2
\]

which introduces the so-called mixing length \( \ell \) which is in fact defined by this equation. Consequently it is unnecessary at this point to consider the possible physical significance of the parameter \( \ell \) (which has been discussed previously and clearly has the dimension of length), or to see the motivation for relating the turbulent velocity components to the mean flow in the particular manner indicated. No empiricism has been introduced up to this point, but further progress in the theory requires some knowledge of \( \ell \). Such information, strictly speaking, will not be available until the mechanism of
turbulent eddies is known, and this circumstance requires as assumption to be made about the functional form of \( \ell \) if any progress is to be made on a diffusion theory. Before discussing various possibilities, it is convenient to introduce one further parameter.

In turbulent flows the eddy component \( u' \) is found to be roughly proportional to the local mean flow; consequently the apparent shear stress \( \tau \) is approximately proportional to the square of the mean velocity. A widely used quantity which makes this proportionality exact is the "friction velocity" \( u_* \), defined by the relation

\[
u_* = \sqrt{\frac{\tau}{\rho}} = \sqrt{\frac{\tau_0}{\rho}}
\]

By supposing a mechanism of turbulence which is in some sense similar to the behavior of gaseous molecules, it seems plausible to assume (in crude analogy to kinetic theory) that the quantity \( \ell \) depends on the gradient of the mean velocity but not on the magnitude of this velocity itself. The problems of determining the velocity profile is then one in which only the quantities \( \ell, Z, \rho, u_*, \) and \( \mu \) (the viscosity coefficient) appear. Since the only dimensionless ratios which can be formed from these quantities are \( \ell/Z \) and \( \rho u_*/\mu \), it follows that the variables must appear in these ratios in a formula for the velocity profile. Before proceeding further it is necessary to consider the properties of flow over a smooth surface. Such flows are characterized by the presence of three distinct zones: (a) In the zone immediately adjacent to the surface, the "laminar sub-layer", there is intense shear and small vertical motion; the flow is laminar and viscosity effects are predominant on account of the great shear. (b) Above this is the turbulent boundary layer with smaller velocity gradient and considerable vertical motion. (c) The region in which the viscous effects are negligible compared with the Reynolds, or apparent, stresses.

It has been found that the flow near a plane surface depends on the degree of roughness of the surface and that irregularities of the surface must be compared with the thickness of the laminar sub-layer. Flows in which irregularities are completely submerged in this sub-layer are known as "aerodynamically smooth", whereas those in which the irregularities protrude through the sublayer are known as "aerodynamically rough" flows. Between these two types there is a transitional regime in which the flow is neither smooth nor fully rough.

It follows that since the laminar sublayer is rather shallow, a surface which is in a smooth flow in the technical sense is also smooth in the ordinary sense. A natural first assumption for the turbulent zone in which molecular viscosity is not important is that \( \ell/Z \) is a constant, \( \ell = kZ \), say,
where \( k \) is a dimensionless number. It then follows at once from the definitions that

\[
\frac{du}{dz} = \frac{u_\infty}{kz}
\]

so that

\[
\frac{\bar{u}}{u_\infty} = \frac{1}{k} \log_{e} z + \text{const.}
\]  

(6)

the constant necessarily being determined by a boundary condition -- a matter of some difficulty in general. Since the friction velocity \( u_\infty \) is of the same order of magnitude as the eddy velocities, it is the most natural reference velocity for the friction layer, and a preferable expression for (6) in dimensionless form can therefore be written as

\[
\frac{\bar{u}}{u_\infty} = \frac{1}{k} \log_{e} \left( \frac{\rho u_\infty z}{\mu} \right) + \text{const.}
\]  

(6')

which is, of course, neither more general or less than (6).

The universal constant \( k \) has been found to have a value of approximately 0.4, and this value is now the usually accepted one. Extensive laboratory tests by Nikuradse and others have shown that for flow in smooth pipes the value of the additive constant in (7) is 5.5, so that

\[
\frac{\bar{u}}{u_\infty} = \frac{1}{0.4} \log_{e} \left( \frac{\rho u_\infty z}{\mu} \right) + 5.5
\]

\[
= \frac{1}{0.4} \log_{e} \left( \frac{\rho u_\infty z}{\mu} \right), \quad \text{approximately}
\]  

(7)

According to this the boundary condition is that \( \bar{u} \) vanishes at \( z = \frac{1}{\rho g u_\infty} \), approximately, and the formula is inapplicable for smaller values of \( z \).

For flow over a rough surface it is to be anticipated that viscosity will play a minor role in determining the total resistance and the velocity profile. Ignoring the viscosity altogether and assuming the velocity vanishes at some height \( Z_0 \), the profile equation becomes

\[
\frac{\bar{u}}{u_\infty} = \frac{1}{k} \left( \frac{z}{Z_0} \right)
\]  

(8)

where now \( Z_0 \), which is termed the "roughness length", must be determined from the experiment. From experiments with pipes roughened by uniform grains of sand \( Z_0 \) is found to be approximately 1/30th of the sand grain di-
ameter; values for other flows have been tabulated by Sheppard and others.

A different treatment given by Rossby and Montgomery who assume that the mixing length depends on the roughness, by putting

\[ l = k(z+Z_0) \]

so that the differential equation for the profile is

\[ \frac{1}{u_*} \frac{d\bar{u}}{dz} = \frac{1}{k} \frac{1}{z+Z_0} \]

thus the velocity gradient is finite at the surface \( z=0 \), and the profile is

\[ \frac{\bar{u}}{u_*} = \frac{1}{k} \log_\epsilon(z+Z_0) + \text{const} \]

and the condition \( \bar{u}=0 \) at \( z=0 \) is satisfied by writing this as

\[ \frac{\bar{u}}{u_*} = \frac{1}{k} \log_\epsilon\left(\frac{Z+Z_0}{Z_0}\right) \] \( (9) \)

The profile given by (9) has the advantage over (8) that it confines the effects of roughness to the layers near the surface, for \( Z_0 \) is generally much smaller than the values of \( Z \) over which (9) is applicable. At relatively great heights (9) is very nearly the same as (8), which is the more usual form.

When the roughness length \( Z_0 \) decreases toward zero, surface irregularities are absorbed by the laminar sub-layer and the surface becomes a smooth one, in the sense of the definition given above. Neither of the profile equations (8) or (9) for rough flow reduces to the smooth flow form (7), however, as \( Z_0 \to 0 \). It is desirable to have one expression for the mean velocity profile which includes flow over any type of surface; rough flow for finite values of the roughness parameter \( Z_0 \), smooth flow for vanishing \( Z_0 \). Sutton has introduced the parameter \( N=\frac{U_*Z_0}{u_*} \) which he denotes the "macroviscosity" and by which the desired unification of the theory is effected. The same parameter is then used in diffusion theory and is, in fact, suggested by Sutton to be the most significant measure of the quality of a flow, more appropriate than the roughness length for problems of turbulent diffusion.

Equation (8) can be written

\[ \frac{\bar{u}}{u_*} = \frac{1}{k} \log_\epsilon\left(\frac{u_*Z}{N}\right) \] \( (8') \)

while (9) is

\[ \frac{\bar{u}}{u_*} = \frac{1}{k} \log_\epsilon\left(\frac{u_*Z+N}{N}\right) \] \( (9') \)
these are the forms of the velocity profile according to the formulations of Prandtl (eq. (8')) and Rossley and Montgomery (eq. (9')). These can be generalized to include the case of aerodynamically smooth flow, by writing

$$\frac{\overline{v}}{u_*} = \frac{1}{K} \log_e \left( \frac{u_* z}{N + v/9} \right) \quad (8'')$$

and

$$\frac{\overline{v}}{u_*} = \frac{1}{K} \log_e \left( \frac{u_* z + N}{N + v/9} \right) \quad (9'')$$

For smooth flows $N = 0$ and these reduce to the forms (7). For fully rough flows $N$ is very much greater than $v$; the profile equations (8'') and (9'') are then indistinguishable from the classical forms (8) and (9).

The treatment of diffusion over rough surfaces can now be considered. In the light of the generalization affected by the introduction of the macroviscosity in the profile equations (8'') and (9''), the analogous extension of diffusion theory appears quite naturally.

For this purpose our point of departure is the form of the correlation coefficient $R(\xi)$ between the eddy velocities which affect a group of particles at times $t$ and $t + \xi$. It is derived by considering that such a correlation must depend in the first instance on the density $\rho$ and viscosity $\mu$ of the fluid, the eddying energy, and the times $\xi$. Since the only dimensionless ratio involving these quantities is

$$\frac{\nu}{W^2 \xi}$$

where $\nu = \mu/\rho$,

and since one anticipates that as $\xi \to 0$, $R \to 1$, and as $\xi \to \infty$, $R \to 0$, the correlation coefficient may be tentatively assumed to be of the form

$$R(\xi) = \left( \frac{\nu}{\nu + W^2 \xi} \right)^n \quad (10)$$

where $n$ is a positive number. When the number $n$ is determined by fitting to the observed velocity profile the power law $\overline{u} = u \left( \frac{z}{z_1} \right)^{(2-n)}$

then (10) is well confirmed for diffusion processes over smooth surfaces. Its direct application to diffusion over rough surfaces has failed to furnish results which are quantitatively correct.

In fully rough flows the kinematic viscosity $\nu$ must be of no im-
portance; previous considerations then suggest that one should write

\[ R(\xi) = \left( \frac{N + \nu}{N + \nu + w^2 \xi} \right)^n, \quad n > 0 \]  

(10')

This generalization has the already familiar properties of suitable reduction in the case of smooth flow (10) and diminishing importance of kinematic viscosity \( \nu' \) for rough flows (in which \( N \gg \nu' \)).

A vertical diffusion coefficient corresponding to the coefficient (10') can be written in the form of a power of \( Z + Z_0 \). For sufficiently simple diffusion problems the exact solution of the resulting diffusion equations can be obtained; approximate solutions are possible more generally. Comparison of the exact solution with observations, for the two-dimensional problem, have been made by Deacon and found to be most favorable. Similarly favorable resulting have been obtained by comparing the generalized form of the diffusion coefficient (10') with direct observations.

It thus appears that for problems of turbulent flow and diffusion near rough surfaces the introduction of the macroviscosity \( N \) marks a definite improvement of the theory. A similar treatment is therefore suggested for problems of particle sedimentation and diffusion.
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